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A NOTE ON THE ENTIRE CYCLIC COHOMOLOGY OF A FINITE DIMENSIONAL NONCOMMUTATIVE SPACE

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ABSTRACT. We present sufficient conditions for entire cyclic cohomology to reduce to ordinary cyclic cohomology. These conditions are characteristic for finite dimensional (noncommutative) spaces.

1. **Introduction.** Cyclic cohomology is an extension of de Rham cohomology to arbitrary (noncommutative) algebras. This cohomology plays a central role in noncommutative differential geometry [C1] and has found important applications in ordinary differential geometry, see *e.g.*, [CGM]. It is believed that to handle infinite dimensional situations, like the noncommutative structures arising from supersymmetric quantum field theory, a different cohomological setup is relevant. This cohomology, called *entire cyclic cohomology*, was introduced in [C2] and further studied in [JL0], [KKL], [KL] and references therein.

In this note we address the problem of the relationship between entire cyclic cohomology and ordinary cyclic cohomology. We describe a set of sufficient conditions under which the two cohomologies are isomorphic. A typical sign of "finite dimensionality" in noncommutative differential geometry is that Hochschild cohomology of the corresponding algebra vanishes beyond a certain dimension (known as the *Hochschild cohomological dimension*). As a consequence of Connes' long exact sequence, the cyclic cohomology groups stabilize starting with this dimension, and the limiting groups capture all the cohomological information. The main result of this note shows that under some additional technical assumptions, these limiting groups are isomorphic to the entire cyclic cohomology groups.

The paper is organized as follows. In Section 2 we introduce the notion of an entire mixed complex. This is an entire version of the notion of a mixed complex of [K]. Examples of entire mixed complexes are the entire cyclic complex and its equivariant generalization. In Section 3 we formulate and prove the retraction theorem, which reduces entire cyclic cohomology to cyclic cohomology. Section 4 contains a simple application of the theorem, namely the computation of entire cyclic cohomology of the algebra of square matrices with complex entries.

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2. **Entire mixed complexes.**

(A). Let k be a Banach algebra over $\mathbb{C}$. Recall [J], [BD] that M is called a *left Banach k-module*, if (α) M is a complex Banach space; (β) M is a left k-module; (γ) there is a constant D such that

$$\|xm\| \leq D\|x\| \, \|m\|, \tag{2.1}$$

for all $x \in k$, $m \in M$. Let C^n, $n = 0, 1, 2, \ldots$, be a sequence of left Banach k-modules satisfying the following uniformity condition: There is a constant L such that the constant D is (2.1) can be chosen to be equal to L^n, for all $x \in k$, $m \in C^n$.

Let $b_n \colon C^n \longrightarrow C^{n+1}$, $n \geq 0$, and $B_n \colon C^n \longrightarrow C^{n-1}$, $n \geq 1$, be continuous homomorphisms of left Banach k-modules obeying the following algebra:

$$b_{n+1}b_n = B_{n-1}B_n = b_{n-1}B_n + B_{n+1}b_n = 0. \tag{2.2}$$

We can thus think of b_n as a coboundary operator and of B_n as a boundary operator. We require also that there is a constant C such that

$$\|b_n\| \leq Cn, \quad n \geq 0, \tag{2.3}$$

and

$$\|B_n\| \leq Cn, \quad n \geq 1. \tag{2.4}$$

(B). The entire mixed complex $(\mathcal{C}^*, \partial)$ is defined as follows. We consider the space $\mathcal{C}$ of all sequences $f = (f_0, f_1, f_2, \ldots)$, $f_n \in C^n$ such that for all $\eta > 0$,

$$\|f_n\|_\eta := \sum_{n \geq 0} \sqrt{n!}\|f_n\|\eta^n < \infty. \tag{2.5}$$

Observe that for $x \in k$, $xf := (xf_0, xf_1, \ldots)$, we have

$$\|xf\|_\eta \leq \|f\|_{L\eta}\|x\|, \tag{2.6}$$

i.e., $xf \in \mathcal{C}$. We set

$$\partial(f_0, f_1, \ldots, f_n, \ldots) := (B_1f_1, b_0f_0 + B_2f_2, \ldots, b_nf_n + B_{n+2}f_{n+2}, \ldots). \tag{2.7}$$

