Bull. Austral. Math. Soc. 54H20, 58F13, 26A18
Vol. 51 (1995) [395–415]

STRANGE TRIANGULAR MAPS OF THE SQUARE

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We show that continuous triangular maps of the square I^2, $F:(x,y)\mapsto (f(x), g(x,y))$, exhibit phenomena impossible in the one-dimensional case. In particular: (1) A triangular map F with zero topological entropy can have a minimal set containing an interval $\{a\}\times I$, and can have recurrent points that are not uniformly recurrent; this solves two problems by S.F. Kolyada.

(2) In the class of mappings satisfying $\mathrm{Per}(F) = \mathrm{Fix}(F)$, there are non-chaotic maps with positive sequence topological entropy and chaotic maps with zero sequence topological entropy.

1. Introduction

Let $I=[0,1]$ be the unit interval. By *triangular map* we always mean a *continuous* map $F : I^2 \to I^2$ of the form $F(x,y) = \big(f(x), g(x,y)\big) = \big(f(x), g_x(y)\big)$. The map f of the corresponding dynamical system is called the *base* for F; and g_x is a map from the *layer* $I_x = \{x\}\times I$ to I. Recall that recent results by Kolyada and Sharkovsky show that the dynamical systems generated by triangular maps are essentially more complex than one-dimensional systems, regardless of the fact that for both the same version of Sharkovsky's theorem [**14**] on the coexistence of cycles is valid [**9**]; for other references, see [**1**] and the survey paper [**10**].

In the present paper we first give a solution to the following two problems from [**10**]:

P_1: Can a minimal set of a triangular map contain an interval?

P_2: Is the condition $h(F)=0$ equivalent to the property that any recurrent point of F is uniformly recurrent, in the class of triangular maps of the square?

(Here $h(F)$ denotes the topological entropy of F.) In Section 2 we provide examples showing that the answer to P_1 is positive and to P_2 negative. (See Theorem 1 below.) In Section 3 we exhibit other examples of strange triangular maps showing that some conditions characterising chaos in the sense of Li and Yorke in the one-dimensional

Received 14th June, 1994
Work supported by M.U.R.S.T. Research funds (40%) and C.N.R. Most of the work on this paper was done while the third author was a Visiting Professor at the University of Milan. The support of this institution is gratefully acknowledged.

case are not equivalent even in the class of triangular maps satisfying $\mathrm{Per}(F) = \mathrm{Fix}(F)$; it turns out that such maps can have positive sequence topological entropy without being chaotic (Theorem 2).

Throughout our paper we use standard terminology as, for example, in [**10**] (see also [**2**] or [**15**]). Some basic notion and results are stated in the text.

2. Solution of Kolyada's problems

Theorem 1. *There are triangular maps $F_1, F_2 : I^2 \to I^2$ with the following properties:*

(i) *Both F_1, F_2 have zero topological entropy.*

(ii) *F_1 has a minimal set M containing an interval of the form $\{a\} \times I$.*

(iii) *F_2 has a recurrent point which is not uniformly recurrent.*

Remark 1. Recall that M is a minimal set for F if, for any $z \in M$, the ω-limit set $\omega_F(z)$ of z is the whole of M. Note that the projection of any minimal set for a triangular map F onto the x-axis is a minimal set for the base map f, and hence, a Cantor set or a cycle [**15**]. However, the second case implies $h(F) > 0$, as can be easily verified (see also [**9**]), so our minimal set M is contained in $Q \times I$, where Q is a minimal ω-limit set of the Feigenbaum's type.

Proof: The proof of our theorem has three parts. First we introduce a general construction of triangular maps F with zero topological entropy, and then in the next two parts, by specifying parameters, we prove (ii) and (iii).

Part I. General construction. Let $f : I \to I$ be a continuous map of type 2^∞ having a unique infinite ω-limit set Q such that $f \mid Q$ is one–to–one. In this case Q is homeomorphic to the space $\{0,1\}^N$ of sequences of two symbols equipped with a metric ρ of pointwise convergence defined, for example, by $\rho(\alpha,\beta) = \max\{1/i \,;\, \alpha(i) \neq \beta(i)\}$ for any distinct $\alpha = \{\alpha(i)\}$ and $\beta = \{\beta(i)\}$ in $\{0,1\}^N$. Moreover, f acts on Q as the adding machine, that is, for $\alpha \in \{0,1\}^N, f(\alpha) = \alpha + 1000\cdots$ where the addition is modulo 2 from left to right; for example, $f(101100\cdots) = 011100\cdots$, $f(11100\cdots) = 00010\cdots$, et cetera. For details the reader is referred to [**3**] where the representation of Q is described; the concept of the adding machine was indicated in [**13**]. Clearly $\omega_f(x) = Q$ for any $x \in Q$; hence Q is a minimal set for f.

Our general map F with free parameters will be monotonic on any layer I_x. This will imply $h(F) = 0$. Indeed, we have (see [**9**])

$$\sup\{h(F, I_x) \,;\, x \in I\} + h(f) \geqslant h(F),$$

where $h(F, I_x)$ denotes the topological entropy of the map $F : I^2 \to I^2$ with respect to the compact subset I_x, that is, the entropy $h(F, I_x)$ is computed only from trajectories starting from I_x. But since F^i is monotonic on I_x for any i, we clearly have $h(F, I_x) = 0$, and of course, $h(f) = 0$ since f is of type 2^∞. Thus, $h(F) = 0$. To have F monotonic on all layers, it suffices to define its restriction to the set $Q \times I$, monotonic on all layers, and then extend it continuously and monotonically to the whole of I^2.

So we shall define F on $Q \times I$. As already mentioned, we may assume without loss of generality that Q is the space $\{0,1\}^N$ with the metric ρ as above. For any $k \geqslant 1$ and any $\alpha \in \{0,1\}^2$, let $\varphi(k,\alpha)$ be a monotonic continuous map $I \to I$ with the following properties:

$$\varphi(k,10) \circ \varphi(k,00) = \varphi(k,11) \circ \varphi(k,01) = Id \tag{1}$$

where Id is the identity map of I, that is, $\varphi(k,1i)$ is a left–inverse (but not necessarily a right–inverse) of $\varphi(k,0i), i = 0,1$, and

$$\|\varphi(k,\alpha) - Id\| = \delta_k \to 0 \quad \text{as} \quad k \to \infty. \tag{2}$$

We shall call any $\varphi(k,\alpha)$ a map of *rank* k.

Let $x \in Q$, $x = x(1)x(2)\cdots$; we shall call $x(3i)$ the i-th *control digit* of x. Define g_x as follows. If all control digits of x are equal to 1, let g_x be the identity map. Otherwise let the first zero control digit of x be the k-th one. Then put

$$g_x = \varphi\big(k, x(3k-2)x(3k-1)\big).$$

Now let $\rho(u,v) < 1/3k$, for some $u, v \in Q$. If there exists $i \leqslant k$ with $u(3i) = 0$ $(= v(3i))$ then $g_u = g_v$, otherwise $\|g_u - Id\| \leqslant \delta_m$ and $\|g_v - Id\| \leqslant \delta_m$ for some $m > k$. Hence, in any case $\|g_u - g_v\| \leqslant 2\delta_m$ which by (2) implies $\lim_{u \to v} \|g_u - g_v\| = 0$ and hence implies the continuity of $g_x(y)$ in $Q \times I$.

Next we prove some identities for F. Let $\underline{0}$ be the zero sequence in Q and let $y_0 \in I$. Denote by y_j the second coordinate of $F^j(\underline{0}, y_0)$, for $j \geqslant 0$. Let $m \leqslant 4 \cdot 8^k$. Then

$$y_m = \varphi_m \circ \varphi_{m-1} \circ \cdots \circ \varphi_1(y_0) \tag{3}$$

where every φ_i is a map of rank $\leqslant k+1$ since during the first $4 \cdot 8^k = 2^{3k+2}$ iterations, the $(k+1)$-th control digit of $f^i(\underline{0})$ is zero. If $m \geqslant 2^2$ then $\varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$ is the block

$$\beta_1 = \varphi(1,11) \circ \varphi(1,01) \circ \varphi(1,10) \circ \varphi(1,00) = Id$$

(see (1)) and this block β_1 repeats in (3) periodically with period 8 (since $f^i(\underline{0})$ begins with three zeros periodically with period 8). Hence, in (3) there are, from the right to left, the block β_1, then a block of four maps, then β_1 again, et cetera.

