Canad. Math. Bull. Vol. **60** (3), 2017 pp. 484–489
http://dx.doi.org/10.4153/CMB-2016-066-x

A Note on Lawton's Theorem

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Abstract. We prove Lawton's conjecture about the upper bound on the measure of the set on the unit circle on which a complex polynomial with a bounded number of coefficients takes small values. Namely, we prove that Lawton's bound holds for polynomials that are not necessarily monic. We also provide an analogous bound for polynomials in several variables. Finally, we investigate the dependence of the bound on the multiplicity of zeros for polynomials in one variable.

1 Introduction and Statement of Results

In this brief note we are dealing with Theorem 1 of Lawton's seminal paper [5]. Lawton applied this theorem to a establish relation between Mahler's measure of multivariate and univariate polynomials and thus proved the corresponding Boyd conjecture. This idea was further generalized by Issa and Lalin in [4]. The theorem itself has many citations and was presented in at least two monographs [2, 6]. We are recalling it here for the convenience of the reader.

Theorem *Let $P(x) \in \mathbb{C}[x]$ be a monic polynomial and let k be the number of nonzero coefficients of P. Then if $k \geq 2$, there is a positive constant C_k that depends only on k such that*

$$\mu\left\{ x \in [0,1) : |P(e^{2\pi i x})| < v \right\} \leq C_k v^{\frac{1}{k-1}}$$

for every real $v > 0$.

Here μ denotes the ordinary one-dimensional measure of the corresponding set. In a remark following the theorem Lawton conjectured that a similar bound also holds for polynomials that are not necessarily monic if we allow the constant C_k to be dependent not only on k but also on the height of P, *i.e.*, on the maximum modulus of its coefficients.

This conjecture turned out to be true, and we prove it here.

Theorem 1.1 *Lawton's theorem holds for polynomials that are not necessarily monic. If $k \geq 2$ and $h = h(P)$ denotes the maximum modulus of the coefficients of P, then for every real $v > 0$ we have*

$$\mu\left\{ x \in [0,1) : |P(e^{2\pi i x})| \leq v \right\} \leq C_k (v/h)^{\frac{1}{k-1}}$$

with $C_k = (k-1)\left(\frac{12\sqrt{2}}{\pi}\right)^{\frac{k-2}{k-1}}$.

Received by the editors July 10, 2016; revised August 1, 2016.
Published electronically November 29, 2016.
AMS subject classification: **11R09**, 11R06.
Keywords: polynomial, Mahler measure.

Initially our interest in this conjecture arose from an attempt to find a lower bound for Mahler's measure of a polynomial with k coefficients in terms of its height. In fact, Theorem 1.1 almost immediately implies that

$$M(P) \geq \frac{\|P\|^{\frac{1}{k+1}}}{\exp((k-1)^2)},$$

where $\|P\|$ denotes the Euclidean norm of the vector of coefficients of P. However, in [1, Theorem 1] a stronger result was established in a different way, so we will not elaborate on this bound here. Recall that one of the equivalent definitions of Mahler's measure of P is given by

$$M(P) = \exp\Big(\int_0^1 \log|P(e^{2\pi i x})|dx\Big).$$

Lawton's theorem allows one to bound the measure of the sets near zeros of $P(z)$, where the term $\log|P(z)|$ is unbounded. It was the key in the proof of Boyd conjecture. The results presented here may be useful in further investigation in that direction.

The exponent $1/(k-1)$ of v seems to be related to the fact that the multiplicity of any zero of a polynomial with k nonzero coefficients cannot exceed $k-1$; see Hajós [3]. For example, it is best possible for polynomials of the form $(x^m-1)^{k-1}$ whose every zero have multiplicity $k-1$. We were able to refine Lawton's theorem for polynomials whose zeros have multiplicity smaller than $k-1$; however, our constant depends heavily on the polynomial P and is valid only for sufficiently small v. We have the following theorem.