Then ∂ is a continuous homomorphism of $\mathcal{C}$ into itself and obeys

$$\partial^2 = 0. \tag{2.8}$$

We now write $\mathcal{C} \cong \mathcal{C}^e \oplus \mathcal{C}^0$, where $\mathcal{C}^e$ is the space of all sequences $(f_0, f_2, \ldots, f_{2j}, \ldots)$, and where $\mathcal{C}^0$ is the space of all sequences $(f_1, f_3, \ldots, f_{2j+1}, \ldots)$. Then ∂ induces two continuous homomorphisms $\partial \colon \mathcal{C}^0 \longrightarrow \mathcal{C}^e$, and $\partial \colon \mathcal{C}^e \longrightarrow \mathcal{C}^0$. The complex

$$\cdots \longrightarrow \mathcal{C}^e \xrightarrow{\partial} \mathcal{C}^0 \xrightarrow{\partial} \mathcal{C}^e \longrightarrow \cdots \tag{2.9}$$

is called *an entire mixed complex*. We denote its cohomologies by H^e and H^0.

(C). Let G be a finite group, and let $k = R(G)$ be its group ring. Let $\mathcal{A}$ be a unital Banach algebra with a G-action. The equivariant entire cyclic complex $\left(\mathcal{C}^*_G(\mathcal{A}), \partial\right)$ of [KKL] is then an example of an entire mixed complex. The special cases of $G = \{1\}$ and $G = \mathbb{Z}_2$ were discussed in [C2] and [JLO].

3. **The retraction theorem.**

(A). Let $(\mathcal{C}^*, \partial)$ be an entire mixed complex. By $(\mathcal{C}_0^*, \partial)$ we denote the subcomplex of $(\mathcal{C}^*, \partial)$ consisting of cochains with finitely many nonvanishing components (note that $(\mathcal{C}_0^*, \partial)$ is a mixed complex in the sense of [K]). We equip $\mathcal{C}_0^*$ with the topology of a direct sum. Let $i: \mathcal{C}_0^* \longrightarrow \mathcal{C}^*$ denote the natural embedding. Clearly, $i\partial = \partial i$. A continuous k-homomorphism $r: \mathcal{C}^* \longrightarrow \mathcal{C}_0^*$ is called an *approximate chain homomorphism*, if there are continuous k-homomorphisms $\rho: \mathcal{C}_0^* \longrightarrow \mathcal{C}^*$ and $\sigma: \mathcal{C}^* \longrightarrow \mathcal{C}_0^*$ such that

$$r\partial - \partial r = \partial \rho = \sigma \partial. \tag{3.1}$$

A continuous k-homomorphism $r: \mathcal{C}^* \longrightarrow \mathcal{C}_0^*$ is called a *retraction*, if the following conditions are satisfied:

(α) r is an approximate chain homomorphism,

(β) there exist two continuous k-homomorphisms $s, t: \mathcal{C}^* \longrightarrow \mathcal{C}^*$ such that

$$ir - I = \partial s + t\partial, \tag{3.2}$$

(γ) there exist two continuous k-homomorphisms $u, v: \mathcal{C}_0^* \longrightarrow \mathcal{C}_0^*$ such that

$$ri - I = \partial u + v\partial. \tag{3.3}$$

The following proposition is an immediate consequence of this definition.

PROPOSITION 3.1. *If there is a retraction between $\mathcal{C}^*$ and $\mathcal{C}_0^*$, then $H^*(\mathcal{C}) \cong H^*(\mathcal{C}_0)$, algebraically and topologically.*

Below we formulate a set of sufficient conditions under which an entire mixed complex $\mathcal{C}^*$ can be retracted to $\mathcal{C}_0^*$.

(B). Let $(\mathcal{C}^*, \partial)$ be an entire mixed complex and let $d(\mathcal{C}^*)$ denote the smallest integer such that for each $n \geq d(\mathcal{C}^*)$ there is a k-homomorphism $J_n: \mathcal{C}^n \longrightarrow \mathcal{C}^{n-1}$ with the following properties:

(α) J_n is continuous and

$$\|J_n\| \leq K, \tag{3.4}$$

uniformly in n,

(β) for $n \geq d(\mathcal{C}^*)$,

$$b_{n-1}J_n + J_{n+1}b_n = I_n, \tag{3.5}$$

where I_n denotes the identity on $\mathcal{C}^n$.