Similarly, if $m \geqslant 2^5$ then $\varphi_{32} \circ \cdots \circ \varphi_1$ is the block β_2 of the form

$$\beta_2 = \varphi^4(2,11) \circ \beta_1 \circ \varphi^4(2,01) \circ \beta_1 \circ \varphi^4(2,10) \circ \beta_1 \circ \varphi^4(2,00) \circ \beta_1$$

which, by $\beta_1 = Id$ and (1), again gives $\beta_2 = Id$. Block β_2 repeats in (3) periodically with period $2^6 = 64$. By induction we get that the block β_k of the first $m_k = 2^{3k-1}$ maps in (3) amounts to the identity; so

$$y_{m_k} = y_0 \qquad \text{for } k \geqslant 1 \tag{4}$$

and for large m the structure of (3) is

$$\cdots \circ \beta_k \circ \beta_k'' \circ \beta_k \circ \beta_k' \circ \beta_k \tag{5}$$

where $\beta_k', \beta_k'', \cdots$ are blocks of the same length m_k as β_k. Since the maps of rank less than $k+1$ are organised into blocks β_i, $i \leqslant k$, that by (4) can be cancelled and since the number of appearances of $\varphi(k+1,00)$ in (3) for $m = 2m_k$ is equal to the number of n's satisfying $0 \leqslant n < 2m_k$ for which the first k control digits in $f^n(\underline{0})$ are ones, that is $2m_k/2^k = 2^{2k}$, we get

$$y_{2m_k} = \varphi^{2^{2k}}(k+1,00)(y_0), \qquad k \geqslant 1 \tag{6}$$

and by (5) and (4), $y_{3m_k} = y_{2m_k}$. Similarly, $y_{4m_k} = y_0 = y_{5m_k}$ and

$$y_{6m_k} = \varphi^{2^{2k}}(k+1,01)(y_0), \qquad k \geqslant 1. \tag{7}$$

Thus we have constructed, under conditions (1) and (2) in particular, a triangular map $F : I^2 \to I^2$ of zero topological entropy satisfying the identities (4)–(7).

Now we specify parameters to get F_1 and F_2.

Part II. Proof of (ii). Let $\{r_k\}_{k=1}^{\infty}$ be a sequence of rational numbers from $(0,1)$, dense in I and such that

$$\lim_{k\to\infty} r_{k+1}^{1/2^{2k}} = 1. \tag{8}$$

It is easy to construct such a sequence. Put $\delta_{k+1} = 1 - r_{k+1}^{1/2^{2k}}$ and, for any $t \in I$,

$$\begin{aligned}
\varphi(k,00) &= (1-\delta_k)t + \delta_k \\
\varphi(k,10) &= \max\{0, (t-\delta_k)/(1-\delta_k)\} \\
\varphi(k,01) &= (1-\delta_k)t \\
\varphi(k,11) &= \min\{1, t/(1-\delta_k)\}
\end{aligned}$$

Note that the $\varphi(k,\alpha)$'s satisfy (1) and that $\|\varphi(k,\alpha) - Id\| = \delta_k$ which, by (8), implies (2). Let F_1 be the function F from part I with the specification as above. Now (6) and (7) can be rewritten as

$$(9) \qquad y_{2m_k} = r_{k+1}(y_0 - 1) + 1, \qquad k \geqslant 1$$

$$(10) \qquad y_{6m_k} = r_{k+1} y_0, \qquad k \geqslant 1.$$

For any $y_0 \in [0,1]$ the set $\{y_{2m_k}, y_{6m_k}\}_{k=1}^{\infty}$ is dense in I; indeed, it is dense in $[0, y_0]$ by (10) and in $[y_0, 1]$ by (9). And since both $f^{2m_k}(\underline{0})$ and $f^{6m_k}(\underline{0})$ have zeros at the first $3k$ places, we have $\lim f^{2m_k}(\underline{0}) = \lim f^{6m_k}(\underline{0}) = \underline{0}$ for $k \to \infty$ and consequently

$$(11) \qquad \omega_{F_1}(z) \supset \{\underline{0}\} \times I \qquad \text{for any } z \in \{\underline{0}\} \times I.$$

Put $M = \omega_{F_1}(\underline{0}, 0)$, and let $w = (u, v) \in M$. Since $F_1^i(w)$ visits any neighbourhood U of $\{\underline{0}\} \times I$ (note that $\{f^i(u)\}$ is dense in Q), $\omega_{F_1}(w)$ contains a point from $\{\underline{0}\} \times I$ and consequently, by (11), $\omega_{F_1}(w) \supset M$. Thus M is a minimal set for F_1 containing the interval $\{\underline{0}\} \times I$, which proves (ii).

PART III. PROOF OF (iii). For any k, let ν_k and $r_{k+1} = \nu_{k+1}^{2^{2k}}$ be numbers such that

$$(12) \qquad 0 < \nu_k < 1, \quad \lim_{k \to \infty} \nu_k = 1 \quad \text{and} \quad \lim_{k \to \infty} r_{k+1} = 0.$$

For $t \in I$ put

$$(13) \qquad \varphi(k, 00) = \varphi(k, 11) = t^{\nu_k}$$

and

$$(14) \qquad \varphi(k, 01) = \varphi(k, 10) = t^{1/\nu_k}.$$

By (12)–(14), both (1) and (2) are satisfied. Let F_2 be the function F from part I with the specification as above. Formulas (6) and (7) reduce to

$$(15) \qquad y_{2m_k} = y_0^{r_{k+1}}, \qquad k \geqslant 1$$

$$(16) \qquad y_{6m_k} = y_0^{1/r_{k+1}}, \qquad k \geqslant 1.$$

Put $y_0 = 1/2$, and $w = (\underline{0}, y_0)$. By (4), w is recurrent, that is, $w \in \omega_{F_2}(w)$. On the other hand, w is not uniformly recurrent. To see this assume the contrary. Take $U = Q \times [0, 3/4]$ as a neigbourhood of w in the F_2-invariant set $Q \times I$. Let $m = m(U)$ be such that for any $i \geqslant 0$ there is a $j < m$ with $F_2^{i+j}(w) \in U$. Let k be such that $m_k > m$ and let $s > k$. Take $i = 2m_s$. Then $f^i(\underline{0})$ has zeros at the first $3s$ places, hence by (15), $y_{2m_s+j} = \varphi_j \circ \cdots \circ \varphi_2 \circ \varphi_1 \big(y_0^{r_s+1}\big)$ where the maps $\varphi_1, \cdots, \varphi_j$ have rank $\leqslant k$ (see the definition of F_2 on $Q \times I$), if $j < m$. Note that all maps $\varphi(k, \alpha)$, for all k and α, commute. Hence, it is easy to see that

$$y_{2m_s+j} \geqslant \varphi^{2^{2(k-1)}}(k, 10) \circ \cdots \circ \varphi^{2^2}(2, 10) \circ \varphi(1, 10)\big(y_0^{r_s+1}\big)$$

since only the maps $\varphi(i, 10) = \varphi(i, 01)$ among the maps of rank i are pushing $y_0^{r_s+1}$ to the left, and since, after cancellation, the number of $\varphi(i+1, \alpha)$'s among $\varphi_1, \cdots, \varphi_j$, cannot exceed 2^{2i}. By (7) and (16) we get

$$y_{2m_s+j} \geqslant y_0^{r_s+1/(r_1 \cdots r_k)}.$$

But because of (12), $\inf\limits_{s>k} r_{s+1}/(r_1 \cdots r_k) = 0$. This means that for some $s > k$, $y_{2m_s+j} > 3/4$, hence $y_{2m_s+j} \notin U$ for any $j < m$, a contradiction. Thus, the point $w = (\underline{0}, 1/2)$ is recurrent but not uniformly. $\square$

REMARK 2. Our construction of maps F_1, F_2 above was inspired by Kolyada's example of a triangular map F of type 2^∞ with positive topological entropy [**10**]. Of course, in the latter case not every g_x is monotonic. However, it seems that the use of the binary representation of Q as in our theorem would simplify Kolyada's construction.

