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DEFORMATIONS AND EQUITOPOLOGICAL DEFORMATIONS OF STRONGLY PSEUDOCONVEX MANIFOLDS

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§ 1. Introduction

One of the main problems in complex analysis has been to determine when two open sets D_1, D_2 in C^n are biholomorphically equivalent. In [26] Poincaré studied perturbations of the unit ball B_2 in C^2 of a particular kind, and found necessary and sufficient conditions on a first order perturbation that the perturbed domain be biholomorphically equivalent to B_2. Recently Burns, Shnider and Wells [7] (cf. also Chern-Moser [9]) have studied the deformations of strongly pseudoconvex manifolds. They proved that there is no finite-dimensional deformation theory for M if one keeps track of the boundary.

In view of this, we have the following definition.

DEFINITION. Let M and M' be two strongly pseudoconvex manifolds with A and A' as its maximal compact analytic set. M is said to be holomorphically equivalent to M' if there exist open neighborhoods U and U' of A and A' respectively and biholomorphic map φ: $U \to U'$ such that $\varphi(A) = A'$.

The natural question one can raise is to determine geometric conditions which imply that M and M' are holomorphically equivalent.

Let M be a strongly pseudoconvex manifold with a one-dimensional exceptional set A. Let Θ be the holomorphic tangent sheaf to M. The general Kodaira-Spencer [18] theory shows that $H^1(M, \Theta)$ corresponds to first order infinitesimal deformations of M and that $H^2(M, \Theta)$ represents the obstructions to formally extending deformations to higher order. $H^1(M, \Theta)$ is finite-dimensional since M is strongly pseudoconvex. $H^2(M, \Theta) = 0$ since A is one-dimensional. Because of the result of Burns, Schneider and Wells we mentioned above, given a deformation of M and a compact K in M,

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we shall only worry about the deformation near K. Then in [23] Laufer proved that M has a versal deformation $\omega: \mathfrak{M} \to Q$ with Q a manifold of dimension $\dim H^1(M, \Theta)$ in case either (i) M is of arbitrary dimension and a sufficiently small neighborhood of A [23, Theorem 2.2 and Theorem 2.5] or (ii) M is of dimension two [23, Theorem 2.8].

Let U be a compact complex submanifold of a complex manifold w. Recall that U is a stable submanifold of W in the sense of Kodaira (cf. [17]) if and only if for any complex fibre manifold $p: \mathscr{W} \to B$ such that $p^{-1}(o) = W$ for a point $o \in B$, there exist a neighborhood N of o in B and a fibre submanifold $\mathscr{U}$ with compact fibres of the complex fibre manifold $\mathscr{W}/N$ such that $\mathscr{U} \cap W = U$. This means that no small deformation of the complex structure of W makes U disappear. In [16], [17] stability of compact submanifolds of complex manifolds and other related topics are studied.

We now restrict exclusively to dimension two strongly pseudoconvex manifold M. We will assume that irreducible components A_i's of the maximal compact analytic set A are nonsingular and have normal crossing M. Furthermore there is no $\boldsymbol{CP}^1$ in M with self-intersection number -1 such that blowing down this curve will not destroy the above properties of the maximal compact analytic set. In [23] (see also [15]), it was proved that there exists a semi-universal deformation of $M = \tau^{-1}(o)$ $\tau: \mathfrak{M}' \to Q'$ such that all irreducible components of A are stable under τ (cf. § 2.2). Let $\mathscr{S}$ be the sheaf of germs of vector fields which are tangential to A. Then $\dim H^1(M, \mathscr{S})$ is the number of moduli of equitopological deformation of M. (cf. § 2. 2). It is a natural question to ask for formulas for $\dim H^1(M, \Theta)$ and $\dim H^1(M, \mathscr{S})$. In this paper, we obtain interesting formulae for both $\dim H^1(M, \Theta)$ and $\dim H^1(M, \mathscr{S})$.

MAIN THEOREM. *Let M be a 2-dimensional strongly pseudoconvex manifold as above (i.e., M is a minimal good resolution of normal surface singularity). Suppose the maximal compact analytic set A can be blown down to a hypersurface singularity $o \in \{f = 0\} \subseteq \boldsymbol{C}^3$. Then*

$$\dim H^1(M, \Theta) = -\frac{7K^2 - 5\chi_T(A)}{6} + \tau - \frac{5}{6}(1 + \mu)$$

$$\dim H^1(M, \mathscr{S}) = -\frac{7K^2 - 5\chi_T(A)}{6} + \tau - \frac{5}{6}(1 + \mu) - \sum \dim H^1(A_i, N_{A_i})$$

where $\quad K =$ *canonical divisor of M with support on A*

$$\begin{aligned}
\chi_T(A) &= \textit{topological Euler number of } A \\
\mu &= \textit{Milnor number of } V \textit{ at the origin} \\
&= \dim C\{x, y, z\} \Big/ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
\tau &= \textit{number of moduli of } V \textit{ at } o, \\
&= \dim C\{x, y, z\} \Big/ \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\
N_{A_i} &= \textit{normal bundle of } A_i \textit{ in } M.
\end{aligned}$$