If no such integer exists, we set $d(\mathcal{C}^*) = \infty$. An immediate consequence of (3.5) is that the cohomologies of b_n vanish for $n \geq d(\mathcal{C}^*)$.

The following theorem is the main result of this note.

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THEOREM 3.2. *Let (C^*, ∂) be a mixed entire complex with $d(C^*) < \infty$. Then there exists a retraction r between (C^*, ∂) and (C_0^*, ∂).*

The rest of this section is devoted to the proof of this theorem. We proceed in steps.

(C). Let $r\colon C^* \longrightarrow C_0^*$ be defined as follows. For $f \in C^*$ we set

$$(3.6)\qquad (rf)_n := \begin{cases} 0, & \text{if } n > n_0, \\ f_n + \sum_{j\geq 1}(-1)^j B_{n+1}J_{n+2}\cdots B_{n+2j-1}J_{n+2j}f_{n+2j}, & \text{if } n = n_0 - 1, n_0, \\ f_n, & \text{if } n < n_0 - 1, \end{cases}$$

where $n_0 = d(C^*)$. We show that the series defining rf converges absolutely. Indeed, for $n = n_0 - 1, n_0$,

$$\begin{aligned} \|(rf)_n\| &\leq \sum_{j\geq 0}(CK)^j(n+1)(n+3)\cdots(n+2j-1)\|f_{n+2j}\| \\ &\leq \zeta^n(n!)^{-1/2}\sum_{j\geq 0}\big((n+2j)!\big)^{1/2}\eta^{2j}\|f_{n+2j}\| \leq \zeta^n(n!)^{-1/2}\|f\|_\eta, \end{aligned}$$

where ζ and η are constants. It is clear that $(rf)_n$ is a k-homomorphism. Therefore, $(rf)_n \in C^n$. To simplify the notation we will sometimes suppress the subscripts and write

$$(rf)_n = \sum_{j\geq 0}(-1)^j(BJ)^j f_{n+2j}.$$

(D). We assert that r is an approximate chain homomorphism. For $f \in C^*$ we set

$$(3.7)\qquad (\rho f)_n = \begin{cases} -\sum_{j\geq 0}(-1)^j(BJ)^j f_{n+2j}, & \text{if } n = n_0, \\ -\sum_{j\geq 0}(-1)^j(BJ)^{j+1} f_{n+2j+2}, & \text{if } n = n_0 - 2, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$(3.8)\qquad (\sigma f)_n = \begin{cases} -\sum_{j\geq 0}(-1)^j(BJ)^j f_{n+1+2j}, & \text{if } n = n_0 + 1, \\ -\sum_{j\geq 0}(-1)^j(BJ)^{j+1} f_{n+1+2j}, & \text{if } n = n_0 - 1, \\ 0, & \text{otherwise}. \end{cases}$$

Estimates similar to those of (C) show that $(\rho f)_n, (\sigma f)_n \in C^n$. We assert that $r\partial - \partial r = \partial\rho = \sigma\partial$. Clearly, $(r\partial f)_n - (\partial rf)_n = (\partial\rho f)_n = (\sigma\partial f)_n = 0$, if $|n - n_0| > 1$. If $n = n_0 - 1$, then

$$(3.9)\quad (r\partial f)_{n_0-1} - (\partial rf)_{n_0-1} = \sum_{j\geq 1}(-1)^j(BJ)^j bf_{n_0-2+2j} + \sum_{j\geq 1}(-1)^j(BJ)^j bf_{n_0+2j} = (\sigma\partial f)_{n_0-1}.$$

On the other hand, we can write (3.9) as

$$-BJbf_{n_0} + \sum_{j\geq 1}(-1)^j(BJ)^j(B - BJb)f_{n_0+2j} = -Bf_{n_0} - bBJf_{n_0} - \sum_{j\geq 1}(-1)^j(BJ)^j bBJf_{n_0+2j}.$$

Using the identity

$$(3.10)\qquad (BJ)^j bBJ = b(BJ)^{j+1},$$

valid for $j \geq 1$, we can rewrite this expression as

$$-Bf_{n_0} - b\sum_{j\geq 0}(-1)^j(BJ)^{j+1}f_{n_0+2j} = -B\sum_{j\geq 0}(-1)^j(BJ)^jf_{n_0+2j} - b\sum_{j\geq 0}(-1)^j(BJ)^{j+1}f_{n_0+2j}$$
$$= (\partial\rho f)_{n_0-1},$$

as asserted. If $n = n_0$, then

$$(3.11)\qquad \begin{aligned}(r\partial f)_{n_0} - (\partial rf)_{n_0} &= \sum_{j\geq 0}(-1)^j(BJ)^jbf_{n_0-1+2j} + \sum_{j\geq 0}(-1)^j(BJ)^jBf_{n_0+1+2j}\\ &\quad - b\sum_{j\geq 0}(-1)^j(BJ)^jf_{n_0-1+2j}.\end{aligned}$$