3. Chaotic and nonchaotic triangular maps

We shall say that a map $F : I^2 \to I^2$ is *chaotic* (in the sense of Li and Yorke) if there are points $u, v \in I^2$ such that

$$(17) \qquad 0 = \liminf_{n\to\infty} |F^n(u) - F^n(v)| < \limsup_{n\to\infty} |F^n(u) - F^n(v)|$$

where $|u - v|$ denotes the distance in I^2.

A set $S \subset I^2$ is a *scrambled set* for F if (17) is satisfied for any distinct u and v in S. If, however, there exists $\varepsilon > 0$ such that, for any distinct u and v in S,

$$(18) \qquad \limsup_{n\to\infty} |F^n(u) - F^n(v)| \geqslant \varepsilon,$$

S is called an *ε-scrambled set* for F.

The next theorem gives a survey of some main conditions characterising chaos for continuous maps of the interval.

THEOREM A. *For a continuous map $f : I \to I$ the following conditions are equivalent:*

(i) *f is chaotic.*

(ii) *f has an uncountable scrambled set.*

(iii) *f has a non–empty perfect ε-scrambled set for some $\varepsilon > 0$.*

(iv) *f has an infinite ω-limit set with two points non–separable by periodic neighbourhoods.*

(v) *f has a trajectory that cannot be asymptotically approximated by periodic trajectories.*

(vi) *f has a positive sequence topological entropy $h_A(f)$ with respect to a sequence A.*

(vii) $\omega(f) \neq \{x \in I : \lim_{n\to\infty} f^{2^n}(x) = x\}$.

(viii) $IsR(f) \neq \omega(f)$.

(ix) *$f \mid \omega(f)$ is not stable in the sense of Ljapunov.*

(x) *$f \mid \Omega(f)$ is not stable in the sense of Ljapunov.*

In the above theorem $h_A(f)$ denotes the sequence topological entropy with respect to an increasing sequence $A = \{n(i)\}_{i=1}^{\infty}$ of positive integers, defined for a continuous map f of a compact metric space X as follows [7]: For $m > 0$ and $\varepsilon > 0$, a set $E \subset X$ is an (A, m, f, ε)-span, if for any $x \in X$ there is some $y \in E$ such that $\left|f^{n(j)}(x) - f^{n(j)}(y)\right| < \varepsilon$ for $1 \leqslant j \leqslant m$. Let $S(A, m, f, \varepsilon)$ be an (A, m, f, ε)-span with the minimal possible number of points. Then

$$h_A(f) = \lim_{\varepsilon\to 0} \limsup_{m\to\infty} \frac{1}{m} \log \#S(A, m, f, \varepsilon). \tag{19}$$

(In particular, $h_{\mathbf{N}}(f) = h(f)$ if $\mathbf{N}$ is the set of positive integers). In Theorem A, $\Omega(f)$ is the set of non–wandering points, and $IsR(f)\,(= AP(f))$ is the set of isochronically recurrent (or almost periodic) points, that is of the points y such that for any neighbourhood U of y there is an $n > 0$ such that $f^{in}(y) \in U$ for any i.

HISTORICAL REMARKS TO THEOREM A. Condition (ii) is essentially the original notion of chaos by Li and Yorke [**12**] from 1975. Equivalence of (ii),(iii),(iv) and (v) was proved in 1986 in [**16**] for maps with zero topological entropy and in [**8**] for other maps. Keeping chronological order, (i) $\Leftrightarrow$ (ii) was proved in [**11**], (vii) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (ix) in [**15**] (however, parts of these equivalences are contained in Sharkovsky's papers from 1965–1968), (i) $\Leftrightarrow$ (ix) in [**4**], (ix) $\Leftrightarrow$ (x) in [**5**], and (i) $\Leftrightarrow$ (vi) in [**6**]. Note that [**8**] contains other conditions not displayed here.

In this section we exhibit examples indicating that for triangular maps, conditions from Theorem A are not equivalent. We shall consider only triangular maps $F : I^2 \to I^2$ with linear base and with fixed layer I_0, that is, maps such that $(x, y) \mapsto (\lambda x, g_x(y))$ and $g_0 = Id$, where $\lambda \in (0, 1)$ is fixed. Denote the class of such maps by $\mathcal{T}_\lambda$. Our main result in this section is the following

THEOREM 2. *There is $F \in \mathcal{T}_\lambda$ with positive sequence topological entropy, but satisfying no other condition from Theorem A. Moreover any ω-limit set of F is a one–point set.*

REMARK 3. It is easy to see that none of the conditions (vii), (viii), (ix), (x) can be satisfied for $F \in \mathcal{T}_\lambda$. Clearly, any F in $\mathcal{T}_\lambda$ has zero topological entropy (see also [9]).

REMARK 4. Surprisingly, the map F_1 from Theorem 1 satisfies all conditions from Theorem A. To see this it suffices to use (4) and (9) to construct a perfect 1-scrambled set $S \subset Q \times \{0\}$ supporting positive sequence entropy. To prove (v), (vii) and (viii) consider the point $u = (\underline{0}, 0)$, for (iv) points u and $v = (\underline{0}, 1)$. We omit the details.

We have also

THEOREM 3. *There is $F \in \mathcal{T}_\lambda$ such that*

(i) *F has a two–point scrambled set,*
(ii) *F has no scrambled set with more than two points,*
(iii) *F has an infinite ω-limit set with no two points separable by periodic neighbourhoods,*
(iv) *F has a trajectory not approximable by periodic trajectories,*
(v) *$h_A(F) = 0$ for any sequence A.*

From Theorems 2 and 3, the function constructed in Remark 6, and $F(x,y) = (\lambda x, y)$ we deduce immediately the following.

COROLLARY. *In the class $\mathcal{T}_\lambda$, condition (vi) of Theorem A is independent of conditions (i), (iv) and (v) of the same theorem.*

We conjecture that in $\mathcal{T}_\lambda$ also conditions (ii) and (iii) are independent of (vi). (See also the examples at the end of our paper.)

Note that by Remark 3, in $\mathcal{T}_\lambda$ we cannot prove more.

It is known that any scrambled set for a continuous map of the interval must have empty interior. Contrary to this, we provide examples indicating that in $\mathcal{T}_\lambda$ there is a wide scale of possible scrambled sets between a two–point scrambled set and a non–empty open 1-scrambled set. (See Examples 1–4 below.)

The next two lemmas will be used later to construct some other examples; moreover Lemma 1 is used for the proof of Theorem 2.

LEMMA 1. *Let $\delta \in (0, 1/2)$, $\sigma \in (0,1)$ and let K, L, M be disjoint compact sets such that $K, L \subset (\lambda, 1)$ and $M \subset (\lambda, 1]$. Then there is a positive integer $s = s(\delta, \sigma)$ and a portion $\tau = \tau(K, L, M, \delta, \sigma)$ of a map $F \in \mathcal{T}_\lambda$, defined on $[\lambda^{2s}, 1] \times I$, such that:*

(i) *$\|\tau - \psi\| \leqslant \delta$, where ψ is the corresponding portion of the map $(x,y) \mapsto (\lambda x, y)$.*

(ii) *For any* $(x,y) \in [\lambda,1] \times I$

$$\tau^{2s}(x,y) = \left(\lambda^{2s}x, y\right) = \psi^{2s}(x,y).$$

(iii) *For any* $(x,y) \in M \times I$ *and any* $i \leqslant 2s$,

$$\tau^{i}(x,y) = \left(\lambda^{i}x, y\right) = \psi^{i}(x,y).$$

(iv)
$$\tau^{s}(K \times I) \subset f^{s}(K) \times [0,\sigma]$$
$$\tau^{s}(L \times I) \subset f^{s}(L) \times [1-\sigma,1].$$

PROOF: Let $\varphi(k,\alpha)$, $\alpha \in \{0,1\}^2$, be the same functions as in Part II of the proof of Theorem 1, with $\delta_k = \delta$. We recall the identity (1). Denote $\varphi(k,\alpha)$ by φ_α.