Theorem 1.2 *Let $P(x) \in \mathbb{C}[x]$ be a nonconstant polynomial and let*

$$m = \max\{\text{multiplicity}(\alpha) : P(\alpha) = 0\}.$$

There are effectively computable constants B_P and C_P such that

$$\mu\big\{x \in [0,1) : |P(e^{2\pi i x})| \leq v\big\} \leq C_P v^{\frac{1}{m}} \text{ for } 0 \leq v \leq B_P.$$

Note 1.3 The constants can be calculated as follows. Since $\gcd(P, P^{(m)}) = 1$, there are polynomials Q_1 and Q_2 such that $PQ_1 + P^{(m)}Q_2 = 1$. We can take

$$B_P = 1/2L(Q_1) \quad \text{and} \quad C_P = m\frac{6\sqrt{2}}{\pi}\big(2L(Q_2)\big)^{1/m},$$

where $L(Q_i)$, for $i \in \{1,2\}$ denotes the length of the corresponding polynomial, that is the sum of absolute values of its coefficients. Note that with the extra condition $\deg Q_1 = d-m-1$ and $\deg Q_2 = d-1$, where $d = \deg P$, the polynomials Q_1 and Q_2 are uniquely determined by P.

We also provide an analog of Theorem 1.1 for polynomials in several variables. For this we introduce following notation. Let $\mathbf{z}_l = (z_1, \dots, z_l) \in \mathbb{C}^l$ and $P(\mathbf{z}_l) = \sum_{\alpha\in\mathcal{J}} a_\alpha \mathbf{z}^\alpha \in \mathbb{C}[\mathbf{z}_l]$, where $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathcal{J}$ are distinct vectors of exponents in $\mathbb{Z}^l$. For all i, $0 \leq i \leq l$ define

$k_i = k_i(P) =$ *the number of terms of P considered as a polynomial in z_i with polynomial coefficients in the remaining variables.*

https://doi.org/10.4153/CMB-2016-066-x Published online by Cambridge University Press

With this notation, we have the following theorem.

Theorem 1.4 *Let $P(\mathbf{z}_l)$ be a polynomial with complex coefficients. If P has at least two monomials, then*

$$\mu_l\left\{\mathbf{z}\in\mathbb{T}^l : |P(\mathbf{z})|\le v\right\}\le C(k_1,\dots,k_l)(v/h)^{1/(\sum_{i=1}^l(k_i-1))},$$

where $C(k_1,\dots,k_l)=C_{k_1}+\cdots+C_{k_l}$ with C_{k_i}, $1\le i\le l$ defined as in Theorem 1.1.

Here μ_l is the l-dimensional measure, $h=h(P)$, and $\mathbb{T}$ is parameterized by $z=e^{2\pi i\theta}$, so that $\mu_l(\mathbb{T}^l)=1$.

Note 1.5 The quantity $\sum_{i=1}^l(k_i-1)$ might be larger or smaller than the number of monomials in P. In general, if P has k monomials, then

$$\max\{k_i : 1\le i\le l\}\le k\le\prod_{i=1}^l k_i.$$

2 Proof of Theorem 1.1

We begin with a simple observation that we can limit the proof to polynomials of height 1. Indeed, if $h=h(P)=1$, the inequality in the conclusion of the theorem reduces to

$$\mu\left\{x\in[0,1) : |P(e^{2\pi ix})|\le v\right\}\le C_k v^{\frac{1}{k-1}}. \tag{2.1}$$

From this we can obtain the inequality for polynomials with height h by observing that $h^{-1}P(z)$ has height 1, so that (2.1) gives

$$\mu\left\{x\in[0,1) : |P(e^{2\pi ix})|\le v\right\}=\mu\left\{x\in[0,1) : |h^{-1}P(e^{2\pi ix})|\le(v/h)\right\}\le C_k(v/h)^{\frac{1}{k-1}}.$$