Using the identity

$$(3.12)\qquad (BJ)^jb - b(BJ)^j = (BJ)^{j-1}B,$$

valid for $j \geq 1$, we note that the first and third terms on the right side of (3.11) add up to

$$\sum_{j\geq 1}(-1)^j(BJ)^{j-1}Bf_{n_0-1+2j} = -\sum_{j\geq 0}(-1)^j(BJ)^jBf_{n_0+1+2j},$$

and thus $(r\partial f)_{n_0} - (\partial rf)_{n_0} = 0 = (\partial\rho f)_{n_0} = (\sigma\partial f)_{n_0}$. Finally, let $n = n_0 + 1$. Then

$$(r\partial f)_{n_0+1} - (\partial rf)_{n_0+1} = -b\sum_{j\geq 0}(-1)^j(BJ)^jf_{n_0+2j} = (\partial\rho f)_{n_0+1}.$$

On the other hand, using (3.10) we find that

$$-b\sum_{j\geq 0}(-1)^j(BJ)^jf_{n_0+2j} = -\sum_{j\geq 0}(-1)^j(BJ)^jBf_{n_0+2+2j} - \sum_{j\geq 0}(-1)^j(BJ)^jbf_{n_0+2j}$$
$$= (\sigma\partial f)_{n_0+1},$$

as asserted.

(E). Now we verify that i and r satisfy conditions (β) and (γ) of (A). To prove (3.2), we set for $f \in C^*$,

$$(3.13)\qquad (sf)_n := \begin{cases}0, & \text{if } n \leq d(C^*) - 1,\\ \sum_{j\geq 1}(-1)^jJ(BJ)^{j-1}f_{n+2j-1}, & \text{if } n \geq d(C^*),\end{cases}$$

and

$$(3.14)\qquad (tf)_n := \begin{cases}0, & \text{if } n \leq d(C^*),\\ \sum_{j\geq 1}(-1)^jJ(BJ)^{j-1}f_{n+2j-1}, & \text{if } n \geq d(C^*) + 1.\end{cases}$$

We assert that s and t are continuous k-homomorphisms of C^* into itself. Indeed, for $n \geq d(C^*)$,

$$\begin{aligned}\|(sf)_n\| &\leq \sum_{j\geq 1}C^{j-1}K^j(n+2)(n+4)\cdots(n+2j)\|f_{n+2j-1}\|\\ &\leq (n!)^{-1/2}\eta^n\sum_{j\geq 1}\zeta^j\big((n+2j-1)!\big)^{1/2}\|f_{n+2j-1}\|,\end{aligned}$$

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where η and ζ are constants. Choosing $R > \eta\zeta^{1/2}$ we find that

$$\sum_{n>d(C^*)}(n!)^{1/2}R^n\|(sf)_n\| \leq \sum_{n\geq 0}(R/\eta)^n\sum_{j\geq 1}\zeta^j\big((n+2j-1)!\big)^{1/2}\|f_{n+2j-1}\|$$
$$\leq \text{const}\sum_{m\geq 0}\zeta^{m/2}(m!)^{1/2}\|f_m\|,$$

i.e.,

(3.15) $$\|sf\|_R \leq \text{const}\,\|f\|_{\zeta^{1/2}}.$$

As a consequence, s exists and is continuous. Similar estimates show the existence and continuity of t. It remains to show that the algebraic identity (3.2) holds. There is nothing to prove if $n < d(C^*) - 1$. If $n = d(C^*) - 1$, then

$$\big((\partial s + t\partial)f\big)_n = (\partial sf)_n = (Bsf)_n$$
$$= \sum_{j\geq 1}(-1)^j(BJ)^jf_{n+2j} = \big((ir - I)f\big)_n,$$

as required. A similar calculation shows that the identity holds for $n = d(C^*)$. If $n > d(C^*)$, then

$$\big((\partial s + t\partial)f\big)_n = b(sf)_{n-1} + B(sf)_{n+1} + \big(t(bf)\big)_n + \big(s(Bf)\big)_n$$
$$= \sum_{j\geq 1}(-1)^jbJ(BJ)^{j-1}f_{n+2j-2} + \sum_{j\geq 1}(-1)^j(BJ)^jf_{n+2j}$$
$$+ \sum_{j\geq 1}(-1)^jJ(BJ)^{j-1}bf_{n+2j-2} + \sum_{j\geq 1}(-1)^j(BJ)^{j-1}Bf_{n+2j}.$$