Define $s = s(\delta,\sigma)$ as the first integer such that

$$\varphi_{00}^{s}(0) = 1-(1-\delta)^{s} \geqslant 1-\sigma; \tag{20}$$

note that we have also

$$\varphi_{01}^{s}(1) = (1-\delta)^{s} \leqslant \sigma. \tag{21}$$

For $0 \leqslant i \leqslant s-1$, define g_x on the interval $[\lambda^{i+1},\lambda^{i}]$ by $g_x = \varphi_{01}$ if $x \in f^{i}(K)$, $g_x = \varphi_{00}$ if $x \in f^{i}(L)$, and $g_x = Id$ if $x \in f^{i}(M) \cup \{\lambda^{i},\lambda^{i+1}\}$. If (a,b) is a component of $(\lambda^{s},1] \setminus \bigcup_{i=0}^{s-1}\left(f^{i}(K \cup L \cup M) \cup \{\lambda^{i}\}\right)$, for $x \in (a,b)$ let $\{g_x\}$ be a family of left–invertible continuous functions $I \to I$ depending continuously and monotonically on x and such that

$$\lim_{x \to a^{+}} g_x = g_a, \qquad \lim_{x \to b^{-}} g_x = g_b.$$

In the following we call such a family a *connecting family* (between g_a and g_b in the above case). Thus we have defined g_x for $x \in [\lambda^{s},1]$. Define now g_x on $\left[\lambda^{2s},\lambda^{s}\right)$ in the following way: if $x \in \left[\lambda^{s+i},\lambda^{s+i-1}\right)$ and $\overline{x} = \lambda^{-2i+1}x$, $1 \leqslant i \leqslant s$, g_x is the inverse to $g_{\overline{x}}$ when $g_{\overline{x}}$ belongs to a connecting family, while $g_x = \varphi_{10}$ or φ_{11} if $g_{\overline{x}} = \varphi_{00}$ or φ_{01} respectively. Define τ on $[\lambda^{2s},1] \times I$ by $\tau(x,y) = (\lambda x, g_x(y))$. Since $\|\varphi_\alpha - Id\| = \delta$ for any $\alpha \in \{0,1\}^2$ we have $\|g_x - Id\| \leqslant \delta$ for any g_x in a connecting family. This implies (i). Properties (ii) and (iii) are immediate and (iv) follows by (20) and (21). ∎

REMARK 5. With little changes it is possible to prove a more general version of Lemma 1 including also the case when K, L, M are subsets of $[\lambda,1]$ and both λ and 1 belong to one of the sets K, L, M.

Lemma 2. *Let $\sigma \in (0,1/2)$ and let $K, L \subset I$ be disjoint non-empty compact intervals with $\max K < \min L$. Then there is a positive integer $p = p(\sigma)$ and a portion $\pi = \pi(K,L,\sigma)$ of a map $F \in \mathcal{T}_\lambda$, defined on $[\lambda^{2p+2},1] \times I$, such that:*

(i) *$\|\pi - \psi\| < \sigma$ where ψ is the corresponding portion of the map $(x,y) \mapsto (\lambda x, y)$.*

(ii) *For any $(x,y) \in [\lambda,1] \times I$*

$$\pi^{2p+2}(x,y) = \left(\lambda^{2p+2}x, y\right) = \psi^{2p+2}(x,y).$$

(iii)
$$\pi^p([\lambda,1] \times K) \subset [\lambda^{p+1},\lambda^p] \times [0,\sigma]$$
$$\pi^p([\lambda,1] \times L) \subset [\lambda^{p+1},\lambda^p] \times [1-\sigma,1].$$

Proof: Let $p = p(\sigma) \geqslant 2$ be an integer such that $(1-\sigma)^p \leqslant \sigma$. Denote $a = \max K$, $b = \min L$. Let $\theta = \theta(\sigma,K,L) : I \to I$ be the continuous function such that $\theta(0) = 0$, $\theta(a) = (1-\sigma)a$, $\theta(b) = (1-\sigma)b + \sigma$ and $\theta(1) = 1$, and linear on the intermediate intervals. Note that θ is strictly increasing, hence a bijection, and that

$$\theta^p(a) \leqslant \sigma, \qquad \theta^p(b) \geqslant 1-\sigma. \tag{22}$$

Now define $\{g_x : x \in [\lambda^{2p+2},1]\}$ as a family of bijective mappings $I \to I$ such that

$$g_1 = Id \quad \text{and} \quad g_\lambda = \theta \tag{23}$$

(24) $\{g_x : x \in (\lambda,1)\}$ is a connecting family (as in the proof of Lemma 1)

$$g_x = \theta \quad \text{for } x \in [\lambda^p,\lambda] \tag{25}$$

and let g_x be the unique map satisfying

$$g_x \circ \theta^{p-1} \circ g_{(\lambda^{-p}x)} = \theta^p \qquad \text{for } x \in [\lambda^{p+1},\lambda^p]; \tag{26}$$

in particular $g_x = Id$ for $x = \lambda^{p+1}$. On the intervals $[\lambda^{p+2},\lambda^{p+1}]$, $[\lambda^{2p+1},\lambda^{p+2}]$ and $[\lambda^{2p+2},\lambda^{2p+1}]$ g_x is defined with the same procedure as in (23)–(26), but applied in the reverse order and with the corresponding inverse functions (see Lemma 1). Then (i) and (ii) immediately follow from the definition of $\{g_x\}$; (iii) follows from (22). ∎

In the next proofs of Theorems 2,3 and subsequent examples the following notation will be useful. Let F_1, F_2 be two portions of triangular maps defined on $[\lambda^{n_1},1] \times I$ and $[\lambda^{n_2},1] \times I$, respectively, and such that

$$F_1(1,y) = F_2(1,y) = (\lambda,y) \qquad \text{and} \qquad F_i(\lambda^{n_i},y) = \left(\lambda^{n_i+1},y\right),\ i = 1,2.$$

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With the symbol $[F_1,F_2]$ we denote the portion of a triangular map defined on $[\lambda^{n_1+n_2},1]\times I$ such that its restriction on $[\lambda^{n_1},1]\times I$ is $F_1(x,y)$ and its restriction on $[\lambda^{n_1+n_2},\lambda^{n_1}]\times I$ is obtained from $F_2(\lambda^{-n_1}x,y)$ by multiplying the base function by λ^{n_1}; in a similar way we define $[F_1,F_2,\cdots,F_k]$. Finally, if F^∞ is the identity map on I_0, we denote by $[F_1,F_2,\cdots]$ the map $F:I^2\to I^2$ given by $\left(\lim_{k\to\infty}[F_1,F_2,\cdots,F_k]\right)\cup F^\infty$.

Proof of Theorem 2: We apply Lemma 1. Let $F=[F_1,F_2,\cdots]$ where $F_i=\tau(K_i,L_i,M_i,\delta_i,1/4)$ is defined on the rectangle $[\lambda^{2s(\delta_i,1/4)},1]\times I$, $\lim\delta_i=0$ and K_i,L_i,M_i are suitable compact sets. Then by (i) of Lemma 1, F is in $\mathcal{T}_\lambda$. Denote $k(0)=0$ and $k(i)=2s(\delta_1,1/4)+\cdots+2s(\delta_i,1/4)$.

