SETS AND SUBSERIES

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Suppose $\alpha_n \neq 0$ for all n. By replacing a set of terms from the series $\sum \alpha_n$ by 0's, we form a subseries of the original series. These subseries can be put in one to one correspondence with the non-terminating binary expansion of the points on the real line segment (0, 1). The point $\xi = .b_1 b_2 \ldots b_n \ldots$ shall correspond to a subseries if and only if $b_n = 0$ whenever the nth term of the original series has been replaced by 0 and $b_n = 1$ whenever the nth term is retained. We can now speak of sets of subseries of the first category, measure zero, etc.

If we have a series $\sum a_n$ whose terms $\{a_n\}$ form a positive monotone decreasing sequence, $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant \ldots$, then it is easy to show by a Dedekind section the existence of an index p such that $\sum a_n{}^q$ converges for $q > p$ and diverges for $q < p$. The object of this note is to show that if the terms of a series are a positive monotone decreasing sequence, the set of subseries that have an index differing from the original series is of the first category.

We first prove a lemma.

LEMMA. *If $\{c_n\}$ is a positive monotone decreasing sequence, $\sum c_n$ diverges,* $\limsup nc_n \neq 0$, *then the set of subseries that converges is of the first category.*

Proof. We can assume that $2n(m)c_{2n(m)} > k$ where $\{n(m)\}$ is a subsequence of $\{n\}$. However, this implies $n(m)c_{n(m)} > \frac{1}{2}k$ and $\nu c_\nu > \frac{1}{2}k$ for $n(m) \leqslant \nu \leqslant 2\,n(m)$. We can associate with this sequence a regular summation method A, defined by

$$t_m = \frac{1}{n(m)} \sum_{\nu=n(m)}^{2\,n(m)} s_\nu \,.$$

Suppose now that $c'_n = c_n$ if the term is retained for the subseries and $c'_n = 0$ otherwise. For any subseries $\sum c'_n$ we shall choose a sequence of 0's and 1's, $\{s'_n\}$, so that $s'_n = 1$ if $c'_n \neq 0$ and $s'_n = 0$ otherwise. There is an evident correspondence between the sequence $\{s'_n\}$ and the point corresponding to the subseries. We now see that if $\sum c'_n$ converges, $\{s'_n\}$ must be A summable to 0. For if $t_{m_\mu} \geqslant \lambda > 0$ for an infinite set $\{\mu\}$, then

$$\sum_{\nu=n(m_\mu)}^{2\,n(m_\mu)} c'_\nu > \lambda\, n(m_\mu)\, c_{2n(m_\mu)} > \lambda \frac{k}{2}$$

and the series diverges.

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However, it has been shown by Hill that the set of points on (0, 1) in the binary expansion corresponding to such a set of 0's and 1's is of the first category.

We are now ready to prove our theorem.

THEOREM. *If $\{a_n\}$ is a positive monotone decreasing sequence, the set of subseries of $\sum a_n$ with a different index is of the first category.*

Proof. The index for the series $\sum a_n$ may be assumed to be 1. For if the index is p, $\sum a_n^p$ will have an index 1 and the set with a different index will be the same for $\sum a_n$ and $\sum a_n^p$.

If $r < 1$, then lim sup $n\, a_n^r > 0$, for if

$$\lim_{n \to \infty} n\, a_n^r = 0,$$

then $a_n = 0(n^{-1/r})$ and for $r < q < 1$, $\sum a_n^q$ would converge, contrary to our assumption. Hence the set of subseries of $\sum a_n^r$ that converges for any r is of the first category.

If a subseries has an index less than 1, then it will belong to the set E_ν of convergent subseries of

$$\sum a_n^{1-1/\nu}, \qquad \nu = 2, 3, \ldots$$

for some ν. The set of all subseries with index less than 1 will be contained in the union of these sets and so will be of the first category.

REFERENCE

1. J. D. Hill, *Summability of sequences of* 0'*s and* 1'*s*, Annals of Math., *46* (1945), 556–62.

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