Shifting the summation index in the first and third sums we can rewrite this expression as

$$-bJf_n - Jbf_n + \sum_{j\geq 1}(-1)^j\{-bJ(BJ)^j + (BJ)^j - J(BJ)^jb + J(BJ)^{j-1}B\}f_{n+2j}$$
$$= -f_n + \sum_{j\geq 1}(-1)^jJ\{b(BJ)^j - (BJ)^jb + (BJ)^{j-1}B\}f_{n+2j},$$

which by (3.12) equals $-f_n$. But $\big((ir - I)f\big)_n = -f_n$, for $n > d(C^*)$, and (3.2) is proven.

To prove (3.3), we define u and v to be the restrictions of s and t to C_0^*, respectively, and repeat the above arguments.

(F). As an immediate consequence of Proposition 3.1 and Theorem 3.3, we obtain

COROLLARY 3.3. *If* $(\mathcal{C}, \partial)$ *is an entire mixed complex with* $d(\mathcal{C}^*) < \infty$, *then*

$$H^*(\mathcal{C}) \cong H^*(\mathcal{C}_0), \tag{3.17}$$

algebraically and topologically.

4. **Application: entire cyclic cohomology of** $\mathrm{Mat}_N(\mathbb{C})$.

(A). In this section we apply the retraction theorem to compute the entire cyclic cohomology of the algebra $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$ of $N \times N$ matrices with complex entries. Let $\big(\mathcal{C}^*(\mathcal{A}), \partial\big)$ denote the entire cyclic complex of $\mathcal{A}$ [C2].

LEMMA 4.1. $d\big(\mathcal{C}^*(\mathcal{A})\big) = 1.$

PROOF. Let $\{E_{\alpha\beta}\}$, $0 \leq \alpha, \beta \leq N$, be the standard basis for $\mathrm{Mat}_N(\mathbb{C})$, $(E_{\alpha\beta})_{\alpha'\beta'} := \delta_{\alpha\alpha'}\delta_{\beta\beta'}$. Recall that

$$E_{\alpha\beta}E_{\alpha'\beta'} = \delta_{\beta\alpha'}E_{\alpha\beta'}. \tag{4.1}$$

We set for $m_j \in M_N(\mathbb{C})$ and $n \geq 1$,

$$(J_nf_n)(m_0, m_1, \ldots, m_{n-1}) := \frac{1}{N} \sum_{\alpha,\beta,\gamma} (m_0)_{\alpha\beta} f_n(E_{\alpha\gamma}, E_{\gamma\beta}, m_1, \ldots, m_{n-1}). \tag{4.2}$$

Clearly

$$\begin{aligned} |(J_nf_n)(m_0, m_1, \ldots, m_{n-1})| &\leq \frac{1}{N} \sum_\gamma \sum_\alpha \|f_n\| \, \|E_{\alpha\beta}\| \, \Big\| \sum_\beta (m_0)_{\alpha\beta} E_{\gamma\beta} \Big\| \prod_{1 \leq j \leq n-1} \|m_j\| \\ &\leq C \|f_n\| \prod_{1 \leq j \leq n-1} \|m_j\|, \end{aligned}$$

