THE EXTREME COVERINGS OF 4-SPACE BY SPHERES

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The object of this paper is to apply, in the case $n = 4$, the results of Barnes and Dickson [1] concerning extreme coverings of n-dimensional Euclidean space by equal spheres whose centres form a lattice.

The reader is referred to [1] for a complete background on the problem. Terms and notations used will be as in that paper.

It will be shown that every extreme quaternary form is equivalent to a multiple of one of the following forms:

$$(1.1)\quad f_1(x) = 2\sum_{i=1}^{4} x_i^2 - \sum_{i<j} x_i x_j,$$

$$(1.2)\quad f_2(x) = 2\sum_{i=1}^{4} x_i^2 + 2\alpha x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4 + 2(1-\alpha)x_3x_4,$$

where $\alpha = (5-\sqrt{13})/2$, and

$$(1.3)\quad \begin{aligned} f_3(x) = {} & (3-\gamma)(x_1^2+x_2^2)+(2+2\beta)(x_3^2+x_4^2) \\ & +2\gamma x_1x_2 - 2\beta x_3x_4 - 2\sum_{\substack{i=1,2\\ j=3,4}} x_i x_j, \end{aligned}$$

where β, γ are the solutions of

$$(1.4)\quad 81\beta^5+234\beta^4-84\beta^3-601\beta^2-156\beta+252 = 0,$$

$$(1.5)\quad \gamma = \frac{(18\beta^2+39\beta+10)\beta}{(\beta+2)(3\beta+14)},$$

for which $\beta \simeq 0.544$, $\gamma \simeq 0.499$.

It will also be shown that $f_1(x)$ is an absolutely extreme form.

Delone and Ryskov [2] have already published a proof that this form is absolutely extreme when $n = 4$ but the highly condensed argument given by them has steps (e.g. an argument by symmetry) which the author was unable to follow.

The results of [1] which will be used here are:

THEOREM 1. *Let $f(\boldsymbol{x}) = \boldsymbol{x}'A\boldsymbol{x}$ be an interior form and $F(\boldsymbol{x}) = \boldsymbol{x}'A^{-1}\boldsymbol{x}$ its inverse. Then f is extreme if and only if F is expressible in the form,*

$$F(\boldsymbol{x}) = \sum_{\boldsymbol{v}} \lambda_{\boldsymbol{v}} \left[\sum_{i=1}^{n} c_i(\boldsymbol{l}_i'\boldsymbol{x})^2 - (\boldsymbol{v}'\boldsymbol{x})^2 \right] \tag{1.6}$$

where $\boldsymbol{v}$ runs over a set of vertices which contains only one vertex from each set of $2(n+1)$ vertices congruent to a given maximal vertex or its negative,

$$\lambda_{\boldsymbol{v}} \geqq 0 \text{ for all } \boldsymbol{v}, \tag{1.7}$$

$\boldsymbol{l}_1, \boldsymbol{l}_2 \cdots, \boldsymbol{l}_n$ are the integral points used to define $\boldsymbol{v}$ and $c_1, \cdots, c_n$ are defined by

$$\boldsymbol{v} = \sum_{i=1}^{n} c_i \boldsymbol{l}_i. \tag{1.8}$$

THEOREM 2. *If f is an extreme form in the interior of a Voronoï cone Δ, then every extreme form in Δ is a multiple of f.*

THEOREM 3. *If f is an extreme form in the interior of a Voronoï cone Δ, then f and Δ have the same group of automorphisms.*

2. Voronoï's cones for quaternary forms

Voronoï [4] showed that any quaternary positive definite quadratic form is equivalent to a form belonging to one of 3 cones in the 10 dimensional space of the coefficients of f.

These 3 cones, Δ, Δ', Δ'', are defined as follows:

(i) If $f \in \Delta$, then

$$\begin{aligned} f(\boldsymbol{x}) = {} & \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \lambda_5 (x_1 - x_2)^2 + \lambda_6 (x_1 - x_3)^2 \\ & + \lambda_7 (x_1 - x_4)^2 + \lambda_8 (x_2 - x_3)^2 + \lambda_9 (x_2 - x_4)^2 + \lambda_{10} (x_3 - x_4)^2, \end{aligned} \tag{2.1}$$

where $\lambda_i \geqq 0$ for all i.