Next let $\{a_j\}$ be a decreasing sequence of points with $a_1=1$ and $\lim a_j=\lambda$. For all $i>0$ with $2^{j-1}\leqslant i<2^j$ define $M_i=[a_j,1]$ and $\delta_i=1/2^j$. Since $\lim M_i=(\lambda,1]$, by (iii) of Lemma 1, the trajectory of any point $(x,y)\in I^2$ eventually is in $\bigcup_{k=0}^{\infty}\left(F^k(M_i\times I)\right)$ for some i. Hence this trajectory is eventually constant in the second variable and consequently, F is nonchaotic.

Let $S_j\subset[a_{j+1},a_j)$ be a set of $2^{2^{j-1}}$ points, namely

$$S_j=\{b_l:l=1,\cdots,2^{2^{j-1}}\}.$$

We write the indexes l in base 2: they are all strings of length 2^{j-1} of digits 0 and 1. For all i with $2^{j-1}\leqslant i<2^j$, define K_i as the set of points of S_j having zero at the $(i+1-2^{j-1})$-th place and define $L_i=S_j\setminus K_i$.

We now show that $h_A(F)>0$ for a suitable sequence A. By Lemma 1 for any distinct $a,b\in S_j$ there exists $i=i(a,b)$ satisfying $2^{j-1}\leqslant i<2^j$ and

$$\left|F^{k(i-1)+s(i)}(a,y_1)-F^{k(i-1)+s(i)}(b,y_2)\right|>1/2$$

independently of y_1,y_2, where $s(i)=s(\delta_i,1/4)$ (see Lemma 1). Define

$$A=\{t(i)\}_{i=1}^{\infty}=\{k(i-1)+s(i)\}_{i=1}^{\infty}.$$

Then, if $S(A,m,F,\varepsilon)$ is an (A,m,F,ε)-span with the minimal possible number of points, for $\varepsilon=1/4$ and $m=2^j-1$ we have

$$\#S(A,m,F,\varepsilon)\geqslant\#S_j=2^{2^{j-1}}$$

and so
$$\lim_{\varepsilon\to0}\limsup_{m\to+\infty}\frac{1}{m}\log\#S(A,m,F,\varepsilon)\geqslant\limsup_{j\to+\infty}\frac{1}{2^j-1}\log 2^{2^{j-1}}$$
$$=\lim_{j\to+\infty}\frac{2^{j-1}}{2^j-1}\log 2=\frac{1}{2}\log 2>0$$

∎

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REMARK 6. It is possible to modify the function F of Theorem 2 in order to get a function $F' \in \mathcal{T}_\lambda$ which still has positive sequence topological entropy but satisfies conditions (i), (iv) and (v) of Theorem A. To do this, we take a new decreasing sequence $\{a_j\}$ with $\lim a_j = \mu > \lambda$ and we construct the sets K_i, L_i as in the proof of Theorem 2. Then we define $F' = [F'_1, F'_2, \cdots] \in \mathcal{T}_\lambda$ where $F'_i = \tau(K'_i, L'_i, \emptyset, \delta_i, 1/4)$, $K'_i = K_i \cup [(3\mu+\lambda)/4, \mu]$ and $L'_i = L_i \cup [(\mu+\lambda)/4, (2\mu+\lambda)/4]$.

Obviously if we don't look for a function in $\mathcal{T}_\lambda$ but only for a triangular map, a much simplier example is

$$G(x,y) = (\lambda x, g(y)) \qquad \text{where} \quad g(y) \quad \text{is the tent map.}$$

PROOF OF THEOREM 3: Instead of a map from $\mathcal{T}_\lambda$ we shall construct a topologically conjugated map F defined on $[0,\infty] \times I$ (where ∞ stands for $+\infty$) by $F(x,y) = (x+1, g_x(y))$ if $x < \infty$, and $F(\infty, y) = (\infty, y)$. Moreover we may assume that $[0,\infty] \times I$ is equipped with the metric $|(x_1,y_1) - (x_2,y_2)| = \max\left\{\left|\lambda^{x_1} - \lambda^{x_2}\right|, |y_1 - y_2|\right\}$, which makes $[0,\infty] \times I$ homeomorphic to I^2 with the usual metric. This will simplify our notation and, on the other hand, will not change the properties considered, including the sequence topological entropy, since they are invariant with respect to homeomorphic transformations, see [7].

For any integer $k > 1$ define continuous maps $\mu_k, \nu_k : I \to I$ by

$$\mu_k(t) = \max\{0, t - 1/k\}, \quad \nu_k(t) = \min\{2t, t + 1/k, 1\}, \quad t \in I.$$

Then $\quad \mu_k^i(I) = [0, 1 - i/k]$ for $0 \leqslant i \leqslant k$, $\quad \nu_k^i \circ \mu_k^i(I) = I$ for $0 \leqslant i \leqslant k-1$.

For any positive integer k define a map F_k on $[0,2k] \times I$ by $F_k(x,y) = (x+1, g_{k,x}(y))$ where

$$g_{k,x}(y) = \begin{cases} \max\{0, y + h_k(x)\} & \text{if} \quad x \in [0,k] \\ \min\left\{y + h_k(x)\min\{ky, 1\}, 1\right\} & \text{if} \quad x \in [k, 2k] \end{cases}$$

and the function $h_k(x)$ is piecewise linear and connecting the points $(0,0), (1,-1/k)$, $(k-1,-1/k), (k+1,1/k), (2k-1,1/k)$ and $(2k,0)$. (Note that $g_{k,x} = Id$ for $x = 0, k, 2k$, $g_{k,x}(y) = \mu_k(y)$ for $x \in [1, k-1]$, $g_{k,x}(y) = \nu_k(y)$ for $x \in [k+1, 2k-1]$.)

The map F_k has the following phase diagram (with $k = 7$) in which we depicted the trajectories going through the points of the segments $([0,1] \cup [k,k+1]) \times \{i/k\}$, $0 \leqslant i \leqslant k$ and $[k,k+1] \times \left\{1/\left(k2^i\right)\right\}$, $1 \leqslant i \leqslant k-1$. All trajectories are piecewise linear, except for the trajectory of the segment $[k,k+1] \times \left\{1/\left(k2^{k-1}\right)\right\}$ in the interval $[2k, 2k+1]$ (see Figure 1).

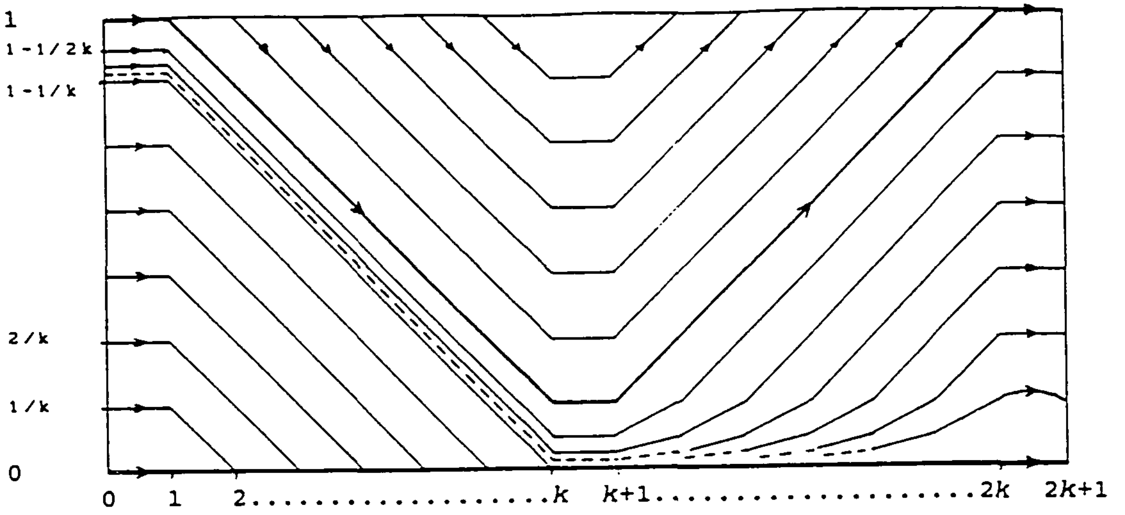

Figure 1

Finally, let

(27) $$k(1) < k(2) < \cdots$$

be a sequence of integers greater than 1 (we shall specify it later) and let $F = [F_{k(1)}, F_{k(2)}, \cdots]$. Clearly F is continuous and topologically conjugate to a map of $\mathcal{T}_\lambda$.