i.e.,

$$\|J_n\| \leq C,$$

uniformly in n. We assert that (3.5) is satisfied for all $n \geq 1$. Indeed,

$$\begin{aligned} &(b_{n-1}J_nf_n + J_{n+1}b_nf_n)(m_0, m_1, \ldots, m_n) \\ &\quad = \sum_{0 \leq j \leq n-1} (-1)^j (J_nf_n)(m_0, \ldots, m_jm_{j+1}, \ldots, m_n) \\ &\qquad + (-1)^n (J_nf_n)(m_nm_0, m_1, \ldots, m_{n-1}) \\ &\qquad + \frac{1}{N} \sum_{\alpha,\beta,\gamma} (m_0)_{\alpha\beta} (b_nf_n)(E_{\alpha\gamma}, E_{\gamma\beta}, m_1, \ldots, m_n) \end{aligned} \tag{4.3}$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{\alpha,\beta,\gamma}(m_0m_1)_{\alpha\beta}f_n(E_{\alpha\gamma},E_{\gamma\beta},m_1,\dots,m_n)\\
&\quad+\frac{1}{N}\sum_{1\le j\le n-1}\sum_{\alpha,\beta,\gamma}(m_0)_{\alpha\beta}f_n(E_{\alpha\gamma},E_{\gamma\beta},m_1,\dots,m_jm_{j+1},\dots,m_n)\\
&\quad+\frac{1}{N}(-1)^n\sum_{\alpha,\beta,\gamma}(m_nm_0)_{\alpha\beta}f_n(E_{\alpha\gamma}E_{\gamma\beta},m_1,\dots,m_n)\\
&\quad+\frac{1}{N}\sum_{\alpha,\beta,\gamma}(m_0)_{\alpha\beta}f_n(E_{\alpha\gamma},E_{\gamma\beta},m_1,\dots,m_n)\\
&\quad-\frac{1}{N}\sum_{\alpha,\beta,\gamma}(m_0)_{\alpha\beta}f_n(E_{\alpha\gamma},E_{\gamma\beta},m_1,m_2,\dots,m_n)\\
&\quad+\frac{1}{N}\sum_{1\le j\le n-1}(-1)^{j+1}\sum_{\alpha,\beta,\gamma}(m_0)_{\alpha\beta}f_n(E_{\alpha\gamma},E_{\gamma\beta},m_1,\dots,m_jm_{j+1},\dots,m_n)\\
&\quad+\frac{1}{N}(-1)^{n+1}\sum_{\alpha,\beta,\gamma}(m_0)_{\alpha\beta}f_n(m_nE_{\alpha\gamma},E_{\gamma\beta},m_1,\dots,m_{n-1}).
\end{aligned}
$$

Using (4.1) we derive the following identities:

$$\frac{1}{N}\sum_{\alpha,\beta,\gamma}(m_0)_{\alpha\beta}E_{\alpha\gamma}E_{\gamma\beta}=m_0, \tag{4.4}$$

$$\sum_{\alpha,\beta,\gamma}(m_0)_{\alpha\beta}f_n(E_{\alpha\gamma},E_{\gamma\beta},m_1,m_2,\dots,m_n)=\sum_{\alpha,\beta,\gamma}(m_0m_1)_{\alpha\beta}f_n(E_{\alpha\gamma},E_{\gamma\beta},m_2,\dots,m_n), \tag{4.5}$$

(4.6)
$$\sum_{\alpha,\beta,\gamma}(m_0)_{\alpha\beta}f_n(m_nE_{\alpha\gamma},E_{\gamma\beta},m_1,\dots,m_{n-1})=\sum_{\alpha,\beta,\gamma}(m_nm_0)_{\alpha\beta}f_n(E_{\alpha\gamma},E_{\gamma\beta},m_1,\dots,m_{n-1}).$$

As a consequence of these identities, the right side of (4.3) equals $f_n(m_0,m_1,\dots,m_n)$ and the lemma is proven.

(B). The above lemma allows us to compute $H^*\big(\mathcal{C}(\mathcal{A})\big)$.

PROPOSITION 4.2. *For any $N \ge 1$,*

$$H^e\Big(\mathcal{C}\big(\mathrm{Mat}_N(\mathbb{C})\big)\Big)\cong\mathbb{C},$$
$$H^0\Big(\mathcal{C}\big(\mathrm{Mat}_N(\mathbb{C})\big)\Big)\cong 0.$$

PROOF. By Corollary 3.4, the entire cyclic cohomology of $\mathrm{Mat}_N(\mathbb{C})$ is isomorphic to the ordinary cyclic cohomology $HC^*\big(\mathrm{Mat}_N(\mathbb{C})\big)$. It is, however, well known that $HC^{2n}\big(\mathrm{Mat}_N(\mathbb{C})\big)\cong\mathbb{C}$, and $HC^{2n+1}\big(\mathrm{Mat}_N(\mathbb{C})\big)\cong 0$.

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