(ii) If $f \in \Delta'$, then

$$\begin{aligned} f(\boldsymbol{x}) = {} & \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \lambda_5 \omega + \lambda_6 (x_1 - x_3)^2 + \lambda_7 (x_1 - x_4)^2 \\ & + \lambda_8 (x_2 - x_3)^2 + \lambda_9 (x_2 - x_4)^2 + \lambda_{10} (x_3 - x_4)^2 \end{aligned} \tag{2.2}$$

where $\lambda_i \geqq 0$ for all i, and

$$\omega = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 + 2x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4. \tag{2.3}$$

(iii) If $f \in \Delta''$, then

$$\begin{aligned} f(\boldsymbol{x}) = {} & \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \lambda_5 \omega + \lambda_6 (x_1 - x_3)^2 + \lambda_7 (x_1 - x_4)^2 \\ & + \lambda_8 (x_2 - x_3)^2 + \lambda_9 (x_2 - x_4)^2 + \lambda_{10} (x_1 + x_2 - x_3 - x_4)^2 \end{aligned} \tag{2.4}$$

where $\lambda_i \geqq 0$ for all i, ω as in (2.3).

The cone Δ is Voronoï's principal domain discussed for n-space by Barnes and Dickson [1] and thus

$$(2.5) \qquad f_1(\boldsymbol{x}) = 2\sum_{i=1}^{4} x_i^2 - \sum_{i<j} x_i x_j$$

is the only extreme form in this domain.

3. The cone Δ''

The form $f_2(\boldsymbol{x})$ has already been shown to be extreme by laborious means [3]. However, we can use the criterion in Theorem 1 to prove the extremeness with considerably less work than before.

Voronoï constructed a table ([4], p. 173) grouping all the vertices of π for the cone Δ'' into types, i.e. sets of congruent vertices and their negatives, and also indicating which integral points define each vertex. Making use of the table it may be easily calculated, that for vertices of types I, II, VI,

$$(3.1) \qquad f_2(\boldsymbol{v}) = \frac{2(1-\alpha+\alpha^2)}{1+\alpha}$$

and for all other vertices

$$(3.2) \qquad f_2(\boldsymbol{v}) = \frac{2}{(2-\alpha)(1+\alpha)}.$$

Substitution of $\alpha = (5-\sqrt{13})/2$ shows that vertices of type I, II, VI are maximal and we can apply the criterion (1.6) to a set of vertices consisting of one of each of these 3 types.

Consider (i) *a type I vertex*:

$$\boldsymbol{l}_1' = (0, 0, 1, 0); \quad \boldsymbol{l}_2' = (0, 0, 0, 1); \quad \boldsymbol{l}_3' = (1, 0, 1, 1); \quad \boldsymbol{l}_4' = (0, 1, 1, 1).$$

Solving $2\boldsymbol{l}_i' A \boldsymbol{v} = f(\boldsymbol{l}_i)$, $(i = 1, --, 4)$, and (1.8) we obtain

$$\boldsymbol{v}' = \frac{1}{1+\alpha}(1-\alpha, 1-\alpha, 1, 1)$$

and

$$\boldsymbol{c}' = \frac{1}{1+\alpha}(2\alpha-1, 2\alpha-1, 1-\alpha, 1-\alpha).$$

Thus

$$(3.3) \quad \sum_{k=1}^{4} c_k(\boldsymbol{l}_k'\boldsymbol{x})^2 = \frac{1}{1+\alpha}\left[(2\alpha-1)x_3^2+(2\alpha-1)x_4^2+(1-\alpha)(x_1+x_3+x_4)^2 + (1-\alpha)(x_2+x_3+x_4)^2\right]$$

and

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$$(3.4) \qquad (\boldsymbol{v}'\boldsymbol{x})^2 = \frac{1}{(1+\alpha)^2}\,[(1-\alpha)x_1+(1-\alpha)x_2+x_3+x_4]^2.$$

(ii) *a type II vertex*:

$$\boldsymbol{l}_1' = (1, 0, 0, 0); \quad \boldsymbol{l}_2' = (0, 1, 0, 0); \quad \boldsymbol{l}_3' = (1, 1, 1, 0); \quad \boldsymbol{l}_4' = (1, 1, 1, 1).$$

This gives

$$\boldsymbol{v}' = \frac{1}{1+\alpha}\,(1, 1, \alpha, 1-\alpha)$$

and

$$\boldsymbol{c}' = \frac{1}{1+\alpha}\,(1-\alpha, 1-\alpha, 2\alpha-1, 1-\alpha).$$

