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Abstract

Let K be a CM number field and G_K its absolute Galois group. A representation of G_K is said to be polarized if it is isomorphic to the contragredient of its outer complex conjugate, up to a twist by a power of the cyclotomic character. Absolutely irreducible polarized representations of G_K have a sign ± 1, generalizing the fact that a self-dual absolutely irreducible representation is either symplectic or orthogonal. If Π is a regular algebraic, polarized, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_K)$, and if ρ is a p-adic Galois representation attached to Π, then ρ is polarized and we show that all of its polarized irreducible constituents have sign $+1$. In particular, we determine the orthogonal/symplectic alternative for the Galois representations associated to the regular algebraic, essentially self-dual, cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ when F is a totally real number field.

1. Introduction

1.1 The sign of a representation

Let L be a field of characteristic 0 or greater than 2. Let G be a group and $g \mapsto g^c$ an involution of G. For ρ a representation $G \to \mathrm{GL}_n(L)$, we define $\rho^\perp : G \to \mathrm{GL}_n(L)$, $g \mapsto {}^t\rho(g^c)^{-1}$. The equivalence class of the representation $\rho^\perp$ only depends on the equivalence class of ρ.

We fix $\chi : G \to L^*$ a character such that $\chi(g) = \chi(g^c)$ for all g. This property ensures that $\rho \mapsto \rho^\perp \chi^{-1}$ is an involution. In the applications, G will be the absolute Galois group of a CM number field K, c the outer automorphism defined by the non-trivial element in $\mathrm{Gal}(K/F)$, where F is the maximal totally real subfield of K, and χ a power of the cyclotomic character.

Let ρ be a semisimple representation $G \to \mathrm{GL}_n(L)$ such that

$$\rho^\perp \simeq \rho\chi. \tag{1}$$

This property is obviously stable by extension of the field of coefficients L.

We shall now attach to any *absolutely irreducible* ρ satisfying (1) an invariant, that we call its *sign*. The invariant can take the values $+1$ or -1. By Schur's lemma, there exists a unique

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(up to a scalar) matrix $A \in \mathrm{GL}_n(L)$ such that

$$\rho^{\perp} = A\rho A^{-1}\chi. \tag{2}$$

Applying this relation twice, we see that $A{}^tA^{-1}$ commutes with $\rho^{\perp}$; hence, by Schur's lemma again, it is a scalar matrix λ. So, ${}^tA = \lambda A$ and $\lambda = \pm 1$. This sign is called the *sign* of ρ (with respect to χ). Note that it is necessarily 1 if n is odd, since there is no invertible antisymmetric matrix in odd dimension.

If $\rho' := Q^{-1}\rho Q$ for some $Q \in \mathrm{GL}_n(L)$, then ρ' satisfies (2) with $A' = {}^tQAQ$, so the sign of ρ only depends on the isomorphism class of ρ. Moreover, it is obvious that it remains unchanged under arbitrary extensions of the coefficient field L. However, it depends on χ in general: if $\rho \simeq \rho \otimes \varepsilon$ for some non-trivial character ε, then the signs of ρ with respect to χ and $\chi\varepsilon$ may differ.[1]

1.2 Galois representations attached to unitary groups

Let F be a totally real field, K a totally imaginary quadratic extension and $c \in \mathrm{Gal}(K/F)$ the non-trivial automorphism. Let Π be a cuspidal automorphic representation for GL_n over K, and assume that Π is polarized, i.e. the contragredient $\Pi^{\vee}$ of Π is isomorphic to $\Pi \circ c$, and that Π_{∞} is algebraic regular (see [CH, General Hypotheses 2.1]).

Under those hypotheses, Shin [Shi11] and the many coauthors of the two-volume book [GRFAbook] have shown the existence of a compatible system of Galois representations attached to Π (see [CH, Theorem 3.2.5]).

Theorem 1.1. *There are a number field $E(\Pi)$ and a compatible system $\rho_{\Pi,\lambda} : G_K \to \mathrm{GL}(n, E(\Pi)_{\lambda})$ of semisimple λ-adic representations, where λ runs through finite places of $E(\Pi)$, such that for all finite primes v of K of residue characteristic prime to $N_{E(\Pi)/\mathbb{Q}}(\lambda)$, and such that Π_v is unramified,*

$$(\rho_{\Pi,\lambda}|_{G_v})^{F-\mathrm{ss}} \simeq \mathrm{L}(\Pi_v \otimes |\bullet|_v^{(1-n)/2}),$$

where G_v is a decomposition group of K at v and $L(\bullet)$ is the local Langlands correspondence.

The given property suffices to characterize uniquely $\rho_{\Pi,\lambda}$ up to isomorphism and implies that $\rho_{\Pi,\lambda}$ satisfies (1); more precisely, let c be a complex conjugation in K, that is, an element of $G_F - G_K$ of order two. We set $g^c = cgc^{-1} = cgc$ for $g \in G_K$: this is an automorphism of order two. For that automorphism, we have

$$\rho^{\perp}(g) = {}^t\rho(g^c)^{-1} \simeq \rho(g)\omega(g)^{n-1},$$

where ω is the cyclotomic character.

The theorem also includes other specifications on $\rho_{\Pi,\lambda}$, including the determination of the Hodge–Tate weights of $\rho_{\Pi,\lambda}$ at places of the same residual characteristic as λ (see also § 1.6 below). This description implies, since Π_{∞} is cohomological, that these weights are distinct integers and hence that $\rho_{\Pi,\lambda}$ is a direct sum of non-isomorphic absolutely irreducible representations of G_K.

[1] For example, let $G \subset \mathrm{GL}_2(L)$ be the normalizer of the diagonal matrices, ρ the natural inclusion and $g^c := g\det(g)^{-1}$. Then ρ has sign -1 for the trivial character, as ${}^t(g^c)^{-1} = wgw^{-1}$ for any $g \in \mathrm{GL}_2(L)$ and for $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. But ρ has sign $+1$ for the order-two character ε which is trivial on the diagonal matrices, as ${}^t(g^c)^{-1} = w'gw'^{-1}\varepsilon(g)$ for $g \in G$ and $w' = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

1.3 The result

The object of this article is to prove the following theorem.

Theorem 1.2. *For every finite prime λ of $E(\Pi)$, every irreducible factor r of $\rho_{\Pi,\lambda}$ that satisfies $r^{\perp} \simeq r \otimes \omega^{n-1}$ has sign $+1$.*

In this statement, it is understood that this sign is computed with respect to the character $\chi := \omega^{n-1}$. It is expected that $\rho_{\Pi,\lambda}$ is absolutely irreducible (this is known if $n \leqslant 3$ by [BR92] and in many cases if $n = 4$ by an unpublished work of Ramakrishnan). If it is so, $\rho_{\Pi,\lambda}$ has only one factor and satisfies (1), and our theorem simply asserts that its sign is $+1$: this is obvious when n is odd, but new when n is even.

The theorem above has an important corollary concerning essentially self-dual Galois representations of a totally real field F. Precisely, let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ such that:

(a) $\Pi^{\vee} \simeq \Pi \otimes \eta$, where η is a Hecke character of F such that $\eta_v(-1)$ does not depend on the real place v of F;

(b) Π_v is cohomological for each real place v of F.

In this case as well, the aforementioned works show that for some coefficient number field $E(\Pi)$ there is a compatible system of λ-adic semisimple representations $\rho_{\Pi,\lambda} : G_F \to \mathrm{GL}_n(E(\Pi)_\lambda)$ which are compatible with the Frobenius-semisimplified local Langlands correspondence twisted by $|\cdot|^{(1-n)/2}$ at each prime not dividing the residue characteristic of λ and unramified for Π (see [CH, Theorem 4.2]). In particular, we have

$$\rho_{\Pi,\lambda}^{\vee} \simeq \rho_{\Pi,\lambda} \otimes \omega^{n-1}\eta_\lambda,$$

where η_λ is the λ-adic realization of η (note that η is necessarily algebraic by (a) and (b)). As Π is cuspidal, $\rho_{\Pi,\lambda}$ is conjecturally irreducible, but, as before, this is not known in general (however, each irreducible constituent of $\rho_{\Pi,\lambda}$ has multiplicity one and is absolutely irreducible). The counterpart of the sign in this situation is the standard alternative orthogonal/symplectic: if $r : G_F \to \mathrm{GL}_d(L)$ is absolutely irreducible and satisfies $r^{\vee} \simeq r \otimes \omega^{n-1}\eta_\lambda$, then the unique G_F-equivariant pairing $r \otimes r \to E(\Pi)_\lambda \omega^{1-n}\eta_\lambda^{-1}$ is either symplectic or orthogonal.