Consequently, in what follows we assume that $h(P)=1$ and limit ourselves to proving (2.1). Define the reciprocal of P in a usual way, $P^*(z)=z^d\bar{P}(z^{-1})$, where d is the degree of P. Let $\mathbb{T}$ denote the unit circle on the complex plane. We parameterize $\mathbb{T}$ by $x\to z=e^{2\pi ix}$, so that $\mu(\mathbb{T})=1$. Clearly, we have

$$\mu\left\{z\in\mathbb{T} : |P(z)|\le v\right\}=\mu\left\{z\in\mathbb{T} : |P^*(z)|\le v\right\}. \tag{2.2}$$

This simple observation is all we need. The rest of the proof is almost identical to Lawton's original proof. We proceed by induction on k. For $k=2$ we can take $C_2=1$. To see this notice that $\mu\{z\in\mathbb{T} : |az^n+b|\le v\}=\mu\{z\in\mathbb{T} : |az+b|\le v\}$; also, without loss of generality we can assume that $b=1$, and $0\le a\le 1$ is a real number. Now consider a circle of radius a centered at $(1,0)$. By elementary considerations we get $C_2=1$; in fact, for $v<1$ we can even take $C_2=1/2$. Assume that (2.1) holds for polynomials with k nonzero coefficients and suppose that P has $k+1\ge 3$ nonzero coefficients. On the induction step, due to (2.2), we have a choice between $f(x)=P(e^{2\pi ix})$ or $f(x)=P^*(e^{2\pi ix})$. Following [5, equations (9), (10), and (11)] we then let $g(x)=\frac{1}{2\pi id}\frac{d}{dx}f(x)$, where $d=\deg P$, so that $g(x)=Q(e^{2\pi ix})$ for some polynomial Q with k coefficients. Let $P(z)=a_1z^{n_1}+\cdots+a_{k-1}z^{n_{k-1}}+a_k$, where $n_1=d>n_2>\cdots>n_{k-1}>0$, and suppose that $h(P)=|a_i|$ for some $1\le i\le k$, and that $a_iz^{n_i}$ is the corresponding monomial. If $n_i/d\ge 1/2$, we choose $f(x)=P(e^{2\pi ix})$, and

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we choose $f(x) = P^*(e^{2\pi i x})$ when $n_i/d < 1/2$. Such a choice guarantees that the differentiation in the next step will not delete a_i and the corresponding monomial is $\max\{\frac{n_i}{d}, 1-\frac{n_i}{d}\}a_i z^{\max\{n_i, d-n_i\}}$. Consequently, $h(Q) \geq \max\{\frac{n_i}{d}, 1-\frac{n_i}{d}\} \geq \frac{1}{2}$.

We can now proceed to apply Lawton's formula (17). For this, following Lawton, set $A = \{z \in \mathbb{T} : |P(z)| \leq v\}$ and $B = \{z \in \mathbb{T} : |Q(z)| \geq u\}$. Hence,

$$\mu\{A \cap B\} \leq \frac{6\sqrt{2}}{\pi}\frac{v}{u},$$

as in the original proof. Further, we can apply induction hypothesis to the polynomial $h(Q)^{-1}Q(z)$ that has height 1. Hence, by (2.1) and the fact that $h(Q) \geq 1/2$, we get

$$\begin{aligned}\mu\big\{z \in \mathbb{T} : |Q(z)| \leq u\big\} &= \mu\big\{z \in \mathbb{T} : |h(Q)^{-1}Q(z)| \leq h(Q)^{-1}u\big\}\\ &\leq \mu\big\{z \in \mathbb{T} : |h(Q)^{-1}Q(z)| \leq 2u\big\} \leq C_k(2u)^{\frac{1}{k-1}}.\end{aligned}$$

Thus, analogously [5, (17)], we get

$$\mu\big\{z \in \mathbb{T} : |P(z)| \leq v\big\} \leq C_k(2u)^{\frac{1}{k-1}} + \frac{6\sqrt{2}}{\pi}\frac{v}{u},$$

where u is replaced by $2u$ in the first term on the right. Following Lawton we choose u that minimizes the expression on the right-hand side:

$$u = 2^{-\frac{3k-1}{2k}}\Big(\frac{\pi}{24}\frac{C_k}{k-1}\frac{1}{v}\Big)^{-\frac{k-1}{k}}.$$