The difficulty in obtaining formulae for $\dim H^1(M, \Theta)$ or $\dim H^1(M, \mathscr{S})$ is that they do not depend entirely on the topological data. Actually Example 3.9 below says that even though Milnor numbers (analytic invariants) for the associated singularities are the same, $\dim H^1(M, \Theta)$ can still be different. Our main observation is that the number of moduli corresponding to the associated singularities are different in this case. The crucial point in the proof of our main Theorem is to consider $\dim H^1(M, \Theta)$ and $\dim H^1(M, \mathscr{S})$ simultaneously. As a corollary to the proof of our Theorem, we have the following

COROLLARY. *Let M be a two-dimensional manifold. Let $A = \cup A_i$,* $i = 1, \cdots, n$ be the maximal compact analytic set in M. Suppose the intersection matrix $[A_i \cdot A_j]$ is negative definite, $p(A) := 1 - \dim H^0(A, \mathcal{O}_A) + \dim H^1(A, \mathcal{O}_A) = 0$ and $A \cdot A = -1$. Then

$$\dim H^1(M, \Theta) = 2(n-1)$$

$$\dim H^1(M, \mathscr{S}) = 3n - 2 + \sum_{i=1}^{n} A_i \cdot A_i\,.$$

Deformations of resolutions of two-dimensional singularities have been studied by several authors including Artin [2], Brieskorn [6], Karras [14], [15], Laufer [20], [21], [23], Riemenschneider [27], [29], Schlessinger, and Wahl [34], [35]). A start on the higher-dimensional theory was made by Riemenschneider [29], Lieberman and Rossi [38]. Our result in this paper can be easily generalized to arbitrary dimension. However we can only get the formula for $\sum_{i=1}^{n-1} (-1)^i \dim H^i(M, \Theta)$. We finally remark that our result is true for complete intersection singularities and the proof is exactly the same.

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§2. Deformations of strongly pseudoconvex manifolds and Milnor's fibration

(2.1) Let M be a strongly pseudoconvex two-dimensional manifold with maximal compact analytic set A. Let $A = \cup A_i$, $1 \leq i \leq n$ be the decomposition of A into irreducible components. We shall assume that A_i's are nonsingular and have only normal crossings. We may further assume without loss of generality that A is connected. Let S be a reduced complex space with distinguished point o. A *deformation* of M is a flat holomorphic map $\varphi\colon \mathfrak{M} \to S$ such that the fiber $M_o = \varphi^{-1}(o)$ is isomorphic to M. We are in fact only interested in deformations near the maximal compact analytic set A of M. According to Riemenschneider [29] we may assume therefore that φ is 1-convex.

Let Θ be the tangent sheaf of M. In [23], Laufer constructed a deformation $\omega\colon \mathfrak{M} \to Q$ of M with smooth parameter space Q of dimension equal to $\dim H^1(M, \Theta)$. He also proved that $\omega\colon \mathfrak{M} \to Q$ is in fact a semi-universal deformation of M.

(2.2) Let $\psi\colon \mathfrak{M} \to S$ be a 1-convex deformation of $M = \psi^{-1}(o)$ with S a reduced complex space. ψ is an *equitopological* deformation of M if all irreducible components A_i, $1 \leq i \leq n$ of the maximal compact analytic set A lift to ψ, that is, there are subspaces $\mathscr{A}_i$, $1 \leq i \leq n$, of $\mathfrak{M}$ such that $\lambda_i := \psi/\mathscr{A}_i\colon \mathscr{A}_i \to S$ are flat deformations of A_i for all i. Associated to A is a weighted dual graph Γ (cf. [12], [19]) which completely determines the topology of A and the differentiable nature of the embedding of A into M. Observe that equitopological deformations of M do not change Γ.

(2.3) Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on M. Considering the composition

$$\Theta \longrightarrow \bigoplus_i \Theta \otimes \mathcal{O}_{A_i} \longrightarrow \bigoplus_i \mathcal{O}_{A_i}(A_i) \longrightarrow 0$$

following Kodaira [17], Laufer [23] and Wahl [34], we have the following exact sequence

$$0 \longrightarrow \mathscr{S} \longrightarrow \Theta \xrightarrow{\ \gamma\ } \bigoplus_i \mathcal{O}_{A_i}(A_i) \longrightarrow 0$$

where $\mathscr{S}$ is a locally free sheaf of rank two. The map

$$H^1(M, \Theta) \xrightarrow{\gamma_*} \bigoplus_i H^1(M, \mathcal{O}_{A_i}(A_i))$$

determines the obstruction to lift A_i infinitesimally, see [17]. Since $H^0(M, \mathcal{O}_{A_i}(A_i)) = 0$ for all i, $H^1(M, \mathscr{S}) = \operatorname{Ker}(\gamma_*)$ and $H^1(M, \mathscr{S})$ is the space of first-order deformations of M to which all A_i lift. Therefore, Image (ρ_o: $T_oS \longrightarrow H^1(M, \Theta)$) is contained in $H^1(M, \mathscr{S})$ for equitopological deformations of M.