The sign $\eta_v(-1)$ in (a) will be denoted by $\eta_\infty(-1)$. The signs $\eta_\infty(-1)$ and $\eta_\lambda(c)$ are related as follows: there is a unique $q \in \mathbb{Z}$ such that $\eta|\cdot|^{-q}$ is an Artin character; thus,

$$\eta_\lambda(c) = (-1)^q \eta_\infty(-1).$$

If $\mathfrak{z}$ denotes the central character of Π, then $\mathfrak{z}$ is an algebraic Hecke character of F and $\mathfrak{z}^{-2} = \eta^n$. In particular, η is the square of an algebraic Hecke character when n is odd; thus, $\eta_\infty(-1) = (-1)^q = \eta_\lambda(c) = 1$ in this case.

Corollary 1.3 (Totally real field case). *If n is even and $\eta_\lambda(c) = 1$, then any irreducible constituent r of $\rho_{\Pi,\lambda}$ such that $r^{\vee} \simeq r \otimes \omega^{n-1}\eta_\lambda$ is symplectic. Otherwise, any such constituent is orthogonal.*

Proof. Let K be a totally imaginary quadratic extension of F which is ramified above some finite place v of F whose residue characteristic is prime to the one of λ, and such that Π_v is unramified. Let $\epsilon_{K/F}$ be the non-trivial character of $\mathrm{Gal}(K/F)$ or its associated Hecke character of F. As $\Pi \not\simeq \Pi \otimes \epsilon_{K/F}$ (at v), Arthur–Clozel's base change Π_K of Π to $\mathrm{GL}_n(\mathbb{A}_K)$ is cuspidal. Moreover, for each irreducible constituent r of $\rho_{\Pi,\lambda}$, $r_{|G_K}$ remains absolutely irreducible, still as $r \not\simeq r \otimes \epsilon_{K/F}$ (at v).

By [CHT08, Lemma 4.1.4], we may find some algebraic Hecke character ψ of K such that $\psi \circ N_{K/F} = \eta \circ N_{K/F}$. In particular, $\eta_{\lambda|G_K} = \psi_\lambda (\psi_\lambda{}^\perp)^{-1}$, $\Pi' := \Pi_K \otimes \psi$ is polarized (and algebraic regular) and $\rho_{\Pi',\lambda} = \rho_{\Pi,\lambda|G_K} \otimes \psi_\lambda$. Theorem 1.2 ensures that for each r as in the statement, $r_{|G_K} \otimes \psi_\lambda$ has sign $+1$ with respect to ω^{n-1}. By Lemma 2.1 below, $r_{|G_K}$ has sign $+1$ with respect to $\psi_\lambda(\psi_\lambda^\perp)^{-1}\omega^{n-1} = \eta_{\lambda|G_K}\omega^{n-1}$.

Fix a complex conjugation $c \in G_F$ and choose a matrix realization $r : G_F \to \mathrm{GL}_d(L)$ such that $r(c)$ is diagonal, so that $r(c) = {}^t r(c) = r(c)^{-1}$. For some $P \in \mathrm{GL}_d(L)$,

$$ {}^t r(g)^{-1} = P r(g) P^{-1} \eta_\lambda(g) \omega(g)^{n-1}, \quad \forall g \in G_F. $$

Applying this to c gives

$$ r(c)P = P r(c) \eta_\lambda(c)(-1)^{n-1}. \tag{3} $$

On the other hand, an immediate computation shows that for all $g \in G_K$,

$$ {}^t r(cgc)^{-1} = A r(g) A^{-1} \eta_\lambda(g) \omega(g)^{n-1} $$

with $A = r(c)P$. By the preceding paragraph, we have ${}^t A = A$, so that

$$ {}^t P = (-1)^{n-1} \eta_\lambda(c) P $$

by (3) and the corollary follows as $\eta_\lambda(c) = 1$ for odd n. □

1.4 Historical remarks

The question of the sign of Galois representations attached to polarized automorphic representations of GL_n on a totally real or CM field is out at least since Clozel, building on the work of Kottwitz, proved their existence in many cases in the mid 1990s. More recently, this question has been extensively discussed in [CHT08], where some cases of the above theorem, concerning Galois representations with some constraining properties ensuring they have a nice and workable deformation theory, are proved by a very indirect method; indeed, the whole long and hard paper is written with an unknown sign ϵ and only near the end, after the Taylor–Wiles method has been adapted to unitary groups, is it shown that $\epsilon = -1$ leads to a contradiction.

Theorem 1.2 appears, without its proof, in the concluding remarks of our book [BC09] (see [BC09, Theorem 9.5.1]) that was made public on the arXives in January 2007. We knew the proof that follows then,[2] and told it to a few colleagues, but decided to wait for a more advanced version of the book project [GRFAbook], on which it depends, before writing it.

Meanwhile, one of us, Chenevier, together with Clozel, have found a completely different proof of a special case of Corollary 1.3, namely when $\eta = 1$ and Π is square integrable at some finite place. In this case, $\rho_{\Pi,\lambda}$ is known to be irreducible by works of Harris–Taylor and Taylor–Yoshida, and they show that it is symplectic for n even. Their proof was actually conditional on the computation of some archimedean orbital integrals, which has since been done by Chenevier and Renard in [CR10]. The method of proof used by Chenevier and Clozel in [CC09] is less expensive in difficult tools than ours, using 'simply' the new insight in the trace formula they discovered. However, it does not seem that it can be extended to the case of a CM field, or even to the case of an automorphic representation that does not satisfy any local square-integrability hypothesis.

Let us mention also that in a recent preprint [Gro], Gross introduced a general notion of *odd* Galois representations and conjectured that the expected Galois representations attached to

[2] At least for places λ of residual characteristic p split in K.

definite reductive groups G are odd in his sense. Our theorem proves his conjecture when G is a unitary group attached to a CM extension K/F, in which case it has the following meaning.

Let $\tilde{G}$ be the semidirect product of $\mathrm{Gal}(K/F) = \langle c \rangle = \mathbb{Z}/2\mathbb{Z}$ by $\mathrm{GL}_n(L) \times L^*$ with respect to the order-two automorphism $(x, y) \mapsto (y\,{}^t x^{-1}, y)$ (see [CHT08, ch. I] for similar considerations). Assume that $\rho : G_K \to \mathrm{GL}_n(L)$ satisfies (1) and is absolutely irreducible; fix A a matrix as in (2) and $\epsilon = \pm 1$ the sign of ρ. Consider the morphism $G_K \to \mathrm{GL}_n(L) \times L^*$ defined by $g \mapsto (\rho(g), \chi(g)^{-1})$. A simple computation shows that this map extends to a morphism $\tilde{\rho} : G_F \to \tilde{G}$ if we set $\tilde{\rho}(c) = ({}^t A^{-1}, \epsilon)c$. Assume now that $\rho = \rho_{\Pi,\lambda}$. The map $\tilde{\rho}$ is the analogue in our situation of the map denoted ρ whose existence is conjectured in Gross [Gro, p. 8] and Gross predicts that the conjugation by $\tilde{\rho}(c)$ on $\mathrm{Lie}(\mathrm{GL}_n)$ is a Cartan involution, that is, has the form $X \mapsto -P\,{}^t X P^{-1}$ with P a *symmetric* invertible matrix. In our situation, the conjugation by $\tilde{\rho}(c)$ on the Lie algebra is the map $X \mapsto -{}^t A^{-1}\,{}^t X\,{}^t A$. So, we see that Gross' prediction amounts to 'A is symmetric', which is exactly our theorem.[3]

1.5 Idea of the proof

The idea of the proof is very simple. Assume that we know that the representation $\rho_{\Pi,\lambda}$ is irreducible. Then there is nothing to prove if n is odd. When n is even, we can reduce to the odd case, as follows: descend Π to a unitary group in n variables and transfer the result to an automorphic representation π of a unitary groups in $n+1$ variables which is compact at infinity, using a special case of endoscopic transfer proved by Clozel, Harris, and Labesse. Use eigenvarieties to deform π into a family of automorphic forms whose Galois representations are generically irreducible. For those Galois representations, the sign is $+1$ since their dimension is odd. Specialize this result to deduce that the components of the representation attached to π, including $\rho_{\Pi,\lambda}$, have sign $+1$.