For any set $B \subset [0, \infty] \times I$ denote by $\mathrm{Orb}_F(B)$ the set $\bigcup\limits_{-\infty}^{\infty} F^n(B)$; thus, in our diagram the curves are just the sets $\mathrm{Orb}_F\left([ik, ik+1] \times \{j/k\}\right)$ for $i = 0, 1,\ 1 \leqslant j \leqslant k$, and $\mathrm{Orb}_F\left([k, k+1] \times \left\{1/(k2^i)\right\}\right), 1 \leqslant i \leqslant k-1$. Put

$$K = [0, \infty) \times \{0\} \quad , \quad L = \mathrm{Orb}_F([0,1] \times \{1\})$$

and

(28) $$X = \mathrm{Orb}_F(K), \qquad Y = \mathrm{Orb}_F(L).$$

It is easy to see that K and L are disjoint continuous curves such that

(29) $$F(K) \subset K, \qquad F(L) \subset L.$$

Moreover, X is the set of points in $[0, \infty) \times I$ lying below the curve L and

(30) $$X \cup Y \quad \text{is a decomposition of} \quad [0, \infty) \times I.$$

Now the proof of (i)–(iv) of Theorem 3 is easy. Clearly any set $S = \{u, v\}$ with $u \in K$ and $v \in L$ is a 1-scrambled set, which implies (i). Since $\omega_F(z) = I_\infty$ for any $z \in L$ and $I_\infty = \operatorname{Fix}(F) = \operatorname{Per}(F)$, we get iv) and since there are no F-invariant nonempty open sets in $[0,\infty] \times I$ we have (iii). To prove (ii) let S be a scrambled set. Since $\omega_F(K) = \{(\infty, 0)\}$ and $I_\infty = \operatorname{Fix}(F)$, we have $\#(S \cap (K \cup I_\infty)) \leqslant 1$. By (28)–(30), any $z \in [0,\infty] \times I$ is eventually in $K \cup L \cup I_\infty$, hence to finish the proof if suffices to show that $\#(S \cap L) \leqslant 1$. But this is easy: let $u, v \in S \cap L$, $u = (u_1, u_2), v = (v_1, v_2)$ and let m be the greatest integer with $m \leqslant |v_1 - u_1|$. (In the sequel we call this m the *iterative distance* between u and v.) Then, for every n there is $k(i)$ such that

$$|F^n(u) - F^n(v)| < \lambda^n + \frac{m+1}{k(i)},$$

and so, by (27), $|F^n(u) - F^n(v)| \to 0$ as $n \to \infty$.

All that remains to prove is (v). Put $n(0) = 0$, $n(i) = 2(k(1) + k(2) + \cdots + k(i))$ for $i > 0$, and choose the sequence $\{k(i)\}_{i>0}$ of (27) so that

$$\text{(31)} \qquad k(i) \quad \text{divides} \quad k(i+1) \quad \text{and} \quad \frac{n(i)}{k(i+1)} \longrightarrow 0 \quad \text{monotonically.}$$

Let $A = \{t(i)\}_{i=1}^{\infty}$ be an increasing sequence of positive integers. Fix $m \geqslant 1$ and consider the elements t_i, $1 \leqslant i \leqslant m$. For each i there exists a unique r such that $n(r) \leqslant t_i < n(r+1)$. Denote by

$$r_1 < r_2 < \cdots < r_q, \qquad q \leqslant m$$

all such integers r. Fix $\varepsilon > 0$ and choose $i = i_\varepsilon$ such that $10/k(i+1) < \varepsilon$. Put $p := k(i+1)$. Since in the definition (19), $h_A(F)$ doesn't change by removing a finite number of elements from A, we may assume without loss of generality that

$$\text{(32)} \qquad \frac{n(r_1)}{k(r_1+1)} < \frac{1}{p}\Big(< \frac{\varepsilon}{10}\Big).$$

and $\lambda^{t_1} < \varepsilon$. To prove (v) we need an (A, m, F, ε)-span T with sufficiently small cardinality. Since $\lambda^{t_1} < \varepsilon$, for any $u, v \in [0,\infty] \times I$ and $s \geqslant t_1$ we have

$$\text{(33)} \qquad |F^s(u) - F^s(v)| < \varepsilon \quad \text{if and only if} \quad |F^s(u) - F^s(v)|_y < \varepsilon,$$

where $|u - v|_y$ denotes (the Euclidean) distance between the second coordinates of u and v respectively. Thus, in the sequel when proving that a set is an (A, m, F, ε)-span we may, by (33), without loss of generality consider the metric $|\cdot|_y$ instead of $|\cdot|$.

By the definition of F and (27) we easily get

$$(34) \qquad |F^s(u) - u|_y \leqslant \frac{s}{k(i+1)} \quad \text{whenever } u \in [n(i), \infty] \times I, \text{ for any } s \geqslant 0.$$

Consequently, by (31), (32) and (34),

$$(35) \quad \left|F^{t_s}(u) - F^{t_s}(v)\right|_y < \frac{3}{p} < \varepsilon \quad \text{if } u, v \in [n(i), \infty] \times I,\ |u - v|_y < \frac{1}{p} \text{ and } t_s < n(i).$$

For simplicity, denote by U_j the intervals

$$U_j = [n(r_j), n(r_j + 1)], \qquad j = 1, \cdots, q$$

and by V_j the corresponding complementary intervals, that is,

$$V_0 = [0, n(r_1)), \quad V_j = (n(r_j + 1), n(r_{j+1})),\ 1 \leqslant j \leqslant q - 1, \qquad V_q = (n(r_q + 1), \infty],$$

and let

$$J_0 = \{0\}, \quad J_1 = \left(0, \frac{1}{p}\right), \quad J_j = \left[\frac{j-1}{p}, \frac{j}{p}\right) \quad \text{if} \quad 2 \leqslant j \leqslant p, \quad J_{p+1} = \{1\},$$
$$I_i = [n(i), n(i) + 1] \quad \text{and} \quad K_i = [n(i) + k(i+1), n(i) + k(i+1) + 1].$$

We shall construct a covering of the rectangle $[0, n(r_q + 1)] \times I$ by a finite family of invariant subsets G (that is, $F(G) \subset G$) which have small intersections with layers $I_x = \{x\} \times I$, for suitable points x. Let

$$\begin{aligned} \mathcal{G}_i = \left\{ \mathrm{Orb}_F \left(I_{r_i} \times J_j\right) : 0 \leqslant j \leqslant p - 1\right\} &\cup \left\{ \mathrm{Orb}_F \left(I_{r_i+1} \times J_j\right) : 1 \leqslant j \leqslant p - 1\right\} \\ &\cup \left\{ \mathrm{Orb}_F \left(I_{r_i} \times \left[1 - \frac{1}{p}, 1 - \frac{1}{k(r_i + 1)}\right]\right)\right\} \\ &\cup \left\{ \mathrm{Orb}_F \left(\left(K_{r_i} \times J_j\right) : 2 \leqslant j \leqslant p + 1\right\} \\ &\cup \left\{ \mathrm{Orb}_F \left(\left(K_{r_i} \times J_1\right) \cap Y\right)\right\} \end{aligned}$$

for $1 \leqslant i \leqslant q$,

$$\mathcal{G}_{q+1} = \left\{ \mathrm{Orb}_F \left(I_{r_q+1} \times J_p\right)\right\},$$

and let $\mathcal{G}$ be the union of $\mathcal{G}_i$, $1 \leqslant i \leqslant q + 1$. It is easy to see that $\mathcal{G}$ is a cover of $[0, n(r_q + 1)] \times I$ consisting of invariant sets. Moreover, $G \cap I_x$ is connected for every $x \in [0, n(r_q + 1)]$ and

$$(36) \qquad \mathrm{diam}(G \cap I_x) \leqslant 1/p \quad \text{if } G \text{ is a minimal element of } \mathcal{G} \text{ and } x \in U_i \text{ for some } i.$$