In [20], Laufer has developed a deformation theory for infinitesimal neighborhoods of A in M. Let $D = \sum n_i A_i$, $n_i \geq 1$, be a cycle on A. Let $\mathcal{O}(-D)$ be the sheaf of germs of holomorphic functions on M which vanish at least to order n_i on A_i. By $\mathcal{O}_D$ we denote the quotient sheaf $\mathcal{O}/\mathcal{O}(-D)$. Then the non-reduced complex space $A(D) := (A, \mathcal{O}_D)$ is called an infinitesimal neighborhood of A of order $n = (n_1, \cdots, n_r)$. Let now ψ: $\mathfrak{M} \to B$ be an equitopological deformation over a smooth parameter space B. Then, by multiplying the equations for the deformations of the A_i, ψ gives a deformation of any infinitesimal neighborhood $A(D)$. Let Θ_D be the sheaf of germs of vector fields on $A(D)$ in the sense of Grauert [10]. Observe that $\Theta_D = \mathscr{S}/\Theta(-D)\mathscr{S}$. The following lemma can be found in [23], [15].

LEMMA 2.4. *Let M be a two-dimensional strongly pseudoconvex manifold. Then the map β: $H^1(M, \mathscr{S}) \to H^1(M, \Theta_D)$ is surjective for all positive cycles D on the exceptional set of M. For all sufficiently large cycles D, β is an isomorphism.*

Proof. $H^1(M, \mathcal{O}(-D)\mathscr{S}) \xrightarrow{\sigma} H^1(M, \mathscr{S}) \xrightarrow{\beta} H^1(M, \Theta_D) \longrightarrow 0$ is exact since $H^2(M, \mathcal{O}(-D)\mathscr{S}) = 0$. By [10] σ is the zero map for all sufficiently large D.

(2.5) Let $A(D)$ be a sufficiently large infinitesimal neighborhood of A such that the analytic type of the embedding of A in M is determined by $A(D)$ and such that $H^1(M, \mathscr{S}) \cong H^1(M, \Theta_D)$. Then Laufer [23] (see also Karras [15]) construct a 1-convex equitopological deformation τ: $\mathfrak{M}' \to Q'$ of $M = \tau^{-1}(o)$, with Q' a manifold, such that the Kodaira-Spencer map ρ_o: $T_oQ' \to H^1(M, \mathscr{S})$ is an isomorphism. As an easy consequence of [23, Theorem 2.5], they proved the following theorem.

THEOREM 2.6. *Let M be a strongly pseudoconvex two-manifold. Let τ: $\mathfrak{M}' \to Q'$ be the* 1*-convex equitopological deformation of M as in* (2.5). *Then τ is semi-universal for equitopological deformations with reduced parameter spaces.*

From the above discussion, it is clear that why it is interesting to get formulae for $\dim H^1(M, \mathscr{S})$ *and* $\dim H^1(M, \Theta)$.

(2.7) Now we recall Milnor's results on the topology of hypersurface singularities (cf. [24], [25]).

Let $f\colon U \subseteq \boldsymbol{C}^{n+1} \to \boldsymbol{C}$ be an analytic function on an open neighborhood U of o in $\boldsymbol{C}^{n+1}$. We denote

$$B_\varepsilon = \{z \in \boldsymbol{C}^{n+1} \colon \|z\| \le \varepsilon\}$$
$$S_\varepsilon = \partial B_\varepsilon = \{z \in \boldsymbol{C}^{n+1} \colon \|z\| = \varepsilon\}\ .$$

Then:

THEOREM 2.8. *For* ε *small enough the mapping* $\varphi_\varepsilon\colon S_\varepsilon - \{f = 0\} \to S^1$ *defined by* $\varphi_\varepsilon = f(z)/|f(z)|$ *is a smooth fibration.*

THEOREM 2.9. *For* $\varepsilon > 0$ *small enough and* $\varepsilon \gg \eta > 0$ *the mapping* $\psi_{\varepsilon,\eta}$: $(\operatorname{Int} B_\varepsilon) \cap f^{-1}(\partial D_\eta) \to S^1$ *defined by* $\psi_{\varepsilon,\eta}(z) = f(z)/|f(z)|$, *where* $\partial D_\eta = \{z \in \boldsymbol{C}\colon |z| = \eta\}$, *is a smooth fibration isomorphic to* φ_ε *by an isomorphism which preserves the argument. We call the fibrations of Theorem* 2.8 *and* 2.9 *the Milnor fibrations of* f *at* o.