There are several technical difficulties that make the proof a little bit more indirect: in the current state of science, we do not know that $\rho_{\Pi,\lambda}$ is (absolutely) irreducible, and we cannot descend Π to $\mathrm{U}(n)$ or transfer it to $\mathrm{U}(n+1)$ without supplementary assumptions on K/F and Π. Moreover, we cannot always deform a representation π into a family whose Galois representation is generically irreducible. But this is not a big issue, since, as was already observed in [BC09, § 7.7], we can actually do so in two steps, deforming π into a family whose generic members can themselves be deformed irreducibly. Similarly, the obstacle posed by the conditions on descent and endoscopic transfer can be solved by base-change techniques inspired by the ones used in [CH].

1.6 Notation and conventions

Our general convention will be that the local Langlands correspondence is normalized so that geometric Frobeniuses correspond to uniformizers (and as in [HT01]). If π is an unramified complex representation of $\mathrm{GL}_n(E)$ with E a p-adic local field, or more generally an irreducible smooth representation with a non-trivial vector fixed by an Iwahori subgroup, we shall often denote by $L(\pi)$ the semisimple conjugacy class in $\mathrm{GL}_n(\mathbb{C})$ of the geometric Frobenius in the L-parameter of π.

If K is a field, we shall denote by G_K its absolute Galois group $\mathrm{Gal}(\overline{K}/K)$; when K is a number field and v a place of K, we also write G_v for G_{K_v}.

We shall use the following notions of p-adic Hodge theory. Let us fix E a finite extension of $\mathbb{Q}_p$, $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ and let V be a p-adic representation of G_E of dimension n

[3] When ρ is not assumed to be irreducible any more, note that Theorem 1.2 still implies that we may find some symmetric A such that (2) holds; hence, a $\tilde{\rho}$ as above satisfying Gross' conjecture.

over $\overline{\mathbb{Q}}_p$. To such a representation, Sen attached a monic polynomial $P_{\mathrm{sen}}(T) \in (\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E)[T]$ of degree n, whose roots will be called the Hodge–Tate weights of V (even when they are not natural integers). Our normalization of the Sen polynomial is the one such that the Hodge–Tate weight of the cyclotomic character $\overline{\mathbb{Q}}_p(1)$ is $-1 \in \overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E$. Under the natural identification $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E = \overline{\mathbb{Q}}_p^{\,\mathrm{Hom}(E,\overline{\mathbb{Q}}_p)}$, we shall often write them as a collection $\{k_{i,\sigma}\}$ for all $i \in \{1, \ldots, n\}$ and all $\sigma \in \mathrm{Hom}(E, \overline{\mathbb{Q}}_p)$, ordered so that for each embedding σ, we have

$$k_{1,\sigma} \leqslant k_{2,\sigma} \leqslant \cdots \leqslant k_{n,\sigma}.$$

We shall need to consider various partial sums of those weights, for which the following definitions will be useful. For I a subset of $\{1, \ldots, n\} \times \mathrm{Hom}(E, \overline{\mathbb{Q}}_p)$, we denote by k_I the sum $\sum_{(i,\sigma)\in I} k_{i,\sigma}$. When $I = \{i\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$, we write k_i instead of k_I. Thus, $k_i = \sum_\sigma k_{i,\sigma}$.

Assume now that V is crystalline in the sense of Fontaine. Let $E_0 \subset E$ be the maximal unramified extension of $\mathbb{Q}_p$ inside E and let $\mathbf{v} : \overline{\mathbb{Q}}_p^* \to \mathbb{Q}$ be the valuation *normalized so that* $\mathbf{v}(p) = e$, where e is the absolute ramification index of E. Fontaine attached to V an E_0-vector space $D_{\mathrm{crys}}(V)$ with a semilinear action of the crystalline Frobenius φ (commuting with $\overline{\mathbb{Q}}_p$), and which is free of rank n over $E_0 \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$. If $f = [E_0 : \mathbb{Q}_p]$, then φ^f is $\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} E_0$-linear and commutes with φ, so its characteristic polynomial $P_\varphi(T)$ actually belongs to $\overline{\mathbb{Q}}_p[T]$. This polynomial will be referred to as the *characteristic polynomial* of φ, its roots are the *eigenvalues* of the crystalline Frobenius and their valuations (with respect to $\mathbf{v}$) its *slopes*.[4] With this notation, if the $k_{i,\sigma}$ are the Hodge–Tate weights of V, then the weak admissibility property of $D_{\mathrm{crys}}(V)$ implies in particular that

$$\mathbf{v}(P_\varphi(0)) = \sum_{i,\sigma} k_{i,\sigma}.$$

We can now explain a bit more precisely the p-adic part of Theorem 1.1. Assume that w is a finite place of K with the same residual characteristic as λ, and assume that Π_w is unramified. Let $P_w(T) \in E(\Pi)[T]$ be the characteristic polynomial of $L(\Pi_w|\cdot|^{(1-n)/2})$. Then a refinement of Theorem 1.1 asserts that *$\rho_{\Pi,\lambda}|G_w$ is a crystalline representation and that the characteristic polynomial $P_\varphi \in E(\Pi)_\lambda[T]$ of its crystalline Frobenius coincides with the image of $P_w(T)$ in* $E(\Pi)_\lambda[T]$; see [CH, Theorem 3.2.5(c)].[5]

2. Sorites on the sign

2.1 The notion of a good representation

For a representation $\rho : G_K \to \mathrm{GL}_n(L)$ that is a direct sum of absolutely irreducible and pairwise non-isomorphic representations, and that satisfies (1) for some fixed character χ, we say that ρ is *good* (with respect to χ) if every irreducible factor of ρ satisfying (1) has sign $+1$.

In this language, the theorem amounts to prove that $\rho_{\Pi,\lambda}$ is good, which is good.

2.2 Some trivial lemmas

In this paragraph, $\rho : G_K \to \mathrm{GL}_n(L)$ is a direct sum of absolutely irreducible and pairwise non-isomorphic representations, and satisfies (1).

[4] This definition is slightly different from the usual definition of the slopes of an isocrystal (which are ours divided by $[E : \mathbb{Q}_p]$), but it will be convenient to us.

[5] We will actually use this identity $P_\varphi = P_w$ only under the following extra assumptions, for which it holds by construction: assumptions (H1) and (H2) stated in § 3.1 below on K/F and Π are satisfied.

LEMMA 2.1. *If $\rho : G_K \to \mathrm{GL}_n(L)$ is good with respect to χ, and if $\psi : G_K \to L^*$ is a character, then $\rho\psi$ is good with respect to $\chi\psi^{-1}\psi^{\perp}$.*

In particular, if m is an integer and if $\psi^{\perp} = \psi\omega^{m-n}$, then ρ is good with respect to ω^{n-1} if and only if $\rho\psi$ is good with respect to ω^{m-1}.