Note that all elements of $\mathcal{G}$ are minimal, except for

$$Y_i = \mathrm{Orb}_F\big((K_{r_i} \times J_1) \cap Y\big) \subset Y \quad \text{and} \quad X_i = \mathrm{Orb}_F\big(I_{r_i} \times J_0\big) \subset X, \qquad 2 \leqslant i \leqslant q.$$

By (34) and (36) we easily get

(37)

$$\operatorname{diam}\left(G \cap \left(\left[\alpha, \alpha + \frac{k(r_i+1)}{p}\right] \times I\right)\right) < \frac{3}{p} \quad \text{if } G \in \mathcal{G} \text{ is minimal, } \alpha \in U_i,\ 1 \leqslant i \leqslant q.$$

Now we shall define an (A, m, F, ε)-span T as the union of sets T_G, $G \in \mathcal{G}$, where any T_G is a subset of G and approximates properly the points from G. More precisely, let T_G be a set of minimal cardinality with the following properties:

for any $u \in G \cap (V_j \times I)$ there is a $v \in T_G \cap G \cap (V_j \times I)$ such that

$$|u - v|_y < \frac{1}{p}, \tag{38}$$

and for any $u \in G \cap (U_j \times I)$ there is a $v \in T_G \cap G \cap (U_j \times I)$ satisfying (38) and

$$\text{the iterative distance between } u \text{ and } v \text{ is less than } \frac{k(r_j+1)}{p}. \tag{39}$$

A simple estimation shows that $\#T_G \leqslant (q+1)p + 2qp^2 \leqslant 4mp^2$ and since $\#\mathcal{G} = q(\#\mathcal{G}_1) + 1 = q(3p+1) + 1 \leqslant 4mp$, we have $\#T \leqslant 16m^2p^3$. Now to complete the proof we have to show that actually T is an (A, m, F, ε)-span, since then the sequence entropy is zero:

$$h_A(F) \leqslant \lim_{\varepsilon \to 0} \limsup_{m \to \infty} \frac{1}{m} \log\left(16m^2p^3\right) = \lim_{\varepsilon \to 0} 0 = 0.$$

Assume first that $u \in V_q \times I$. Let v be any point in $T \cap (V_q \times I)$ satisfying (38). Then by (35) we easily get

$$\left|F^{t_s}(u) - F^{t_s}(v)\right|_y < \varepsilon, \tag{40}$$

for $1 \leqslant s \leqslant m$.

Next let $u \in V_i \times I$ with $i < q$. Since $\mathcal{G}$ covers $[0, n(r_q+1)] \times I$, there is a $G \in \mathcal{G}$ containing u such that either G is minimal or $G = \mathrm{Orb}_F\left((K_{r_{i+1}} \times J_1) \cap Y\right)$ or $G = \mathrm{Orb}_F\left(I_{r_{i+1}} \times J_0\right)$. Take as v any point in $T_G \cap (V_i \times I)$ satisfying (38). By (35), to prove that v approximates u at times $t_s, 1 \leqslant s \leqslant m$, it is enough to show that (40) is satisfied when $t_s \geqslant n(r_{i+1})$. So fix such an s and find l such that $t_s \in U_l$. By (31) and (32), the iterative distance α between u and v is less than $k(r_l+1)/p$. Assume

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$F^{t_s}(u), F^{t_s}(v) \in U_l \times I$: if u, v belong to a minimal G, then (40) is true by (37) while in the other cases (40) follows immediately since both points belong to the same curve K or L. In all other situations, take the maximal integer t such that $F^t(u), F^t(v) \in U_l \times I$. Now we have $|F^t(u) - F^t(v)|_y < 3/p$ by (37), $|F^{t_s}(u) - F^t(u)|_y < 1/p$ by (31), (32) and (34) (since $0 < t_s - t < n(r_l)$), and a similar inequality for v, which in view of (32) imply (40).

Finally, let $u \in U_i \times I$ with $i \leqslant q$. In this case there is a $G \in \mathcal{G}$ containing u such that either G is minimal or $G = \mathrm{Orb}_F\left((K_{r_i} \times J_1) \cap Y\right)$ or $G = \mathrm{Orb}_F\left(I_{r_i} \times J_0\right)$. Take as v any point in $T_G \cap (U_i \times I)$ satisfying (38) and (39). If u belongs to one of the last two sets G, the conclusion is obvious. In the other cases the argument is almost the same as for the sets $V_i \times I$. Fix an s and find l such that $t_s \in U_l$. It is enough to consider only the case $l = i$; in the other cases we proceed as above.

If $F^{t_s}(u) \in U_i \times I$ and $F^{t_s}(v) \notin U_i \times I$ then (40) is true by (37), (31) and (32), since the iterative distance between u and v is less than $k(r_i + 1)/p$. Similarly, if $F^{t_s}(v)$ is and $F^{t_s}(u)$ is not in $U_i \times I$. Finally, if neither $F^{t_s}(u)$ nor $F^{t_s}(v)$ is in $U_i \times I$, let $t(u)$ and $t(v)$ be the maximal integers such that $F^{t(u)}(u), F^{t(v)}(v) \in U_i \times I$. Then by (37), (34), (31) and (32) we have

$$\left|F^{t_s}(u) - F^{t(u)+1}(u)\right|_y + \left|F^{t(u)+1}(u) - F^{t(u)}(u)\right|_y < \frac{4}{p}$$

and similarly for u replaced by v. Since G is a minimal element of $\mathcal{G}$, by (36) we obtain $\left|F^{t(u)}(u) - F^{t(v)}(v)\right|_y < 2/p$ and consequently,

$$\left|F^{t_s}(u) - F^{t_s}(v)\right|_y < \frac{4}{p} + \frac{4}{p} + \frac{2}{p} < \varepsilon,$$

which completes the proof of (v), and hence of Theorem 3. $\square$

We conclude this paper with the following four examples.

EXAMPLE 1. There is $F \in \mathcal{T}_\lambda$ having a non-empty open 1-scrambled set S.

Let $\{K_i, L_i\}_{i=1}^\infty$ be an ordering in a sequence of all non-degenerate compact intervals with rational end-points, contained in $(\lambda, 1)$, and such that $K_i \cap L_i = \emptyset$. Let $\{H_i, N_i\}_{i=1}^\infty$ be an analogous sequence in $[0,1]$, with the further condition that $\max H_i < \min N_i$. Define, by Lemma 1,

$$\tau_i := \tau(K_i, L_i, \emptyset, 1/i, 1/i)$$

and, by Lemma 2,

$$\pi_i := \pi(H_i, N_i, 1/i).$$

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We take $F = [\tau_1, \pi_1, \tau_2, \pi_2, \cdots]$. For this function F the strip $S = (\lambda, 1) \times (0,1)$ is a 1-scrambled set. Indeed, let (x_1, y_1), (x_2, y_2) be distinct points in S. If $x_1 \neq x_2$ then $x_1 \in K_i$, $x_2 \in L_i$ and $x_1, x_2 \in K_j$ for infinitely many i and for infinitely many j. By (iv) of Lemma 1 we get

$$(41) \qquad 0 = \liminf_{n\to\infty} |F^n(x_1,y_1) - F^n(x_2,y_2)| < \limsup_{n\to\infty} |F^n(x_1,y_1) - F^n(x_2,y_2)| = 1.$$

If $y_1 \neq y_2$, then we may assume that $y_1 < y_2$ and so $y_1 \in H_i$, $y_2 \in N_i$ and $y_1, y_2 \in H_j$ for infinitely many i and j and, by (iii) of Lemma 2, again (41) is satisfied.

EXAMPLE 2. There is $F \in \mathcal{T}_\lambda$ having a scrambled set S with $\operatorname{Int}(S) \neq \emptyset$ and such that any ε-scrambled set $S_0 \subset S$ with $\varepsilon > 0$ is finite.