COROLLARY 2.10. *The fibers of* φ_ε *have the homotopy type of an* n-*dimensional finite* CW-*complex.*

THEOREM 2.11. *Let* $V_o\colon f = 0$. *For* $\varepsilon > 0$ *small enough,* S_ε *cuts the smooth part of the algebraic set* V_o *transversally. If* o *is an isolated critical point of* f, *then the pairs* $(S_\varepsilon, S_\varepsilon \cap V_o)$ *for any* ε *small enough are diffeomorphic, and* $(B_\varepsilon, B_\varepsilon \cap V_o)$ *is homeomorphic to* $(B_\varepsilon, C(S_\varepsilon \cap V_o))$, *where* $C(S_\varepsilon \cap V_o)$ *is the cone which is the union of real line segments joining* o *and points of* $S_\varepsilon \cap V_o$.

§ 3. Formulae for $\dim H^1(M, \Theta)$ and $\dim H^1(M, \mathscr{S})$

Given a 2-dimensional strongly pseudoconvex manifold M it is easy to see that $\dim H^1(M, \Theta)$ depends not only on the C^∞ structure of M but also on the complex structure of M. In this section we shall prove that $\dim H^1(M, \Theta)$ depends only on the C^∞ structure of M and numerical invariants of the isolated singularity associated to M.

THEOREM 3.1. *Let* $f(x, y, z)$ *be holomorphic in* N, *a Stein neighborhood of* $(0, 0, 0)$ *with* $f(0, 0, 0) = 0$. *Let* $V = N \cap f^{-1}(0)$ *have the origin as its only singular point. Let* $\pi\colon M \to V$ *be the minimal resolution of* V *such that the*

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irreducible components of $A = \pi^{-1}(0, 0, 0) = \bigcup_{i=1}^{n} A_i$ *are nonsingular with normal crossings. Let* Θ *be the tangent sheaf of* M. *Let* $\mathscr{S}$ *be the sheaf of germs of vector fields that are tangential along* A. *Then*

$$\dim H^1(M, \Theta) = -\frac{7K^2 - 5\chi_T(A)}{6} + \tau - \frac{5}{6}(1 + \mu) \tag{3.1}$$

$$\dim H^1(M, \mathscr{S}) = -\frac{7K^2 - 5\chi_T(A)}{6} + \tau - \frac{5}{6}(1 + \mu) - \sum_{i=1}^{n} \dim H^1(A_i, N_{A_i}) \tag{3.2}$$

where $K =$ *canonical divisor of* M *with support on* A

$\chi_T(A) =$ *topological Euler characteristic of* A

$\mu =$ *Milnor number of* V *at the origin*

$$= \dim \boldsymbol{C}\{x, y, z\} \Big/ \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$\tau =$ *number of moduli of* $V = \dim \boldsymbol{C}\{x, y, z\} \Big/ \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$N_{A_i} =$ *normal bundle of* A_i *in* M.

Proof. Any holomorphic function which agrees with f to sufficiently high order defines a holomorphically equivalent singularity at (0, 0, 0), [3]. So we may take f to be a polynomial. Compactify $\boldsymbol{C}^3$ to $\boldsymbol{P}^3$. Let $\overline{V}_t$ be the closure in $\boldsymbol{P}^3$ of

$$V_t = \{(x, y, z) \in \boldsymbol{C}^3 : f(x, y, z) = t\} .$$

By adding a suitably general high order homogeneous term of degree e to the polynomial f, we may additionally assume that $\overline{V}_o$ has $(0, 0, 0) \in \boldsymbol{C}^3$ as its only singularity and that $\overline{V}_t$ is non-singular for small $t \neq 0$. We may also assume that the highest order terms of f define, in homogeneous coordinates, a nonsingular hypersurface of order e in $\boldsymbol{P}^2 = \boldsymbol{P}^3 - \boldsymbol{C}^3$. $\overline{V}_t$ is then necessarily irreducible for all small t. Without loss of generality, we take $N = \boldsymbol{C}^3$. Then $V = V_o$.

Let $\mathscr{I}_t$ be the ideal sheaf of $\overline{V}_t$ in $\boldsymbol{P}^3$. We denote Θ_t and $\Theta_{\boldsymbol{P}^3}$ the tangent sheaves of $\overline{V}_t$ and $\boldsymbol{P}^3$ respectively. From the second fundamental exact sequence

$$\mathscr{I}_o/\mathscr{I}_o^2 \longrightarrow \Omega^1_{\boldsymbol{P}^3}/\overline{V}_o \longrightarrow \Omega^1_{\overline{V}_o} \longrightarrow 0 , \tag{3.3}$$

we get

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(3.4) $$0 \longrightarrow \Theta_o \longrightarrow \Theta_{P^3}/\overline{V}_o \longrightarrow N_{\overline{V}_o} \longrightarrow T^1_{V_o} \longrightarrow 0$$

where $N_{\overline{V}_o}$ is the normal sheaf of $\overline{V}_o$ in P^3 and $T^1_{V_0}$ is the set of isomorphism classes of first order infinitesimal deformations of V_o at the origin (cf. [30]). It is well known that

$$T^1_{V_o} \cong \mathcal{O}_{C^3}\Big/\left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

It follows that

(3.5) $$\chi(\overline{V}_o, \Theta_o) = \chi(\overline{V}_o, \Theta_{P^3}/\overline{V}_o) - \chi(\overline{V}_o, N_{\overline{V}_o}) + \tau$$

where $\chi(\chi, \mathscr{F}) = \sum_i (-1)^i \dim H^i(X, \mathscr{F})$ for any coherent sheaf $\mathscr{F}$ over the compact complex space X.