Proof. Note first that since ρ satisfies (1) for χ, then $\rho\psi$ still satisfies (1) for the character $\chi' = \chi\psi^{-1}\psi^{\perp}$ (which still satisfies $\chi'(g^c) = \chi'(g)$) and is a sum of absolutely irreducible pairwise non-isomorphic factors, namely the $\rho_i\psi$ where the ρ_i are the factors of ρ. Now if ρ_i is an irreducible factor that satisfies (1) for χ', a matrix A that satisfies (2) for ρ_i and χ satisfies also (2) for $\rho_i\psi$ and χ'; hence, the signs of ρ_i and $\rho_i\psi$ are the same, which proves the first assertion. The last part of the lemma follows at once. □

LEMMA 2.2. *If $\rho : G_K \to \mathrm{GL}_n(L)$ is good, and ρ' is a subrepresentation of ρ that satisfies (1), then ρ' is good, too (with respect to the same character).*

Proof. The proof of this is *really* trivial. □

LEMMA 2.3. *Let F' be a totally real extension of F, and $K' = KF'$. If $\rho|G_{K'}$ has the same number of irreducible components as ρ, and if those components are absolutely irreducible and pairwise non-isomorphic, then $\rho|G_{K'}$ is good with respect to $\chi_{|G_{K'}}$ if and only if ρ is good with respect to χ.*

Proof. If ρ_i is an (absolutely) irreducible factor of ρ that satisfies (1), then $\rho_i|G_{K'}$ is still absolutely irreducible by hypothesis, still satisfies (1) and has obviously the same sign as ρ_i. The lemma follows. □

2.3 A specialization result

In this paragraph, $\mathcal{O}$ is a henselian discrete valuation domain with fraction field L and residue field k, such that $2 \in \mathcal{O}^*$. We set also $G = G_K$ and assume that the character $\chi : G_K \to L^*$ actually falls into $\mathcal{O}^*$; thus, it makes sense to talk about condition (1) for k or L-valued representations of G (by a slight abuse of language, we shall also denote by χ the residual character $G_K \to k^*$). A simple but crucial observation for our proof is the following proposition.

PROPOSITION 2.4. *Assume that $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ is such that $\rho \otimes L$ and $\bar{\rho}^{\mathrm{ss}}$ are a sum of absolutely irreducible pairwise non-isomorphic representations and satisfy (1). If $\rho \otimes L$ is good with respect to χ, then so is $\bar{\rho}^{\mathrm{ss}}$.*

Moreover, the converse holds if $\bar{\rho}^{\mathrm{ss}}$ has the same number of irreducible factors as $\rho \otimes L$.

Of course, in this statement $\bar{\rho}^{\mathrm{ss}}$ denotes the semisimplification of the reduction $\bar{\rho} := \rho \otimes_{\mathcal{O}} k$ of ρ.

Proof. Let $\bar{\rho}_1$ be a factor of $\bar{\rho}^{\mathrm{ss}}$ that satisfies (1). Let $\tau_1, \dots, \tau_k$ be the irreducible factors of $\rho \otimes L$. For each of them, we can choose a stable $\mathcal{O}$-lattice, and see them as representations of G over $\mathcal{O}$. We have $\bar{\rho}^{\mathrm{ss}} = \bigoplus_{i=1}^{k} \overline{\tau_i}^{\mathrm{ss}}$, so $\bar{\rho}_1$ appears in exactly one of the $\overline{\tau_i}^{\mathrm{ss}}$, say $\overline{\tau_1}^{\mathrm{ss}}$. Moreover, $\overline{\tau_1}^{\perp}$ is isomorphic to $\overline{\tau_i}\chi$ for some $i \in \{1, \dots, k\}$. But, it follows that $\overline{\tau_i}^{\mathrm{ss}}$ contains $\bar{\rho}_1$ (since $\bar{\rho}_1$ satisfies (1)), so the only possibility is that $i = 1$. In other words, τ_1 satisfies (1) and, replacing ρ by τ_1, we are reduced to prove the lemma with the supplementary assumption that $\rho \otimes L$ is absolutely irreducible. In that case, the proposition is [BC09, Lemma 1.8.8].

Since this proposition is really one of the main tools used in our proof, and since the proof of [BC09, Lemma 1.8.8] is a little bit difficult to separate from the other concerns of [BC09, § 1.8], let us sketch it here for the convenience of the reader, trying to be as pedagogical as possible.

Note first that the basic point that makes the result not obvious is that there is no reason that we can find a matrix A for $\rho \otimes L$ as in (2) with $A \in \mathrm{GL}_n(\mathcal{O})$. *A priori*, we just have $A \in \mathrm{GL}_n(L)$, and it is therefore not possible to reduce (2) mod m.

There is one case, however, where a simple proof is possible: assume that $\bar{\rho}^{\mathrm{ss}}$ is absolutely irreducible. In this case, the representations $\rho^{\perp}$ and $\rho\chi$ over $\mathcal{O}$, being isomorphic over L and residually absolutely irreducible, are isomorphic over $\mathcal{O}$ by a theorem of Serre and Carayol (cf. [Car94]). In other words, we can find a matrix $A \in \mathrm{GL}_n(\mathcal{O})$ such that (2) holds and, reducing this modulo m, we get that $\rho \otimes L$ and $\bar{\rho}$ have the same sign in this case. Note that this proves also the last assertion of the theorem (in all cases!).

The proof of the general case consists in reducing to the residually irreducible case. This is not possible, however, if we keep working with representations of groups only. We have to work in the larger world of representations of algebras instead. As we saw, we may assume that $\rho \otimes L$ is absolutely irreducible, and we set $\bar{\rho}^{\mathrm{ss}} = \bigoplus_i \bar{\rho}_i$.

Let R be the algebra $\mathcal{O}[G]$ and $S = \rho(R) \subset M_n(\mathcal{O})$. We have $S \otimes_{\mathcal{O}} L = M_n(L)$. The algebra S is provided with a natural $\mathcal{O}$-algebra anti-automorphism τ, induced by the one on R defined on $g \in G$ by $g \mapsto \chi(g)^{-1}(g^c)^{-1}$. Explicitly, by (2), we have, for $M \in S$,

$$\tau(M) = {}^tA^{-1t}M^tA, \tag{4}$$

and by our sign assumption ${}^tA = A$: the involution τ is a symmetric involution of the matrix algebra $S \otimes_{\mathcal{O}} L = M_n(L)$.

On the other hand, let $\overline{S}$ denote the image of $k[G]$ in the k-endomorphisms of the representation $\bar{\rho}^{\mathrm{ss}} = \bigoplus_i \bar{\rho}_i$. Then $\overline{S} \simeq \prod_i M_{n_i}(k)$ ($n_i = \dim \bar{\rho}_i$, $\sum_i n_i = n$) and $\overline{S}$ is also provided with a natural k-algebra anti-automorphism τ as above. Moreover, there is a natural surjective $\mathcal{O}$-algebra map $S \to \overline{S}$ which is τ-equivariant.

Let us denote by $\epsilon_i \in \overline{S}$ the central idempotent corresponding to $\bar{\rho}_i$. It is well known that ϵ_i can be lifted as an idempotent e_i of S as $\mathcal{O}$ is henselian and S finite over $\mathcal{O}$. However, we need a more precise lifting result. Let us fix an i such that $\bar{\rho}_i$ satisfies (1); then we have $\tau(\epsilon_i) = \epsilon_i$. What we need is an idempotent e_i in S lifting ϵ_i, such that $\tau(e_i) = e_i$. The existence of such an idempotent is easy to prove: first choose any lift $x \in S$ of ϵ_i and let S_0 be the sub-$\mathcal{O}$-algebra generated by $\frac{1}{2}(x + \tau(x))$. Obviously, τ fixes any element of S_0. The restriction of the natural surjection $S \to \overline{S}$ to S_0 is onto a k-subalgebra of $\overline{S}$ that contains the image of $\frac{1}{2}(x + \tau(x))$, that is, ϵ_i. Thus, defining e_i as a lift of ϵ_i in S_0 does the job. (This result is the trivial case of [BC09, Lemma 1.8.2].) As $\bar{\rho}_i$ is absolutely irreducible and has multiplicity one in $\bar{\rho}^{\mathrm{ss}}$, it actually turns out that the rank of e_i is $n_i = \dim \bar{\rho}_i$, and that $e_i S e_i \simeq M_{n_i}(\mathcal{O})$. Replacing ρ by a conjugate if necessary, we may then assume that e_i is a diagonal idempotent of rank n_i in $M_n(L)$.

Applying (4) to $M = e_i$, we get $Ae_i = {}^te_i{}^tA$, that is, Ae_i is symmetric. In other words, τ induces a symmetric involution on $e_i S e_i \simeq M_{n_i}(\mathcal{O})$. As a consequence, τ also induces a symmetric involution on $\epsilon_i \overline{S} \varepsilon_i = \mathrm{End}_k(\bar{\rho}_i)$, which exactly means that the sign of $\bar{\rho}_i$ is $+1$. $\square$

Remark 2.5. The above proposition, or rather its crucial case [BC09, Lemma 1.8.8], is a theorem of Thompson in the case where G is a finite group, $\chi = 1$ and the involution $g \mapsto g^c$ is the identity: see [Tho84, last theorem].