It leads to

$$\mu\{z \in \mathbb{T} : |P(z)| \leq v\} \leq C_{k+1}v^{\frac{1}{k}}$$

with

$$C_{k+1} = k\Big(\frac{C_k}{k-1}\Big)^{1-\frac{1}{k}}\Big(\frac{12\sqrt{2}}{\pi}\Big)^{\frac{1}{k}}.$$

Finally, we can calculate the constant explicitly as $C_k = (k-1)(\frac{12\sqrt{2}}{\pi})^{\frac{k-2}{k-1}}$. ∎

3 Proof of Theorem 1.2

Recall that $m = \max\{\text{multiplicity}(\alpha) : P(\alpha) = 0\}$. Hence, $\gcd(P, P^{(m)}) = 1$, so by Bezout identity there are polynomials Q_1 and Q_2 such that

$$PQ_1 + P^{(m)}Q_2 = 1. \tag{3.1}$$

Thus, P and $P^{(m)}$ cannot be simultaneously too small. Define

$$A_0 = \big\{z \in \mathbb{T} : |P(z)| \leq v, |P^{(m)}(z)| \leq v_m\big\}.$$

For $v = v_0 < 1/2L(Q_1)$ and $v_m = 1/2L(Q_2)$, equation (3.1) cannot be satisfied, so with these restriction we get

$$\mu(A_0) = 0. \tag{3.2}$$

We now follow Lawton's formulas [5, (12)–(17)] with a slight modification. For $1 \leq j \leq m$ define sets

$$A_j = \big\{z \in \mathbb{T} : |P^{(m-j)}(z)| \leq v_{m-j}\big\} \quad \text{and} \quad B_j = \big\{z \in \mathbb{T} : |P^{(m-j)}(z)| \geq v_{m-j}\big\},$$

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where the quantities $\nu_1, \dots, \nu_{m-1}$ will be determined later. Analogously to Lawton's (15), for $1 \le j \le m-1$ we have

$$\mu(A_{j+1}) \le \mu(A_j) + \mu(A_{j+1} \cap B_j). \tag{3.3}$$

Hence, following Lawton's formula (17) with $A = A_1$ and $B = B_0$, $u = \nu_m$, $\nu = \nu_{m-1}$, and (3.2), (3.3), we get

$$\mu(A_1) \le 0 + \mu(A_1 \cap B_0) \le \frac{6\sqrt{2}}{\pi} 2L(Q_2)\nu_{m-1}. \tag{3.4}$$

Note that $A_m = \{z \in \mathbb{T} : |P(z)| \le \nu\}$, so for $m = 1$ formula (3.4) gives the required bound. Otherwise, (3.3) subsequently gives

$$\mu(A_2) \le \frac{6\sqrt{2}}{\pi}\Big(2L(Q_2)\nu_{m-1} + \frac{\nu_{m-2}}{\nu_{m-1}}\Big), \dots$$
$$\mu(A_m) \le \frac{6\sqrt{2}}{\pi}\Big(2L(Q_2)\nu_{m-1} + \frac{\nu_{m-2}}{\nu_{m-1}} + \dots + \frac{\nu_0}{\nu_1}\Big).$$

The minimum is obtained for $\nu_j = (2L(Q_2))^{-j/m}\nu_0^{(m-j)/m}$ for $1 \le j \le m-1$, and is equal to the constant listed in Note 1.3. ■

