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A REMARK ON MINIMAL LAGRANGIAN DIFFEOMORPHISMS AND THE MONGE-AMPÈRE EQUATION

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We construct a counterexample to a theorem of Jon Wolfson concerning the existence of globally smooth solutions of the second boundary value problem for Monge-Ampère equations in two dimensions, or equivalently, on the existence of minimal Lagrangian diffeomorphisms between simply connected domains in $\mathbf{R}^2$.

In [**6**] Wolfson studied minimal Lagrangian diffeomorphisms between simply connected domains in $\mathbf{R}^2$. He derived several conditions guaranteeing the existence and nonexistence of such maps.

Given two bounded, connected, simply connected domains D_1 and D_2 in $\mathbf{R}^2$ with smooth boundaries ∂D_1 and ∂D_2, Wolfson calls the pair (D_1, D_2) *pseudoconvex* if

$$\min_{\partial D_1} \kappa_1 + \min_{\partial D_2} \kappa_2 > 0, \tag{1}$$

where κ_1, κ_2 denote the curvatures of $\partial D_1, \partial D_2$ respectively relative to the inner normals.

One of his results states that if (D_1, D_2) is a pseudoconvex pair of domains with equal areas, then there is a minimal Lagrangian diffeomorphism $\phi : \overline{D}_1 \to \overline{D}_2$, smooth up to the boundary ([**6**, Theorem 5.1]).

An equivalent statement is that there is a solution $w \in C^\infty(\overline{D}_1)$ of the the second boundary problem for the Monge-Ampère equation

$$\begin{gathered} \det \nabla^2 w = 1 \quad \text{in} \quad D_1, \\ \nabla w \text{ is a diffeomorphism from } \overline{D}_1 \text{ onto } \overline{D}_2, \end{gathered} \tag{2}$$

([**6**, Corollary 6.2]).

These two statements are equivalent in the sense that if ϕ is a minimal Lagrangian diffeomorphism from $\overline{D}_1$ onto $\overline{D}_2$, then after a suitable choice of Lagrangian angle, ϕ can be written as ∇w where w solves (2), and vice versa.

The existence of globally smooth solutions of (2) was proved by Delanoë [**4**] under the assumption that both ∂D_1 and ∂D_2 have positive curvatures. The higher dimensional analogue of Delanoë's result was proved by Caffarelli [**3**] and the author [**5**].

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Here we construct a counterexample to Wolfson's existence result.

THEOREM. *There is a smooth, pseudoconvex pair of domains (D_1, D_2) in $\mathbf{R}^2$ with equal areas such that there is no solution $w \in C^2(\overline{D}_1)$ of (2). Consequently, there is no globally smooth minimal Lagrangian diffeomorphism from $\overline{D}_1$ onto $\overline{D}_2$.*

REMARK. It will be clear from the construction that $\min_{\partial D_1} \kappa_1 + \min_{\partial D_2} \kappa_2$ can be made arbitrarily large.

The key to this construction is the following "obliqueness condition", the proof of which we defer to the end of the paper.

LEMMA. *Let ν_1 and ν_2 denote the inner unit normal vector fields to ∂D_1 and ∂D_2 respectively. Let $w \in C^2(\overline{D}_1)$ be a solution of (2). Then*

$$\nu_1(x) \cdot \nu_2(\nabla w(x)) > 0 \quad \textit{for all} \quad x \in \partial D_1. \tag{3}$$