3. Proof of the main theorem

3.1 Proof of Theorem 1.2 under special hypotheses

We shall first prove the theorem under a set of additional hypotheses on the CM extension K/F, the automorphic representation Π and the place λ.

Let us call p the residual characteristic of λ. Recall that the automorphic representation Π defines an embedding $\iota : E(\Pi) \to \mathbb{C}$. We fix once and for all algebraic closures $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$, as well as some embeddings $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}}_p \to \mathbb{C}$ such that the induced map $\iota_p \iota_\infty^{-1} \iota : E(\Pi) \to \overline{\mathbb{Q}}_p$ factors through $E(\Pi)_\lambda$.

3.1.1 *Some special hypotheses.* (H1) Special Hypotheses 2.2 of [CH], that is:

(H1a) K/F is unramified at all finite places;

(H1b) Π_v is spherical at all non-split non-archimedean places v of K;

(H1c) the degree $[F : \mathbb{Q}]$ is even.

(H2) Hypothesis 1.3 of [CHL], that is, for all real places σ of K, the infinitesimal character of Π_σ is sufficiently far from the walls.[6]

(H3) There[7] is a place v above p in F that splits in K and, for w a place of K above v, Π_w is unramified. Denote by $\{\varphi_1, \ldots, \varphi_n\}$ the eigenvalues of $L(\Pi_w |\cdot|^{(1-n)/2})$. Then the Hodge–Tate weights $\{k_{i,\sigma}\}$ of $\rho_{\Pi,\lambda}|G_w$ and the slopes $\mathbf{v}(\varphi_j)$ are in sufficiently general position in the following sense: if

$$c = \max_{i \in \{1,\ldots,n\}} \min_{j \in \{1,\ldots,n\}} |\mathbf{v}(\varphi_i) - k_j|,$$

then, for all distinct subsets I and J of $\{1, \ldots, n\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ with $|I| = |J| < n[K_w : \mathbb{Q}_p]$, we have

$$|k_I - k_J| > (n+1) \cdot c.$$

In (H3) above, $\mathbf{v} : \overline{\mathbb{Q}}_p^* \to \overline{\mathbb{Q}}$ is the valuation such that $\mathbf{v}(p)$ is the ramification index of p in K_w.

3.1.2 *The theorem.* We want to prove the following theorem.

THEOREM 3.1. *With the supplementary hypotheses (H1), (H2) and (H3), Theorem 1.2 holds.*

The rest of this subsection is entirely devoted to the proof of this theorem.

3.1.3 *Descent and transfer.* Let $m = n$ if n is odd, and $m = n+1$ if n is even, so that m is always odd. Let us call $\mathrm{U}(m)$ the unitary group over F attached to K in m variables that is quasi-split at every finite place of F and compact at every infinite place. Since m is odd, such a group always exists (uniquely up to isomorphism). Actually, $\mathrm{U}(m)$ is simply the standard unitary group attached to the hermitian form $\sum_{i=1}^m N_{K/F}(z_i)$ on K^m (see [BC09, § 6.2.2]).

If n is odd, that is if $n = m$, by hypothesis (H1) and Labesse's base-change theorem [Lab, Theorem 5.4], we can descend Π to a representation π of $\mathrm{U}(m)$ with $\pi_v \simeq \Pi_w$ for every place w of K split over v in F (with the natural identification $\mathrm{U}(n)(F_v) \simeq \mathrm{GL}_n(K_w)$), and such that for

[6] Precisely, this means that the extremal weight of the associated algebraic representation of $\mathrm{GL}_n(K \otimes \mathbb{R})$ does not belong to a wall.
[7] See § 1.6 for the notation used in this assumption.

each complex place w of K above a real place v of F, π_v has the same infinitesimal character as Π_w (under the natural identification $\mathrm{U}(n)(K_w) \simeq \mathrm{GL}_n(\mathbb{C})$).

If n is even, we use a result of endoscopic transfer due to Clozel *et al.* [CHL]. Note first that using $\iota_\infty \iota_p^{-1}$, if $v = ww^c$ is as in (H3) we may identify $\mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p) = \mathrm{Hom}(F_v, \overline{\mathbb{Q}}_p)$ with subsets Σ_v and Σ_w of $\mathrm{Hom}(F, \mathbb{R})$ and $\mathrm{Hom}(K, \mathbb{C})$. Let us first fix

$$\mu : K^* \backslash \mathbb{A}_K^* \to \mathbb{C}^*$$

a Hecke character such that $\mu(c(x))^{-1} = \mu(x)$, and such that for each $s \in \Sigma_v$, $\mu_s(z) = (\sigma_s(z)/\overline{\sigma_s(z)})^{\frac{1}{2}}$, where $\sigma_s \in \Sigma_w$ is associated to s as above. This last assumption implies that $\mu|\mathbb{A}_F^*$ coincides with the sign of K/F, and that μ does not come by base change from a Hecke character of $\mathrm{U}(1)$ (see e.g. [BC09, § 6.9.2]). Such a Hecke character always exists and, as K/F is unramified at all finite places, we can even assume (and we will) that it is unramified at the finite places of K which are either above p or not split above F. Let us choose another Hecke character

$$\chi : K^* \backslash \mathbb{A}_K^* \to \mathbb{C}^*$$

such that $\chi(c(z))^{-1} = \chi(z)$ but assume now that χ descends to $U(1)$, i.e. that for each real place $s \in \Sigma_v$, $\chi_s(z) = \sigma_s(z/c(z))^{-a_s}$ for some $a_s \in \mathbb{Z}$. Assume also that χ is unramified at the finite places of K which do not split over F. Under hypotheses (H1) and (H2), and if all the a_s are big enough, by [CHL, Theorem 4.7] we can transfer Π to an automorphic representation π of $\mathrm{U}(m)$ in such a way that at every place w of K split over a place v in F, we have

$$L(\pi_v) = L(\Pi_w \mu_w) \oplus L(\chi_w). \tag{5}$$

Moreover, for each real place v of F and each complex place w of K above v, the infinitesimal character of π_v is obtained from the one of $\Pi_w \mu_w$ in the obvious way: in terms of the associated Harish-Chandra's cocharacter, it is the direct sum of the one of $\Pi_w \mu_w$ and the one of χ_w.

In both cases (n even or odd), Clozel *et al.* actually constructed a π which is moreover unramified at all the finite places of K which are not split over F (we do not really need this, but this fixes ideas).

3.1.4 *Consequences of (H3).* When $n = m$ is odd, we set $\rho_\pi := \rho_{\Pi,\lambda}$. When n is even, the G_K representation of dimension m attached to π is by definition

$$\rho_\pi := \rho_{\Pi,\lambda}(\mu|\cdot|^{-\frac{1}{2}}) \oplus \chi|\cdot|^{(1-m)/2}.$$

Note that $\mu|\cdot|^{-\frac{1}{2}}$ and $\chi|\cdot|^{(1-m)/2}$ are both algebraic Hecke characters of K. We shall identify them here with their p-adic realizations given by ι_∞ and ι_p and we focus on the place w as in (H3). By assumption, $\mu|\cdot|^{-\frac{1}{2}}$ is actually unramified at the place w, and $\chi|\cdot|^{(1-m)/2}$ is crystalline, and we shall denote by φ_μ and $\varphi_\chi \in \overline{\mathbb{Q}}_p^{\,*}$ their associated Frobenius eigenvalues.

If n is even, so $m = n+1$, we set for each $\sigma \in \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$,

$$k_{m,\sigma} := \frac{m-1}{2} + a_\sigma$$

(where σ is viewed as an element of $\mathrm{Hom}(F, \mathbb{R})$ as above). Thus, in any case, the $k_{i,\sigma}$ for $i = 1, \ldots, m$ and $\sigma \in \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ are the Hodge–Tate weights of $\rho_\pi | G_w$. We shall use for this extended collection $\{k_{i,\sigma}\}$ with all $i \in \{1, \ldots, m\}$ and, for a subset I of $\{1, \ldots, m\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$, the notation k_I analogous to the one in § 1.6.