From the exact sequence

$$0 \longrightarrow \Theta_{P^3}(-e) \longrightarrow \Theta_{P^3} \longrightarrow \Theta_{P^3}/\overline{V}_o \longrightarrow 0$$

we have

(3.6) $$\begin{aligned}\chi(\overline{V}_o, \Theta_{P^3}/\overline{V}_o) &= \chi(P^3, \Theta_{P^3}) - \chi(P, \Theta_{P^3}(-e)) \\ &= \chi(\overline{V}_t, \Theta_{P^3}/\overline{V}_t), \qquad t \neq 0\end{aligned}$$

Let $\overline{V}$ be the hypersurface defined by

$$w^e f\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right) - tw^e = 0$$

in $P^3 \times D_\varepsilon$ where D_ε is a disk of radius ε in C. Since the normal sheaf of $\overline{V}$ in $P^3 \times D_\varepsilon$ is locally free, $N_{\overline{V}}$ is torsion-free and in particular a π-flat coherent analytic sheaf where $\pi\colon \overline{V} \to D_\varepsilon$ is the natural projection. Therefore

(3.7) $$\chi(\overline{V}_o, N_{\overline{V}_o}) = \chi(\overline{V}_t, N_{\overline{V}_t}) \qquad \text{for } t \neq 0\,.$$

On the other hand $\overline{V}_t$ is smooth for $t \neq 0$, so the exact sequence

$$0 \longrightarrow \Theta_t \longrightarrow \Theta_{P^3/\overline{V}_t} \longrightarrow N_{\overline{V}_t} \longrightarrow 0$$

gives

(3.8) $$\chi(\overline{V}_t, \Theta_t) = \chi(\overline{V}_t, \Theta_{P^3}/\overline{V}_t) - \chi(\overline{V}_t, N_{\overline{V}_t})$$

Putting (3.11)–(3.14) together, we have

(3.9) $$\chi(\overline{V}_o, \Theta_o) = \chi(\overline{V}_t, \Theta_t) + \tau, \qquad t \neq 0\,.$$

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Let $\overline{M}$ be the resolution of $\overline{V}_0$ which has M as an open subset. Let $\mathscr{S}$ be the sheaf of germs of vector fields on $\overline{M}$ which are tangential along each $A_i \subseteq A$. As in (2.3), we have

$$0 \longrightarrow \mathscr{S} \longrightarrow \Theta \longrightarrow \bigoplus_{i=1}^{n} N_{A_i} \longrightarrow 0 \tag{3.10}$$

where N_{A_i} is the normal bundle of A_i in M. If A is given by $xy = 0$, where (x, y) is a local coordinate system of M, then $\mathscr{S}$ is generated by $x(\partial/\partial x)$ any $y(\partial/\partial y)$; if A is given by $x = 0$, then $\mathscr{S}$ is generated by $x(\partial/\partial x)$ and $\partial/\partial y$. From (3.10), since the normal bundle of A_i are negative,

$$\Gamma(M, \mathscr{S}) \xrightarrow{\sim} \Gamma(M, \Theta) \tag{3.11}$$

$$\dim H^1(M, \Theta) - \dim H^1(M, \mathscr{S}) = \bigoplus_{i=1}^{n} \dim H^1(A_1, N_{A_i}) . \tag{3.12}$$

In [11], Hironaka has proved that every normal singularity V_o admits an equivariant $\pi\colon M \to V_o$, i.e., one for which $\pi_* \Theta_M \cong \Theta_{V_o}$. For a proof in dimension two, see [8], 1.2; [34], 4.2; and also [31]. Moreover in this case a minimal good resolution in equivariant (cf. [34], 4.2). By Leray spectral sequence, we have

$$\dim H^1(M, \Theta) = \chi(\overline{V}, \Theta_{\overline{V}}) - \chi(\overline{M}, \Theta_{\overline{M}}) = \tau + \chi(\overline{V}_t, \Theta_{\overline{V}_t}) - \chi(\overline{M}, \Theta) \quad t \neq 0 . \tag{3.13}$$

It is easy to see that

$$\chi_T(\overline{M}) = \chi_T(\overline{V}_o) + \chi_T(A) - 1 \tag{3.14}$$

where $\chi_T(X)$ denote the topological Euler characteristic of X.