4 Proof of Theorem 1.4

Again we limit the proof to polynomials with height 1. The general case follows immediately according to the observation at the beginning of the proof of Theorem 1.1. Consequently, we assume that $h(P) = 1$ and proceed by induction on the number of variables l. The base case $l = 1$ is established by Theorem 1.1. Write P as $\sum_{j=1}^{k_l} b_j(\mathbf{z}')z_l^{n_j}$, where $\mathbf{z}' = (z_1, \dots, z_{l-1})$, $b_j(\mathbf{z}')$ are polynomials, and $n_1 > n_2 > \dots > n_l \ge 0$ are integers. For a fixed $\mathbf{z}' \in \mathbb{T}^{l-1}$ define

$$h_1(\mathbf{z}') = \max\big\{|b_j(\mathbf{z}')| : 0 \le j \le k_l\big\}.$$

Further, factor out $h_1(\mathbf{z}')$ so that $P(\mathbf{z}_l) = h_1(\mathbf{z}')\sum_{j=1}^{k_l}(b_j(\mathbf{z}')/h_1(\mathbf{z}'))z_l^{n_j}$. The set E of $\mathbf{z}'$ on which h_1 vanishes has ($(l-1)$-dimensional) measure zero and in what follows we can neglect it. Let $g_{\mathbf{z}'}(z_l) = \sum_{j=1}^{k_l}(b_j(\mathbf{z}')/h_1(\mathbf{z}'))z_l^{n_j}$. Then $P(\mathbf{z}_l) = h_1(\mathbf{z}')g_{\mathbf{z}'}(z_l)$ and for any $0 \le \lambda \le 1$ we have

$$|P(\mathbf{z})| \le \nu \Rightarrow h_1(\mathbf{z}') \le \nu^\lambda \quad \text{or} \quad |g_{\mathbf{z}'}(z_l)| \le \nu^{1-\lambda}.$$

Hence,

$$\mu_l\big\{\mathbf{z} \in \mathbb{T}^l : |P(\mathbf{z})| \le \nu\big\} \le \mu_{l-1}\big\{\mathbf{z}' \in \mathbb{T}^{l-1} : h_1(\mathbf{z}') \le \nu^\lambda\big\} + \mu_1\big\{z_l \in \mathbb{T} : |g_{\mathbf{z}'}(z_l)| \le \nu^{1-\lambda}\big\}.$$

For $\mathbf{z}' \in \mathbb{T}^{l-1} - E$ the polynomial $g_{\mathbf{z}'}(z_l) = \sum_{j=1}^{k_l}(b_j(\mathbf{z}')/h_1(\mathbf{z}'))z_l^{n_j}$ has height 1. Hence, by Theorem 1.1,

$$\mu_1\big\{z_l \in \mathbb{T} : |g_{\mathbf{z}'}(z_l)| \le \nu^{1-\lambda}\big\} \le C_{k_l}\nu^{(1-\lambda)/(k_l-1)}.$$

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On the other hand, since $h(P) = 1$, one of the polynomials b_j, say b_{j_0}, has height 1. By the definition of h_1, we have

$$h_1(\mathbf{z}') \geq |b_{j_0}(\mathbf{z}')| \text{ for every } \mathbf{z}' \in \mathbb{T}^{l-1}.$$

Hence, by induction hypothesis,

$$\mu_{l-1}\left\{\mathbf{z}' \in \mathbb{T}^{l-1} : h_1(\mathbf{z}') \leq \nu^{\lambda}\right\} \leq C(k_1, \ldots, k_{l-1})\nu^{\lambda/\sum_{i=1}^{l-1}(k_i-1)}.$$

Thus,

$$\mu_l\left\{\mathbf{z} \in \mathbb{T}^l : |P(\mathbf{z})| \leq \nu\right\} \leq C_{k_l}\nu^{(1-\lambda)/(k_l-1)} + C(k_1, \ldots, k_{l-1})\nu^{\lambda/\sum_{i=1}^{l-1}(k_i-1)}.$$

From this we get the bound in the conclusion of the theorem by choosing

$$\lambda = \frac{\sum_{i=1}^{l-1}(k_i - 1)}{\sum_{i=1}^{l}(k_i - 1)}. \qquad \blacksquare$$

Acknowledgment The author thanks Chris Smyth for stimulating questions.

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