If n is even, we set $\varphi'_i := \varphi_i\varphi_\mu$ for $i < m$ and $\varphi'_m := \varphi_\chi$. We have $\mathbf{v}(\varphi'_i) = \mathbf{v}(\varphi_i)$ for $i < m$ and $\mathbf{v}(\varphi'_m) = k_m$. When $n = m$ is odd, we shall simply set $\varphi'_i := \varphi_i$. Thus, in both cases, $\varphi'_1, \ldots, \varphi'_m$ are the Frobenius eigenvalues of $L(\pi_w|\cdot|^{(1-m)/2})$.

If n is even, we make precise now our choice of χ. We assume that the $k_{m,\sigma} = a_\sigma + (m-1)/2$ are all big with respect to the k_i and $\mathbf{v}(\varphi'_i)$ for $i = 1, \ldots, n$, and also that they are set sufficiently far apart so that any non-trivial sum of the form $\sum_{\sigma\in S} \pm k_{m,\sigma}$, where $S \subset \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$, is big in the same sense as above. This is of course always possible. With those assumptions we have the following lemma.

LEMMA 3.2. (i) *The representation π_v is a fully induced unramified principal series, and the eigenvalues of $L(\pi_w|\cdot|^{(1-m)/2})$ are $\varphi'_1, \ldots, \varphi'_m$.*

(ii) *We have $c = \max_{i\in\{1,\ldots,m\}} \min_{j\in\{1,\ldots,m\}} |\mathbf{v}(\varphi'_i) - k_j|$ and, for all distinct subsets I and J of $\{1, \ldots, m\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ and with $|I| = |J| < m[K_w : \mathbb{Q}_p]$, we have $|k_I - k_J| > m \cdot c$.*

Proof. By (H3) and, if n is even, by (5), π_v is unramified. Moreover, the eigenvalues of $L(\pi_w|\cdot|^{(1-m)/2})$ are $\varphi'_1, \ldots, \varphi'_m$, and no quotient of those eigenvalues is equal to q, the cardinal of the residue field of K_w. Indeed, a well-known result of Jacquet–Shalika asserts that for $i = 1, \ldots, n$, the complex numbers $q^{(1-m)/2}\iota_\infty\iota_p^{-1}(\varphi'_i)$ are $<q^{1/2}$ in absolute value, and $q^{(1-m)/2}\iota_\infty\iota_p^{-1}(\varphi'_m)$ has norm 1 by construction. Hence, π_v is a full unramified principal series by Zelevinski's theorem, which is (i).

For (ii), there is nothing to prove if $n = m$ is odd. Assume that n is even, so that $m = n + 1$. Let us note that for $i = m$, we have $\min_{j\in\{1,\ldots,m\}} |\mathbf{v}(\varphi'_i) - k_j| = 0$ since $\mathbf{v}(\varphi'_m) = k_m$. For $i \leqslant n$, the minimum $\min_{j\in\{1,\ldots,m\}} |\mathbf{v}(\varphi'_i) - k_j| = 0$ is not realized for $j = m$ because k_m is much too big.

Hence,

$$\begin{aligned}\max_{i\in\{1,\ldots,m\}} \min_{j\in\{1,\ldots,m\}} |\mathbf{v}(\varphi'_i) - k_j| &= \max_{i\in\{1,\ldots,n\}} \min_{j\in\{1,\ldots,n\}} |\mathbf{v}(\varphi'_i) - k_j| \\ &= c.\end{aligned}$$

It remains to prove that $|k_I - k_J| > mc = (n+1)c$. Let I_0 (respectively J_0) be the subset of I (respectively of J) of pairs (i, σ) with $i = m$. If $I_0 = J_0$, then $k_I - k_J = k_{I-I_0} - k_{J-J_0}$ and, since $I - I_0$, $J - J_0$ are distinct subsets of the same cardinality of $\{1, \ldots, n\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$, the desired inequality comes directly from (H3). If $I_0 \neq J_0$, $k_I - k_J$ contains, in addition to a bounded number of terms $\pm k_{i,\sigma}$ for $i \leqslant n$, a non-trivial sum of the form $\sum_S \pm k_{\sigma,m}$, where $S \subset \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$; hence, $|k_I - k_J|$ is again greater than mc. □

3.1.5 *Eigenvarieties and their families of Galois representations.* We are ready now to start the deformation argument. Let $U = \prod_v U_v \subset \mathrm{U}(m)(\mathbb{A}_{F,f})$ be a compact open subgroup such that $\pi^U \neq 0$, and assume that U_v is an Iwahori subgroup for the place v of (H3) and that U_v is hyperspecial for all places v of F that are not split in K.

From now on, we shall reserve the notation v for the place of F of hypothesis (H3), and w for one of the places of K above v. We shall denote by d the degree of the field $F_v = K_w$ over $\mathbb{Q}_p$. To U, the place v and (ι_p, ι_∞), we can attach by [Che, Theorem 1.6] (see also [Che04] and [Eme06]) an eigenvariety $X = X_{U,v,(\iota_\infty,\iota_p)}$ for the group $\mathrm{U}(m)/F$, which is a reduced rigid analytic space over $\mathbb{Q}_p$ of equidimension[8] md.

[8] It is not necessary here to let the weights corresponding to the other possible places of F above p move, but we could have, and the eigenvariety would then have dimension $m[K : \mathbb{Q}]$.

By Labesse's base-change theorem [Lab, Corollary 5.3], if π' is any automorphic representation of $\mathrm{U}(m)$ which is unramified outside the split finite places of K/F, then π' admits a base change to GL_m/K (which is strong at each finite place and cohomological at each archimedean place with the expected infinitesimal character); hence, a Galois representation by Theorem 1.1. As explained in [BC09, ch. 7.5] (or in [Che04]), this is enough to equip X with a continuous m-dimensional pseudocharacter $T: G_K \to \mathcal{O}(X)$ of dimension m. The eigenvariety X and this T satisfy a number of properties and we will only list below the ones we shall need. If $x \in X(\bar{\mathbb{Q}}_p)$, we denote by T_x the evaluation of T at x and ρ_x the semisimple representation $G_K \to \mathrm{GL}_m(\bar{\mathbb{Q}}_p)$ of trace T_x. There are:

(i) Zariski-dense and accumulation subsets $Z^{\mathrm{reg}} \subset Z \subset X(\overline{\mathbb{Q}}_p)$ of *classical points*;

(ii) a set of dm analytic functions[9] $\kappa_{1,\sigma}, \ldots, \kappa_{m,\sigma}$, where σ runs over the embeddings $K_w \to \overline{\mathbb{Q}}_p$;

(iii) a set of locally constant functions $s_1, \ldots, s_m : X(\bar{\mathbb{Q}}_p) \to \mathbb{Q}$

satisfying the following conditions:

(a) if $z \in Z$, $\rho_z|G_w$ is crystalline;

(b) if $z \in Z$, the ordered Hodge–Tate weights of $\rho_z|G_w$ are $\{\kappa_{i,\sigma}\}$;

(c) let C be any real number and $Z_C := \{z \in Z^{\mathrm{reg}}, |\kappa_I(z) - \kappa_J(z)| > C$ for all distinct subsets I, J of $\{1, \ldots, m\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ such that $|I| = |J| < md\}$. Then Z_C is Zariski-dense and accumulation in X.

Moreover, the classical points z in Z correspond to pairs $(\pi(z), \mathcal{R}(z))$, where $\pi(z)$ is an automorphic representation of $\mathrm{U}(m)$ such that $\pi(z)^U \neq 0$ and $\mathcal{R}(z) = (\varphi_1, \ldots, \varphi_m)$ is an accessible refinement[10] of $\pi(z)_w|\cdot|^{(1-m)/2}$, in the following sense: ρ_z is the Galois representation attached to the base change of $\pi(z)$ to GL_m/K by Theorem 1.1 and, for each $i = 1, \ldots, m$, $\mathbf{v}(\varphi_i) = s_i(z) + \kappa_i(z)$.

(d) If $z \in Z$ parameterizes $(\pi(z), \mathcal{R}(z) = (\varphi_1, \ldots, \varphi_m))$, then for all i we have $\mathbf{v}(\varphi_i) = s_i(z) + \kappa_i(z)$.