Recall from Theorem 2.9 that the intersection of V_t with the open ε-ball is diffeomorphic with the fiber F_o. So the manifold with boundary $V_t \cap B_\varepsilon$ is connected, with 2nd Betti number equal to μ, and with Euler number

$$\chi_T(V_t \cap B_\varepsilon) = 1 + \mu .$$

Since the two manifolds $V_t \cap B_\varepsilon$ and $\overline{V}_t - \operatorname{Int} B_\varepsilon$ have union $\overline{V}_t$ and intersection K_t, we have the Euler number of $\overline{V}_t$

$$\begin{aligned}\chi_T(\overline{V}_t) &= \chi_T(V_t \cap B_\varepsilon) + \chi_T(\overline{V}_t - \operatorname{Int} B_\varepsilon) - \chi(K_t) \\ &= 1 + \mu + \chi_T(\overline{V}_o - \operatorname{Int} B_\varepsilon) - \chi_T(V_o \cap S_\varepsilon)\end{aligned}$$

by the differentiable triviality of the family $\{V_t\}$ away from $(0, 0, 0) \in \boldsymbol{C}^3$. Hence

$$\begin{aligned}\chi_T(\overline{V}_t) &= 1 + \mu + \chi_T(\pi^{-1}(\overline{V}_0 - \operatorname{Int} B_\varepsilon)) + \chi_T(\pi^{-1}(V_0 \cap B_\varepsilon)) \\ &\quad - \chi_T(V_0 \cap S_\varepsilon) - \chi_T(\pi^{-1}(V_0 \cap B_\varepsilon)) \\ &= 1 - \mu + \chi_T(\overline{M}) - \chi_T(A)\end{aligned}$$

since $\pi^{-1}(V_0 \cap B_\varepsilon)$ contracts to A. Thus we have

$$\chi_T(\overline{V}_t) - \chi_T(\overline{M}) = 1 + \mu - \chi_T(A), \qquad t \neq 0 . \tag{3.15}$$

Let

$$\omega = \frac{dx \wedge dy}{\partial f/\partial z} = \frac{dy \wedge dz}{\partial f/\partial x} = \frac{dz \wedge dy}{\partial f/\partial y}$$

is a non-zero holomorphic 2-form on $V_o - \{o\}$. $\pi^*(\omega)$ extends to a meromorphic 2-form on M with a pole set contained in A. K, the divisor of ω and also called the canonical divisor, may be characterized topologically by the adjunction formula [32]

$$A_i \cdot K = -A_i \cdot A_i + 2g_i - 2 + 2\delta_i$$

where A_i is an irreducible component of A, g_i is the genus of A_i, and δ_i is the "number" of nodes and cusps on A_i.

ω, defined above, is a non-zero holomorphic 2-form on V_t, $t \neq 0$, and on $V_o - \{o\}$. Let $K_{\infty,t}$ be the part of the divisor of ω on $\overline{V}_t$ which is supported on $\overline{V}_t - V_t$, for t small. $K_{\infty,t} \cdot K_{\infty,t}$ is independent of t since the family $\{\overline{V}_t\}$ is differentiably trivial away from $(0,0,0) \in C^3$. Let $K_\infty \cdot K_\infty$ denote this constant value for $K_{\infty,t} \cdot K_{\infty,t}$. Riemann-Roch Theorem [13] says

$$\begin{aligned}\chi(\overline{V}_t, \Theta_t) &- \chi(\overline{M}, \Theta) \\ &= \tfrac{1}{6}(7K_\infty \cdot K_\infty - 5\chi_T(\overline{V}_t)) - \tfrac{1}{6}(7K_\infty \cdot K_\infty + 7K \cdot K - 5\chi_T(\overline{M})) \\ &= -\frac{7K^2}{6} - \frac{5}{6}(\chi_T(\overline{V}_t) - \chi_T(\overline{M})) \\ &= -\frac{7K^2}{6} - \frac{5}{6}(1 + \mu - \chi_T(A)) \\ &= -\frac{7K^2 - 5\chi_T(A)}{6} - \frac{5}{6}(1 + \mu) .\end{aligned} \tag{3.16}$$

Put this into (3.13), and we have

$$\dim H^1(M, \Theta) = -\frac{7K^2 - 5\chi_T(A)}{6} + \tau - \frac{5}{6}(1 + \mu) .$$

Hence from (3.12)

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$$\dim H^1(M, \mathscr{S}) = -\frac{7K^2 - 5\chi_T(A)}{6} + \tau - \frac{5}{6}(1 + \mu) - \sum_{i=1}^{n} \dim H^1(A_i, N_{A_i})$$

Q.E.D.

We recall a proposition by Laufer [23, Proposition 4.14].

LEMMA 3.2 (Laufer). *Let M be a strongly pseudoconvex two-dimensional manifold with exceptional set A. Let $\pi: M' \to M$ be a quadratic transformation of M at a singular point p of A. Then there is a canonical map $\pi_1: H^1(M', \Theta') \to H^1(M, \Theta)$. π_1 is onto with kernel of dimension 2. The canonical map $\pi_2: \Gamma(M', \mathscr{S}') \to \Gamma(M, \mathscr{S})$ is an isomorphism.*

From Theorem 3.1 and Lemma 3.2, we have the following Theorem 3.3.