PROOF OF THEOREM: Consider the spiral γ in $\mathbf{R}^2$ given in polar coordinates (r, θ) on $\mathbf{R}^2$ by $r(t) = 1 + t$, $\theta(t) = t$ for $t \in [0, \infty)$. This is a convex curve that starts at $(1, 0)$ and spirals in the anticlockwise direction around the origin. The curvature of γ relative to the inward pointing normal vector field ν (that is, towards the origin) is bounded between 0 and $3/\sqrt{8} < 2$. For $L > 0$ to be fixed later let

$$\Gamma_L = \operatorname{Image}\left(\gamma\big|_{[0,L]}\right).$$

For small $\varepsilon > 0$, also to be fixed later, let

$$D_{\varepsilon,L} = \left\{x \in \mathbf{R}^2 : \operatorname{dist}(x, \Gamma_L) < \varepsilon\right\}$$

be the ε-neighbourhood of Γ_L. The area of $D_{\varepsilon,L}$ is approximately $2\varepsilon\widetilde{L} + \pi\varepsilon^2$, where

$$\widetilde{L} = \int_0^L \sqrt{1 + (1+t)^2}\, dt$$

is the length of Γ_L. Furthermore, $\partial D_{\varepsilon,L} \in C^{1,1}$ and the curvature of $\partial D_{\varepsilon,L}$ with respect to the inner normal vector field is bounded from below by $-2 + O(\varepsilon)$. By smoothing $\partial D_{\varepsilon,L}$ near the two semicircular parts of its boundary we obtain a connected, simply connected domain $\widetilde{D}_{\varepsilon,L}$ with $\partial\widetilde{D}_{\varepsilon,L} \in C^\infty$ and such that

$$\operatorname{area}(\widetilde{D}_{\varepsilon,L}) = 2\varepsilon\widetilde{L}, \qquad \text{curvature of } \partial\widetilde{D}_{\varepsilon,L} \geqslant -3.$$

We now fix $L \geqslant 4\pi$, so that Γ_L winds around the origin at least twice and the Gauss map of Γ_L covers every point in $\mathbf{S}^1$ at least twice. We then let $B = B_r(0)$ with $r > 0$ chosen so that

$$\operatorname{area}(B) = \operatorname{area}(\widetilde{D}_{\varepsilon,L}).$$

Obviously $r \to 0$ as $\varepsilon \to 0$ for fixed L, so by making ε small enough we can make the curvature of ∂B as large as we want, say greater than 4. Then $(D_1, D_2) := (\widetilde{D}_{\varepsilon,L}, B)$ is a pseudoconvex pair.

We now claim that there is no C^1 diffeomorphism ψ from $\overline{D}_1$ onto $\overline{D}_2$ such that

$$\nu_1(x) \cdot \nu_2\big(\psi(x)\big) > 0 \quad \text{for all} \quad x \in \partial D_1. \tag{4}$$

Suppose on the contrary that there is such a diffeomorphism, and let $\xi = (-r, 0)$, $\eta = (r, 0)$ where $r > 0$ is as above. Let γ^+ and γ^- denote the closed upper and lower semicircles of ∂D_2. Let $\widehat{\xi} = \psi^{-1}(\xi)$, $\widehat{\eta} = \psi^{-1}(\eta)$ and $\widehat{\gamma}^+ = \psi^{-1}(\gamma^+)$, $\widehat{\gamma}^- = \psi^{-1}(\gamma^-)$. From the construction of D_1 and the fact that $\psi|_{\partial D_1}$ is a diffeomorphism from ∂D_1 onto ∂D_2, we see that the Gauss map of at least one of the curves $\widehat{\gamma}^+$ and $\widehat{\gamma}^-$ must cover $\mathbf{S}^1$. If this curve is $\widehat{\gamma}^+$, then (4) implies that $(0, -1)$ does not belong to $\nu_1(\widehat{\gamma}^+)$, while if the curve is $\widehat{\gamma}^-$, then (4) implies that $(0, 1)$ does not belong to $\nu_1(\widehat{\gamma}^-)$. In either case we obtain a contradiction. ∎

Proof of Lemma: This result is proved in [5]. Since the proof is short we include it here for the convenience of the reader.