The subset $Z^{\mathrm{reg}} \subset Z$ parameterizes refined automorphic representations $(\pi, \mathcal{R})$ satisfying some additional properties, and for our concerns here we shall simply assume that they are those $(\pi, \mathcal{R})$ such that π_v is unramified and such that for each real place s inducing v via ι_p and ι_∞, the infinitesimal character of π_s is sufficiently far from the walls. Under this latter condition, and by [Lab, Corollary 5.3] again, the base change of an automorphic representation of $\mathrm{U}(m)$ is not necessarily cuspidal, but always associated to a decomposition $m_1 + \cdots + m_r = m$ and an r-tuple $(\pi_1, \ldots, \pi_r)$ of cuspidal (polarized, cohomological) automorphic representations π_i of $\mathrm{GL}_{m_i}(\mathbb{A}_K)$; moreover, each π_i satisfies property (H2) in dimension m_i and is unramified at v. In particular, for a $z \in Z^{\mathrm{reg}}$, the characteristic polynomial of the crystalline Frobenius of $\rho_z|G_w$ coincides with the polynomial $P_w(T)$ associated to $\pi_w|\cdot|^{(1-m)/2}$ by the refinement of Theorem 1.1 recalled in § 1.6, and we also have the following.

(d′) If $z \in Z^{\mathrm{reg}}$, then the m slopes of the crystalline Frobenius of $\rho_z|G_w$ are the $s_i(z) + \kappa_i(z)$ for $i = 1, \ldots, m$.

[9] Again, we shall use for this collection $\{\kappa_{i,\sigma}\}$ and, for a subset I of $\{1, \ldots, m\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ (respectively an $i \in \{1, \ldots, m\}$), the notation κ_I (respectively κ_i) analogous to the one in § 1.6.

[10] Recall that a *refinement* of an irreducible smooth representation ρ of $\mathrm{GL}_m(K_w)$ such that $\rho^I \neq 0$ for I an Iwahori subgroup is an ordering $(\varphi_1, \ldots, \varphi_m)$ of the eigenvalues of $L(\rho)(\mathrm{Frob}_w)$. It is said to be *accessible* if ρ appears as a *subrepresentation* of the induced representation $\mathrm{Ind}_B^{\mathrm{GL}_m(K_w)} \chi \delta_B^{1/2}$, where B is (say) the upper Borel subgroup, δ_B the modulus character and χ the (unramified) character of the diagonal torus sending $(x_1, \ldots, x_m)$ to $\prod_{i=1}^m \varphi_i^{\mathbf{v}(x_i)}$ (see [BC09, § 6.4.4]).

3.1.6 *Choice of a refinement.* Going back to the representation π introduced above, if we choose an accessible refinement $\mathcal{R}$ of π_v, there is a point $z_0 \in Z$ corresponding to $(\pi, \mathcal{R})$.

Lemma 3.3. *There exists a refinement $\mathcal{R}$ of π_v such that the pseudocharacter T is generically irreducible in a neighbourhood of the corresponding point z_0.*

(This means that there exists an affinoid neighbourhood Ω of z_0 such that for any x in some Zariski-dense subset of Ω, the representation ρ_x is irreducible.)

Proof. We shall eventually show that the conclusion holds for $T|G_w$. Note that, by construction, for all $\sigma \in \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ and $i \in \{1, \ldots, m\}$, $\kappa_{i,\sigma}(z_0) = k_{i,\sigma}$. Let us first renumber the $\varphi'_i \in \overline{\mathbb{Q}}_p^*$ so that $|\mathbf{v}(\varphi'_i) - k_i| = \min_j |\mathbf{v}(\varphi'_i) - k_j|$. By Lemma 3.2(ii), there is one and only one way to do so, and this being done we have $\mathbf{v}(\varphi'_1) < \mathbf{v}(\varphi'_2) < \cdots < \mathbf{v}(\varphi'_m)$ (strict inequalities). Then consider a transitive permutation σ of $\{1, \ldots, m\}$. We choose the refinement

$$\mathcal{R} = (\varphi'_{\sigma(1)}, \ldots, \varphi'_{\sigma(m)}).$$

Since π_v is a full unramified principal series by Lemma 3.2(i), all the refinements of π_v are accessible, so π together with $\mathcal{R}$ defines a point z_0.

Before proving the irreducibility property of the lemma, let us observe a combinatorial property of this refinement. We have by definition $\kappa_i(z_0) = k_i$ and $s_i(z_0) = \mathbf{v}(\varphi'_{\sigma(i)}) - k_i$. We claim that for any non-empty proper subset $I \subset \{1, \ldots, m\}$,

$$\sum_{i \in I} s_i(z_0) \neq 0. \tag{6}$$

Indeed, we compute

$$\begin{aligned}
\left|\sum_{i \in I} s_i(z_0)\right| &= \left|\sum_{j \in J} \mathbf{v}(\varphi'_j) - \sum_{i \in I} k_i\right| \quad \text{where } J = \sigma(I) \\
&= \left|\left(\sum_{j \in J} k_j - \sum_{i \in I} k_i\right) + \sum_{j \in J}(\mathbf{v}(\varphi'_j) - k_j)\right| \\
&\geqslant \left|\left(\sum_{j \in J} k_j - \sum_{i \in I} k_i\right)\right| - \sum_{j \in J} |\mathbf{v}(\varphi'_j) - k_j| \\
&> mc - mc \quad \text{by Lemma 3.2(ii) as } I \neq J \\
&= 0.
\end{aligned}$$

Let us choose now some affinoid neighbourhood Ω of $z_0 \in X$ on which the s_i are constant and in which Z_C is Zariski-dense for $C = \sum_{i=1}^m |s_i(z_0)|$. We claim that for every point z of $Z_C \cap \Omega$, $\rho_z|G_w$ is irreducible. Indeed, if it was not, it would have a subrepresentation of dimension $0 < r < m$ and, by the weak admissibility of $D_{\mathrm{crys}}(\rho_z|G_w)$, there would exist a subset $I \subset \{1, \ldots, m\}$ of cardinal r, and a subset $J \subset \{1, \ldots, m\} \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p)$ with $|J| = rd$, such that

$$\sum_{i \in I} (\kappa_i(z) + s_i(z)) = \kappa_J(z)$$

(here we use that $z \in Z^{\mathrm{reg}}$ and property (d′) of eigenvarieties). Since $z \in Z_C$, we see at once that $I \times \mathrm{Hom}(K_w, \overline{\mathbb{Q}}_p) = J$. But, this implies that

$$0 = \sum_{i \in I} s_i(z),$$

a contradiction with (6) as $s_i(z) = s_i(z_0)$ for all i. □

3.1.7 *End of the proof.* Let $\Omega \subset X$ be the neighbourhood defined above of the point z_0, and let A be a complete discrete valuation ring, with a map of $\operatorname{Spec} A$ to the spectrum of the rigid local ring $\mathcal{O}_{z_0}$ of X at z_0 which sends the special point of $\operatorname{Spec} A$ to z_0 and the generic point to the generic point of any irreducible component of Ω containing z_0. Let us call L the fraction field of A and m its maximal ideal. By pulling back the pseudocharacter T over A, we get a representation $\rho: G_K \to \mathrm{GL}_m(A)$ such that $\rho \otimes L$ is absolutely irreducible and satisfies (1) (for $\chi = \mathbb{Q}_p(m-1)$) and

$$\bar{\rho}^{\mathrm{ss}} = \begin{cases} \rho_{\Pi,\lambda} & \text{if } m = n, \\ \rho_{\Pi,\lambda}(\mu|\cdot|^{-1/2}) \oplus (\chi|\cdot|^{(1-m)/2}) & \text{if } m = n+1. \end{cases}$$

Since $\rho \otimes L$ is absolutely irreducible and satisfies (1), it has a sign that can only be $+1$. Hence, it is good, and so is $\bar{\rho}^{\mathrm{ss}}$ by Proposition 2.4, with respect to ω^{m-1}. Hence, $\rho_{\Pi,\lambda}$ is good with respect to ω^{n-1} by Lemmas 2.2 and 2.1, as $\psi := \mu|\cdot|^{-1/2}$ satisfies $\psi^\perp = \psi\omega$. □

3.2 Weakening of the hypothesis (H3), removal of (H2)

We consider the following variant of (H3):

(H4) each place of F above p splits in K and, if w is a place of K above p, then Π_w has a non-zero vector invariant by an Iwahori subgroup of $\mathrm{GL}_n(K_w)$.