THEOREM 3.3. *Let $f(x, y, z)$ be holomorphic in N, a Stein neighborhood of $(0, 0, 0)$ with $f(0, 0, 0) = 0$. Let $V = N \cap f^{-1}(o)$ have the origin as its only point. Let $\pi: M \to V$ be any resolution of V which is obtained as follows: let $\pi_0: M_0 \to V$ be the minimal resolutions of V, let $\pi_1: M_1 \to M_0$ be a quadratic transformation of M at a singular point p_1 of the exceptional set in $M_0, \cdots,$ let $\pi_r: M_r \to M_{r-1}$ be a quadratic transformation of M_{r-1} at a singular point p_r of the exceptional set in M_{r-1}. Then $M = M_r$ and $\pi = \pi_0 \circ \pi_1 \circ \cdots \pi_r$. Let Θ be the tangent sheaf of M. Then*

$$\dim H^1(M, \Theta) = -\frac{7K^2 - 5\chi_T(A)}{6} + \tau - \frac{5}{6}(1 + \mu)\,.$$

If the irreducible components of $A = \pi^{-1}(0, 0, 0) = \bigcup_{i=1}^{n} A_i$ are nonsingular with normal crossings, then

$$\begin{aligned}\dim H^1(M, \mathscr{S}) &= -\frac{7K^2 - 5\chi_T(A)}{6} + \tau - \frac{5}{6}(1 + \mu) \\ &\quad - \sum_{i=1}^{n} \dim H^1(A_i, N_{A_i}) \\ &= \textit{number of moduli of equitopological deformation of } M.\end{aligned}$$

THEOREM 3.4. *Let M be a two-dimensional manifold. Let $A = \cup A_i$, $i = 1, \cdots, n$ be the maximal compact analytic set in M. Suppose the intersection matrix $[A_i \cdot A_j]$ is negative definite,*

$$p(A) := 1 - \dim H^0(A, \mathcal{O}_A) + \dim H^1(A, \mathcal{O}_A) = 0, \qquad \textit{and } A \cdot A = -1\,.$$

Then

$$\dim H^1(M, \Theta) = 2(n - 1)$$

$$\begin{aligned}\dim H^1(M, \mathscr{S}) &= 2(n-1) + \sum_{i=1}^{n} A_i \cdot A_i + n \\ &= 3n - 2 + \sum_{i=1}^{n} A_i \cdot A_i .\end{aligned}$$

Proof. By Artin's result, we know that A is an exceptional curve of the first kind. So $\pi\colon M \to \boldsymbol{C}^2$ is just a point modification of $\boldsymbol{C}^2$ at origin. It is easy to see that $K^2 = -n$ and $\chi_T(A) = n + 1$. By the proof of Theorem 3.1, we get

$$\dim H^1(M, \Theta) = -\frac{7K^2 - 5\chi_T(A)}{6} - \frac{5}{6} - \dim \operatorname{coker} \rho \tag{3.17}$$

and

$$\dim H^1(M, \mathscr{S}) = -\frac{7K^2 - 5\chi_T(A)}{6} - \frac{5}{6} - \dim \operatorname{coker} \rho - \sum_{i=1}^{n} \dim H^1(A_i, N_{A_i}) \tag{3.18}$$

where $\rho\colon \pi_* \Theta_M \to \Theta_{\boldsymbol{C}^2}$ is the inclusion (cf. (1.2) of [8]) and N_{A_i} is the normal bundle of A_i. Easy computation shows that the image of ρ is those sheaf of germs of vector fields on $\boldsymbol{C}^2$ which vanish at origin. Hence coker ρ has dimension 2. From (3.17), we have

$$\dim H^1(M, \Theta) = -\frac{7(-n) - 5(n+1)}{6} - \frac{5}{6} - 2 = 2n - 2$$

$$\begin{aligned}\dim H^1(M, \mathscr{S}) &= 2n - 2 - \sum_{i=1}^{n} H^1(A_i, N_{A_i}) \\ &= 2n - 2 - \left[-\sum_{i=1}^{n}(A_i \cdot A_i + 1)\right] \\ &= 3n - 2 + \sum_{i=1}^{n} A_i \cdot A_i .\end{aligned}$$

The following corollary is first proved by Laufer [23, Corollary 2.6].

COROLLARY 3.5. *Let M be a two-dimensional manifold. Let A be a submanifold of M which is a compact Riemann surface of genus* 0 *with $A \cdot A = -1$. Let $\lambda\colon \mathfrak{M} \to S$ be a deformation of $M = \lambda^{-1}(o)$. Then in a neighborhood of A in $\mathfrak{M}$, λ is the trivial deformation.*

Let M be a strongly pseudoconvex manifold. In [37], we introduce a bunch of numerical invariants associated to M. In the following we introduce the γ invariant for M.

DEFINITION 3.6. Let M be a strongly pseudoconvex manifold of dimension 2. Suppose the irreducible components of the maximal compact analytic set A in M are nonsingular and have normal crossings. Let π: $M \to V$ be the bimeromorphic morphism such that π blows down A to an isolated singularity. Let Θ_M, Θ_V be tangent sheaves of M and V respectively. Then γ is defined to be $\dim \operatorname{coker} \rho$ where ρ: $\pi_* \Theta_M \to \Theta_V$ is the natural inclusion. (cf. (1.2) of [8]).