Thus

$$(3.5) \qquad \sum_{k=1}^{4} c_k(\boldsymbol{l}_k'\boldsymbol{x})^2 = \frac{1}{1+\alpha}\,[(1-\alpha)(x_1^2+x_2^2+\{x_1+x_2+x_3+x_4\}^2) + (2\alpha-1)(x_1+x_2+x_3)^2]$$

and

$$(3.6) \qquad (\boldsymbol{v}'\boldsymbol{x})^2 = \frac{1}{(1+\alpha)^2}\,(x_1+x_2+\alpha x_3+(1-\alpha)x_4)^2.$$

(iii) *a type VI vertex*:

$$\boldsymbol{l}_1' = (1, 0, 0, 0); \quad \boldsymbol{l}_2' = (0, 1, 0, 0); \quad \boldsymbol{l}_3' = (1, 1, 0, 1); \quad \boldsymbol{l}_4' = (1, 1, 1, 1).$$

This will yield the same results as type II but with x_3 and x_4 permuted.

With $\boldsymbol{v}$ running over the above set of 3 vertices, let $\lambda_{\boldsymbol{v}} = (1+\alpha)^2$ for all $\boldsymbol{v}$, then the right-hand side of (1.6) becomes $\boldsymbol{x}'B_2\boldsymbol{x}$, where

$$B_2 = \begin{bmatrix} 2\alpha(2-\alpha) & \alpha^2+4\alpha-3 & \alpha(2-\alpha) & \alpha(2-\alpha) \\ \alpha^2+4\alpha-3 & 2\alpha(2-\alpha) & \alpha(2-\alpha) & \alpha(2-\alpha) \\ \alpha(2-\alpha) & \alpha(2-\alpha) & 2\alpha(2-\alpha) & 3-2\alpha-2\alpha^2 \\ \alpha(2-\alpha) & \alpha(2-\alpha) & 3-2\alpha-2\alpha^2 & 2\alpha(2-\alpha) \end{bmatrix}.$$

As $\alpha = (5-\sqrt{13})/2$, then $\alpha^2-5\alpha+3 = 0$ and using this, we obtain

$$(3.7) \qquad B_2 = \alpha(2-\alpha)\begin{bmatrix} 2 & 1-\alpha & 1 & 1 \\ 1-\alpha & 2 & 1 & 1 \\ 1 & 1 & 2 & \alpha \\ 1 & 1 & \alpha & 2 \end{bmatrix} = kA_2^{-1}.$$

Thus $\boldsymbol{x}'B_2\boldsymbol{x} = kF_2(\boldsymbol{x})$ and (1.6) is satisfied. By Theorem 2 this is the only extreme form in Δ''.

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The form $f_2(x)$ was found by making use of Theorem 3. The group $G(\Delta'')$ of automorphisms of Δ'' contains the transformations: (i) $x_1 \leftrightarrow x_2$, (ii) $x_3 \leftrightarrow x_4$, (iii) $x_1 \to x_1 - x_3$, $x_2 \to x_4$, $x_3 \to x_1$, $x_4 \to x_4 - x_2$. By Theorem 3, these are also automorphisms of any extreme form f in the interior of Δ''. The family of forms in Δ'' satisfying this condition was then found to be those forms of the same shape as $f_2(x)$ with $\frac{1}{2} < \alpha < 1$. The value for α was found by minimizing $\mu(f)$ over α where $\mu(f) = f(v)/d^{1/n}$, v a maximal vertex and $d = d(f) = \det A$.

4. The cone Δ'

Using Voronoï's table ([4], p. 169) for Δ' we find that for vertices of type I, II, V, VI, IX, X

$$f_3(v) = \frac{(1-\gamma)^2}{2} \cdot \frac{\beta+2}{3\beta+2} + \frac{2(\beta+1)^2}{3\beta+2}, \tag{4.1}$$

and for all other vertices,

$$f_3(v) = \frac{(1-\gamma)^2}{2} \cdot \frac{1}{3-2\gamma} + \frac{2(\beta+1)^2}{3\beta+2}. \tag{4.2}$$

When

$$\frac{\beta+2}{3\beta+2} > \frac{1}{3-2\gamma}$$

i.e.