Theorem 3.4. *With the supplementary hypotheses (H1) and (H4), Theorem 1.2 holds.*

We shall argue by induction on $n \geqslant 1$. There is nothing to show if $n = 1$.

Let $\mathrm{U}(n)$ be the n-variable unitary group over F attached to K/F that is quasi-split at every finite place and compact at every infinite place. Hasse's principle shows that this group exists, even if n is even, by condition (H1c) (see e.g. [CH, Lemma 3.1]). Moreover, condition (H1) and Labesse's base-change theorem also ensure that Π descends to an automorphic representation π for $\mathrm{U}(n)$. Again, π is unramified at non-split finite places of K/F and, for each complex place w of K above a real place v of F, π_v has the same infinitesimal character as Π_w (under the natural identification $\mathrm{U}(n)(K_w) \simeq \mathrm{GL}_n(\mathbb{C})$).

Let $U = \prod_v U_v \subset \mathrm{U}(n)(\mathbb{A}_{F,f})$ be a compact open subgroup such that $\pi^U \neq 0$, and assume that $U_v = I_v$ for each place v above p and that U_v is hyperspecial for each place v of K that is not split over F. Let X be the eigenvariety attached to U, to *all the finite places of F above p* (in the setting of [Che, § 1.1], S_p is the set of all the places above p) and ι_∞, ι_p. Now X has equidimension $n[F:\mathbb{Q}]$ but all that we said for the eigenvarieties of $\mathrm{U}(m)$ in § 3.1.5 also applies to this X with minor changes, the only difference being that there is no preferred place v above p. Precisely, let us fix once and for all a place v of F above p, as well as a place w of K above v. Then (i), (ii), (iii), (a), (b) and (c) hold (with m replaced by n). To be perfectly correct for (i) and (a), we have to make precise that Z (respectively Z^{reg}) parameterizes now the refined automorphic representations $(\pi, \mathcal{R})$ of $\mathrm{U}(n)$ such that $\pi^U \neq 0$ (respectively such that π_v is unramified and such that the infinitesimal character of π_s is sufficiently regular *for all the real places s of F*).

Let $z_0 \in Z$ be the point corresponding to π together with some accessible refinement of π_x for each place x of F above p. As a general fact (see [BC09, § 6.4.4]), there is always such a refinement (for each x) and we choose them anyhow here.

Let c be the maximum of the $|s_i(z_0)|$ and choose $C \in \mathbb{R}$ such that $C > nc$. Let $\Omega \subset X$ be an open affinoid of X containing z_0, in which Z_C is Zariski-dense, and on which the s_i are constant. We claim that for $z \in Z_C$, ρ_z is good. Indeed, let $\Pi(z)$ be Labesse's base change of $\pi(z)$ to $\mathrm{GL}_n(\mathbb{A}_K)$. As $z \in Z^{\mathrm{reg}}$, and as explained in § 3.1.5, there exist a decomposition $n = n_1 + \cdots + n_r$ and cuspidal automorphic representations Π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_K)$, satisfying (H1b), (H2) and unramified at v, such that

$$\rho_z = \bigoplus_{i=1}^{r} \rho_{\Pi_i,\lambda} \otimes \chi_i$$

for some characters $\chi_i : G_K \to \overline{\mathbb{Q}}_p^*$ such that $\chi_i^{\perp} = \chi_i \omega^{n-n_i}$. If $r > 1$, then ρ_z is good by induction and Lemma 2.1. If $r = 1$, then $\Pi(z)$ is cuspidal and it satisfies (H2) and (H3) by construction, so ρ_z is good by Theorem 3.1. (To check (H3), remark that for such a z, and for each $i \in \{1, \ldots, n\}$, we have $\mathrm{Min}_j(s_i(z) + k_i(z) - k_j(z)) = s_i(z) = s_i(z_0)$.)

Let W be any irreducible component of Ω containing z_0, and $\mathrm{Frac}(W)$ its associated function field. As Z_C is Zariski-dense in Ω, we may find a $z \in Z_C \cap W$ such that the pseudocharacters T_z and $T \otimes_{\mathcal{O}(\Omega)} \mathrm{Frac}(W)$ have the same number of irreducible factors. Such factors are necessarily absolutely irreducible here, by [BC09, Theorem 1.4.4(iii)]. Arguing as in the preceding section, let A be a complete discrete valuation ring with a map from $\mathrm{Spec}\, A$ to $\mathrm{Spec}\, \mathcal{O}_z$ which sends the special point of $\mathrm{Spec}\, A$ to z and its generic point to $\mathrm{Frac}(W)$, and let $\rho : G_K \to \mathrm{GL}_n(A)$ be a representation with trace T such that $\rho \otimes_A L$ is a direct sum of absolutely irreducible, distinct representations (use e.g. [BC09, Proposition 1.6.1]). As we saw, $\bar{\rho}^{\mathrm{ss}} = \rho_z$ is good and hence so is $\rho \otimes_A L$ by Proposition 2.4, as well as $\rho \otimes_A \mathrm{Frac}(W)$ for any irreducible component W containing z_0. But, arguing back now at the point z_0 as in the preceding section, we obtain that $\rho_{z_0} = \rho_{\Pi,\lambda}$ itself is good as a specialization of a good representation, and we are done.

3.3 Removal of Hypotheses (H1) and (H4)

We now prove Theorem 1.2.

Lemma 3.5. *Let L be a finite extension of $\mathbb{Q}_p$ and $\rho : G_K \longrightarrow \mathrm{GL}_n(L)$ a continuous representation which is a direct sum of absolutely irreducible representations. There is a finite Galois extension M/K such that for every finite extension K'/K linearly disjoint from M, ρ and $\rho|G_{K'}$ have the same number of irreducible factors, and the irreducible factors of $\rho|G_{K'}$ are absolutely irreducible.*

Proof. We can assume without loss of generality that ρ is absolutely irreducible. In particular, there exist n^2 elements $g_1, \ldots, g_{n^2}$ such that the $\rho(g_i)$ generate $M_n(L)$ as an L-vector space. Since G_K has a basis of neighbourhoods of 1 that are open normal subgroups, and since ρ and the determinant are continuous, there is an open normal subgroup U of G_K such that if for all $i = 1, \ldots, n^2$, $g_i' \in g_i U$, then the $\rho(g_i')$ still generate $M_n(L)$. Set $M = \bar{K}^U$, so M is a finite Galois extension of K.

If K' is a finite extension of K which is linearly disjoint from M, so is its Galois closure. Hence, we may assume that K' is Galois over K. Thus, $\mathrm{Gal}(K'M/K')$ is naturally isomorphic to $\mathrm{Gal}(K/M)$. For every i, choose g_i' in $G_{K'}$ whose image in $\mathrm{Gal}(K'M/K')$ is sent to g_i by the above isomorphism. This implies that $g_i' g_i^{-1} \in U$ and hence the $\rho|G_{K'}(g_i')$ generate $M_n(L)$, and $\rho|G_{K'}$ is absolutely irreducible. □

By [CH, Proposition 4.1.1 and Theorem 4.2.2], for any finite extension M/K there exists a totally real solvable Galois extension F'/F such that $K' = KF'$ is linearly disjoint to M and such that Arthur–Clozel's base change $\Pi_{K'}$ and K'/F' satisfy hypotheses (H1) and (H4). We apply this to an M as in the lemma above. Thus, $(\rho_{\Pi,\lambda})_{|G_{K'}}$ has the same number of irreducible components as $\rho_{\Pi,\lambda}$, and those components are absolutely irreducible. Moreover, those components are pairwise non-isomorphic since $\rho_{\Pi,\lambda}$ has distinct weights. By Theorem 3.4, we know that $(\rho_{\Pi,\lambda})_{|G_{K'}}$ is good. Hence, by Lemma 2.3, $\rho_{\Pi,\lambda}$ is good.

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