The proof of the following theorem is the same as Theorem 3.1.

THEOREM 3.7. *Let M be a strongly pseudoconvex manifold of dimension 2. Suppose the irreducible components of the maximal compact analytic set A in M are nonsingular and have normal crossings. Let π: $M \to V$ be the bimeromorphic morphism, such that π blows down A to an isolated hypersurface singularity. Then*

$$\dim H^1(M, \Theta) = -\frac{7K^2 - 5\chi_T(A)}{6} + \tau - \frac{5}{6}(1 + \mu) - \gamma$$

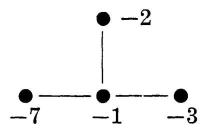
$$\dim H^1(M, \mathscr{S}) = -\frac{7K^2 - 5\chi_T(A)}{6} + \tau - \frac{5}{6}(1 + \mu) - \gamma - \sum \dim H^1(A_i, N_{A_i}) .$$

The following example was suggested to us by Laufer.

EXAMPLE 3.9. Consider the singularities given at (0, 0, 0) by

$$V = \{(x, y, z): z^2 + x^3 + y^7 = 0\}$$
$$V' = \{(x, y, z): z^2 + x^3 + y^7 + kxy^5 = 0, k \neq 0\} .$$

Both V and V' resolve to have exceptional sets A and A' which have the *same weighted dual graph.*



The genera of the compact Riemann surfaces are zero.

$K^2 = -4$	$K'^2 = -4$
$\chi_T(A) = 5$	$\chi_T(A') = 5$
$\mu = 12$	$\mu' = 12$
$\tau = 12$	$\tau' = 11$
$\dim H^1(M, \Theta) = 10$	$\dim H^1(M', \Theta) = 9$
$\dim H^1(M, \mathscr{S}) = 1$	$\dim H^1(M', \mathscr{S}) = 0$

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This example says that $\dim H^1(M, \Theta)$ and $\dim H^1(M, \mathscr{S})$ depend not only on the topological structure of the strongly pseudoconvex manifold but also on the complex structure of the manifold.

PROPOSITION 3.9. *Let M be a strongly pseudoconvex two-dimensional manifold with exceptional set A. Let $\rho\colon M' \to M$ be a quadratic transformation of M at any point p of A. Then*

$$\dim H^1(M', \Theta) - \dim H^1(M, \Theta) = 2 - \dim \operatorname{coker} i$$

where $i\colon \Gamma(M, \Theta_o) \to \Gamma(M, \Theta)$ and Θ_o = sheaf of germs of tangent vector fields which vanish at p.

Proof.(*) Let $A_1 = \rho^{-1}(p)$ and N be the normal bundle of A_1 in M'. Then we have $N \cong \mathcal{O}_{A_1}(-1)$, $A_1 \cong \boldsymbol{CP}^1$ and an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \Theta_{M'} \arrow[r] & \rho^*\Theta_M \arrow[r] & N^* \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\ & & & \Theta_{A_1} \otimes \mathcal{O}_{A_1} N & \end{tikzcd}$$

The associated cohomology sequence

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(M', \Theta_{M'}) \arrow[r] \arrow[dr] & \Gamma(M', \rho^*\Theta_M) \arrow[r] \arrow[d, phantom, "\wr\|"] & \Gamma(M, N^*) \\ & & \Gamma(M, \Theta_M) & \\ \arrow[r] & H^1(M', \Theta_{M'}) \arrow[r] & H^1(M', \rho^*\Theta_{M'}) \arrow[r] \arrow[d, phantom, "\|\wr"] & H^1(M, N^*) = 0 \\ & & H^1(M, \Theta_M) & \end{tikzcd}$$

then gives the proposition immediately.

COROLLARY 3.10. *Let M be a strongly pseudoconvex two-manifold with exceptional set A. Let $\rho\colon M' \to M$ be a quadratic transformation of M at any smooth point p of A. Suppose p is in A_1, an irreducible component of A. If one of the following conditions holds,*

(1) *A_1 is a compact Riemann surface of genus zero, and A_1 intersects with the other irreducible components of A in at least three points.*

(2) *A_1 is a compact Riemann surface of genus one and A_1 intersects with the other irreducible components of A in at least one point.*

(3) *A_1 is a compact Riemann surface of genus ≥ 2.*

(*) The above simplified proof was suggested to us by the referee.

Then $$\dim H^1(M', \Theta) = \dim H^1(M, \Theta) + 2 \, .$$

Proof. Since the global vector fields of M must be tangential to the exceptional set, the automorphism given by integrating along a vector field must map any singular point q of A into itself. Since automorphism of compact Riemann surface of genus one fixing three points must be an identity, statement (1) above follows from Proposition 3.9. As the tangent bundle of compact Riemann surface of genus 1 is trivial, statement (2) follows also from Proposition 3.9. The statement (3) above is trivial because there are no global vector fields on compact Riemann surface of genus ≥ 2. Q.E.D.

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