Fix a and b such that $\lambda < a < b < 1$ and let $\{K_i, L_i\}_{i=1}^{\infty}$ be an ordering in a sequence of all non–degenerate compact intervals $K_i = [a, \alpha_i]$, $L_i = [\beta_i, b]$ with $\alpha_i < \beta_i$ and $\alpha_i, \beta_i \in \mathbf{Q}$. We denote $\sigma_i = \beta_i - \alpha_i$ and define

$$\tau_i = \tau(K_i, L_i, \emptyset, 1/i, 1 - \sigma_i)$$

as in Lemma 1 with the further requirement that the connecting families between φ_{01} and φ_{00} are of affine functions with the same slope as φ_{01} (and φ_{00}) and varying linearly with x. Furthermore let

$$\tau_i^* = \tau(K, \emptyset, \emptyset, 1/i, 1/i)$$

where $K = [a,b]$. Finally define $F = [\tau_1, \tau_1^*, \tau_2, \tau_2^*, \cdots]$. We prove that the strip $S = [a,b] \times I$ has the desired properties. Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ be distinct points in S. By (iv) of Lemma 1 applied to τ_i^* we have the equality in (17). To prove the inequality in (17), consider the following two cases:

CASE A. If $y_1 \neq y_2$ then by (ii) of Lemma 1 applied to τ_i^* we get

$$\limsup_{n\to\infty} |F^n(u) - F^n(v)| \geqslant |y_1 - y_2|.$$

CASE B. If $y_1 = y_2$ and $x_1 < x_2$, there are infinitely many i such that $x_1 \in K_i$, $x_2 \in L_i$ and $|x_1 - x_2|/2 < \sigma_i < |x_1 - x_2|$; then by (iv) of Lemma 1 applied to τ_i we have

$$\limsup_{n\to\infty} |F^n(u) - F^n(v)| \geqslant \sigma_i > \frac{|x_1 - x_2|}{2}.$$

Thus S is a scrambled set. Obviously during the application of the portions τ_i^* the y-distance of the two points $u, v \in S$ cannot increase. The same happens during the application of τ_i when $x_1 = x_2$ or when both points belong either to K_i or to

L_i. A simple computation shows that in all other cases, if $y_1 = y_2$, the choice of the connecting families assures that their y-distance remains less or equal to $2\sigma_i < 2|x_1 - x_2|$. Summarising we obtain

$$\begin{aligned}|F^n(u) - F^n(v)| &\leqslant |F^n(x_1,y_1) - F^n(x_1,y_2)| + |F^n(x_1,y_2) - F^n(x_2,y_2)| \\ &\leqslant |y_1 - y_2| + 3|x_1 - x_2|\end{aligned}$$

and so

$$\limsup_{n\to\infty} |F^n(u) - F^n(v)| \leqslant |y_1 - y_2| + 3|x_1 - x_2|.$$

Assume there exists an ε-scrambled set $S_0 \subset S$, $\varepsilon > 0$ containing infinitely many points. Then there is a Cauchy sequence $\{(x_k,y_k)\} \subset S_0$ and so for n,m great enough $|y_n - y_m| + 3|x_n - x_m| < \varepsilon$, contrary to the definition of an ε-scrambled set.

EXAMPLE 3. There is $F \in \mathcal{T}_\lambda$ having a perfect 1-scrambled set, but no scrambled set S with $\operatorname{int}(S) \neq \emptyset$.

To see this we modify F from Example 1 as follows. Let $Q \subset (\lambda, 1)$ be a Cantor set and let $\{M_i\}_{i=1}^{\infty}$ be an increasing sequence of compact sets with $\lim_{i\to\infty} M_i = [\lambda,1] \setminus Q$. Take $F = [\tau_1, \tau_2, \cdots]$ where $\tau_i = \tau(K_i \cap Q, L_i \cap Q, M_i, 1/i, 1/i]$, and K_i, L_i are as in Example 1. Clearly, $Q \times \{0\}$ is, by (iv) of Lemma 1, a 1-scrambled set. On the other hand, no scrambled set contains more that one point from $\bigcup_{i,n=1}^{\infty} F^n(M_i \times I)$, by (iii) of Lemma 1.

EXAMPLE 4. There is $F \in \mathcal{T}_\lambda$ having an infinite scrambled set and such that any scrambled set is at most countable.

As a first step we construct a function $F \in \mathcal{T}_\lambda$ having a two–point 1-scrambled set, but no three–point scrambled set, simplier than that used in Theorem 3. Let $\varphi(k,\alpha)$, $\alpha = \{0,0\}$ and $\alpha = \{1,0\}$ be the same functions as in Part II of the proof of Theorem 1, with $\delta_k \to 0$ for $k \to \infty$. Let $F = [F_1, F_2, \cdots]$ where the portions F_k are iteratively defined as follows.

For $x \in (\lambda, 1]$ let $\{g_x\}$ be a connecting family between $g_1 = Id$ and $g_\lambda = \varphi(1,10)$. Then choose $g_x = \varphi(1,10)$ for all $x \in \left(\lambda^{s(1)}, \lambda\right]$, where $s(1) = \left[\log(\rho_1)/\log(1-\delta_1)\right] + 2$ and $\rho_1 = 1/2$. For $x \in \left(\lambda^{s(1)+1}, \lambda^{s(1)}\right]$ let $\{g_x\}$ be a connecting family between $\varphi(1,10)$ and $\varphi(1,00)$. For $x \in \left(\lambda^{2s(1)+1}, \lambda^{s(1)+1}\right]$ choose $g_x = \varphi(1,00)$ and finally, for $x \in \left(\lambda^{2s(1)+2}, \lambda^{2s(1)+1}\right]$, let $\{g_x\}$ be a connecting family between $\varphi(1,00)$ and Id. The portion so constructed is F_1. We may now define $\rho_2 = \max\{h(x),\ x \in [\lambda, 1]\}$ where $h(x)$ is the y-component of the point $F_1^{2s(1)+2}(x, \rho_1)$.

Suppose we have defined the portion F_k, $k \geqslant 1$, and we have calculated ρ_{k+1}. We may now define the portion F_{k+1} and the number ρ_{k+1} as for F_k and ρ_k, simply by substituting functions $\varphi(k,\alpha)$ and parameters δ_k, ρ_k with the corresponding functions $\varphi(k+1,\alpha)$ and parameters δ_{k+1}, ρ_{k+1} respectively.

With this function F, for any pair $(x_1,y_1)(x_2,y_2)$ with $x_1,x_2 \in (0,1]$, $y_1,y_2 \in [0,1)$ we have

$$\lim_{n\to\infty} |F^n(x_1,y_1) - F^n(x_2,y_2)| = 0.$$

It follows that any scrambled set cannot contain more than two points; more precisely, S is a scrambled set if and only if

$$S = \{(x_1,y_1),(x_2,y_2)\} \text{ with } x_1 \in (0,1],\ y_1 \in [0,1),\ x_2 \in I,\ y_2 = 1.$$

For this two-point set we have

$$\liminf |F^n(x_1,y_1) - F^n(x_2,y_2)| = 0 < 1 = \limsup |F^n(x_1,y_1) - F^n(x_2,y_2)|.$$

Now, in order to produce our example, we substitute for the functions $\varphi(k,\alpha)$ the functions $\psi(k,\alpha)$ given by:

$$\psi(k,\alpha)(t) = \left(1 - \frac{1}{2^i}\right) + \frac{1}{2^{i+1}}\,\varphi(k,\alpha)\left(2^{i+1}\left(t - \left(1 - \frac{1}{2^i}\right)\right)\right), \quad \psi(k,\alpha)(1) = 1,$$
$$\text{where} \quad t \in \left[1 - \frac{1}{2^i}, 1 - \frac{1}{2^{i+1}}\right), \quad i \geqslant 0.$$

It is easy to see that this new function F has the required properties.

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https://doi.org/10.1017/S0004972700014222 Published online by Cambridge University Press