Let $w \in C^2(\overline{D}_1)$ be a solution of (2). Since D_1 is connected, either $\nabla^2 w > 0$ everywhere or $\nabla^2 w < 0$ everywhere; in either case $\nabla^2 w$ is invertible. Let $h \in C^1(\overline{D}_2)$ be a function such that $h > 0$ in D_2 and $h = 0$, $|\nabla h| = 1$ on ∂D_2. Then $H = h(\nabla w)$ is positive in D_1 and zero on ∂D_1, so

$$\nabla_\tau H = h_{p_k} \nabla_{k\tau} w = 0 \quad \text{on} \quad \partial D_1$$

for any tangential vector field τ on ∂D_1, and

$$\nabla_\nu H = h_{p_k} \nabla_{k\nu} w \geqslant 0 \quad \text{on} \quad \partial D_1,$$

where to simplify notation we write ν rather than ν_1 for the inner unit normal vector field to ∂D_1. Thus

$$\nabla_i H = h_{p_k} \nabla_{ik} w = (\nabla_\nu H)\nu_i \quad \text{on} \quad \partial D_1. \tag{5}$$

Since $\nabla^2 w$ is invertible, we see that

$$\chi := h_{p_k} \nu_k = (\nabla_\nu H) w^{\nu\nu} \quad \text{on} \quad \partial D_1, \tag{6}$$

where $w^{\nu\nu} = w^{ij}\nu_i\nu_j$ and $[w^{ij}] = [\nabla^2 w]^{-1}$. From (5) we also see that

$$h_{p_i} h_{p_k} \nabla_{ik} w = \chi \nabla_\nu H.$$

Combining this with (6) we obtain

$$\chi = \sqrt{w^{ij}\nu_i\nu_j \nabla_{kl} w h_{p_k} h_{p_l}} \quad \text{on} \quad \partial D_1,$$

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which is positive since $w \in C^2(\overline{D}_1)$ with either $\nabla^2 w > 0$ or $\nabla^2 w < 0$ in $\overline{D}_1$. Finally, we observe that $\nabla h|_{\partial D_2} = \nu_2$, so $\chi(x) = \nu_1(x) \cdot \nu_2(\nabla w(x))$ and (3) follows. ∎

REMARK. Brenier [1] has shown that given any two bounded domains $D_1, D_2 \subset \mathbf{R}^n$ with $|D_1| = |D_2|$ and $|\partial D_1| = |\partial D_2| = 0$ (where $|\cdot|$ denotes Lebesgue measure in $\mathbf{R}^n$), there is a convex function u (unique up to constants) such that

$$\det \nabla^2 u = 1 \quad \text{in} \quad D_1, \qquad \nabla u(D_1) = D_2, \tag{7}$$

in a suitable generalised sense, where the equation is interpreted in an integral sense and ∇u is interpreted in the almost everywhere sense. Moreover, Caffarelli [2] has proved the interior regularity of convex Brenier solutions of (7) if D_2 is convex. Thus our example shows that it is the global regularity, not the existence or interior regularity, that may fail under Wolfson's pseudoconvexity condition.

REFERENCES

[1] Y. Brenier, 'Polar factorization and monotone rearrangement of vector valued functions', *Comm. Pure Appl. Math.* **44** (1991), 375–417.

[2] L. Caffarelli, 'The regularity of mappings with a convex potential', *J. Amer. Math. Soc.* **5** (1992), 99–104.

[3] L. Caffarelli, 'Boundary regularity of maps with convex potentials II', *Ann. of Math.* **144** (1996), 453–496.

[4] P. Delanoë, 'Classical solvability in dimension two of the second boundary value problem associated with the Monge-Ampère operator', *Ann. Inst. H. Poincaré Anal. Non Linéaire* **8** (1991), 443–457.

[5] J. Urbas, 'On the second boundary value problem for equations of Monge-Ampère type', *J. Reine Angew. Math.* **487** (1997), 115-124.

[6] J. Wolfson, 'Minimal Lagrangian diffeomorphisms and the Monge-Ampère equation', *J. Differential Geom.* **46** (1997), 335–373.

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