$$2(\gamma-1)+\gamma\beta < 0,$$

vertices of types I, II, V, VI, IX, X are maximal. For the solution of (1.4) and (1.5) being considered this is in fact true. Using one vertex of each of the above types we find that for the form $f_3(x)$, the right hand side of (1.6) with

$$\lambda_v = \frac{(3\beta+2)^2(3-2\gamma)}{2(3\beta^2+4\beta+2\gamma-\gamma^2)} \tag{4.3}$$

for all v, becomes $x'B_3x$, where

$$B_3 = \begin{bmatrix} (3-\gamma)\beta+2(2-\gamma) & 2(1-\gamma)-\gamma\beta & 3-2\gamma & 3-2\gamma \\ 2(1-\gamma)-\gamma\beta & (3-\gamma)\beta+2(2-\gamma) & 3-2\gamma & 3-2\gamma \\ 3-2\gamma & 3-2\gamma & 2(3-2\gamma) & 3-2\gamma \\ 3-2\gamma & 3-2\gamma & 3-2\gamma & 2(3-2\gamma) \end{bmatrix}. \tag{4.4}$$

All of the coefficients of $x'B_3x$ except the coefficients of x_1^2, x_2^2, x_1x_2, appear as in (4.4) without applying any conditions on γ and β. The conditions (1.4), (1.5) imply that the other coefficients are equal to the values given in (4.4).

As $x'B_3x = kF_3(x)$, the criterion (1.6) is satisfied. By Theorem 2, this is the only extreme form in Δ'.

The form $f_3(x)$ was found in the same manner as $f_2(x)$, by applying Theorem 3 and minimizing $\mu(f)$ over the parameters β, γ.

5. Conclusion

We now have three extreme quaternary forms. As there are only 3 inequivalent Voronoï cones of quaternary forms, these must be the only inequivalent extreme forms. Theorem 2 excludes the possibility of any other interior or boundary extreme forms.

It remains now to determine the absolutely extreme form, i.e. the form for which $\mu(f)$ is an absolute minimum.

(i) $f_1(x)$.

It is easily calculated that $\mu(f_1) = (2\sqrt[4]{5})/5 \simeq .598$.

(ii) $f_2(x)$.

From (3.1), $\max f_2(v) = 2(1-\alpha+\alpha^2)/(1+\alpha)$ and we have that $d(f_2) = (1+\alpha)^2(2-\alpha)^2$.

So

$$\mu(f_2) = \frac{2(1-\alpha+\alpha^2)}{(1+\alpha)^{\frac{3}{2}}(2-\alpha)^{\frac{1}{2}}}$$

and, as $\alpha = (5-\sqrt{13})/2$ we obtain $\mu(f_2) \simeq .621$.

(iii) $f_3(x)$.

From (4.1), $\max_v f_3(v) = ((1-\gamma)^2/2)((\beta+2)/3\beta+2)+(2(\beta+1)^2/(3\beta+2))$ and we have that $d(f_3) = (3-2\gamma)(3\beta+2)^2$.

Calculation yields $\gamma \simeq .499$, $\beta \simeq .544$ and $\mu(f_3) \simeq .618$.

Thus $f_1(x)$ is the absolutely extreme quaternary form and the covering density of the corresponding lattice is

$$\theta(\Lambda_1) = J_4(\mu(f_1))^2,$$

where J_4 is the volume of a unit 4-dimensional sphere.

Thus $\theta(\Lambda_1) = 2\pi^2/5\sqrt{5}$.

It is obvious that, as the Voronoï cones become less symmetrical, the group of automorphisms will become smaller, and fewer coefficients will be determined by Theorem 3.

The evidence of previously known extreme forms might have suggested the conjecture that a form is extreme when all its vertices are maximal. However, Δ' shows that this condition is neither necessary nor sufficient as this cone can be shown to contain forms with all vertices maximal.

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References

[1] Barnes, E. S. and Dickson, T. J., 'Extreme coverings of n-space by spheres', *J. Aust. Math. Soc.* 7 (1967), 115–127.

[2] Delone, B. N. and Ryskov, S. S., 'Solution of the problem of the least dense lattice covering of a 4-dimensional space by equal spheres', *Dokl. Akad. Nauk. S.S.S.R.* 152 (1963), 523.

[3] Dickson, T. J., 'An extreme covering of 4-space by spheres', *J. Aust. Math. Soc.* 6 (1966), 179–192.

[4] Voronoï G., 'Recherches sur les paralleloédres primitifs (Part. 2)', *J. rein angew. Math.* 136 (1909), 67–181.

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