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Stability of the weak Pinsker property for flows

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Abstract. An ergodic flow is said to have the weak Pinsker property if it admits a decreasing sequence of factors whose entropies tend to zero and each of which has a Bernoulli complement. We show that this property is preserved under taking factors and $\bar{d}$-limits. In addition, we show that a flow has the weak Pinsker property whenever one ergodic transformation in the flow has this property.

In [**5**], J.-P. Thouvenot defined the weak Pinsker property for ergodic transformations and showed it to be stable under the taking of factors and $\bar{d}$ limits. Our purpose here is to consider the flow version of this property and to prove the corresponding stability results. Throughout this paper all partitions are assumed to be finite, unless the reverse is explicitly stated. All transformations and flows are assumed to have finite entropy. The basic definition is the following:

Definition. An ergodic, measure-preserving, (finite entropy) flow S on a Lebesgue space X has the *weak Pinsker property* if it admits partitions B_n and H_n, $n = 1, 2, 3, \ldots$, such that:

$(H_n)_S \supset (H_{n+1})_S$;

$\lim_{n\to\infty} h(S, H_n) = 0$;

$(B_n)_S \perp (H_n)_S$;

$(B_n)_S \vee (H_n)_S = X$; and

(S, B_n) is Bernoulli.

The spirit of this definition is to offer an alternative to the (failed) Pinsker conjecture. There are no known transformations (and hence (cf. theorem 3 below) no known flows) which fail to satisfy the weak Pinsker property, and many examples of interest can readily be seen to satisfy it. Our main results are the following:

THEOREM 1. *If* (S, X) *has the weak Pinsker property, then every factor of* (S, X) *has the weak Pinsker property.*

THEOREM 2. *If a sequence of processes* $\{(S^{(n)}, P^{(n)})\}_{n=1}^{\infty}$ *converges in* $\bar{d}$ *to a process* (S, P), *and each* $(S^{(n)}, (P^{(n)})_{S^{(n)}})$ *has the weak Pinsker property, then* $(S, (P)_S)$ *has the weak Pinsker property.*

Theorem 1 will be proved by using the techniques and results of **[1]** to adapt the arguments of **[5]** to the flow setting. The same approach can be used to prove theorem 2, but we have available a simpler route. D. J. Rudolph has observed that theorem 1 can be used to obtain the following result, interesting in its own right, and we use this to obtain theorem 2 quickly. Here and in subsequent arguments we make use of the notational conventions of **[1]**, and we refer the reader to § 2 of that paper for a summary of those conventions.

THEOREM 3. *If* (S, X) *is a flow such that for some ergodic* S_{t_0} *in* S, S_{t_0} *has the weak Pinsker property, then* S *has the weak Pinsker property.*

Proof of theorem 3. By theorem 1, it is sufficient to construct a flow $(\hat{S}, \hat{X})$ which has the weak Pinsker property and which has (S, X) as a factor. We let (S', X') be a rotation of period t_0 on the interval $X' = [0, t_0)$, and we set

$$(\hat{S}, \hat{X}) = (S, X) \times (S', X').$$

Let $(B_n)_{S_{t_0}}$ and $(H_n)_{S_{t_0}}$ be the factors of (S_{t_0}, X) given by the definition of the weak Pinsker property. We use them to construct factors $(\hat{B}_n)_{\hat{S}}$ and $(\hat{H}_n)_{\hat{S}}$ of $(\hat{S}, \hat{X})$ as required by the definition. Let $\hat{B}_n$ and $\hat{H}_n$ be defined by

$$\hat{b}_n(x, t) = b_n(S_{-t}x) \quad \text{and} \quad \hat{h}_n(x, t) = h_n(S_{-t}x),$$

for all $(x, t) \in X \times X'$. Since we have $(\hat{H}_n)_{\hat{S}} \subset (\hat{H}_{n-1})_{\hat{S}}$ and $h(\hat{S}, \hat{H}_n) = h(S, H_n)$, it suffices to show that each $(\hat{S}, (\hat{H}_n)_{\hat{S}})$ has a Bernoulli complement in $(\hat{S}, \hat{X})$. Choose $t_1 \in (0, t_0)$ such that $\hat{S}_{t_1}$ is ergodic. By theorem 2′ of **[1]**, it is sufficient to show that $(\hat{S}_{t_1}, (\hat{H}_n)_{\hat{S}})$ has a Bernoulli complement in $(\hat{S}_{t_1}, \hat{X})$. If we introduce the auxiliary partition

$$\hat{P} = \{X \times (0, t_0/2), X \times [t_0/2, t_0)\},$$

we observe that

$$(\hat{H}_n \vee \hat{P})_{\hat{S}_{t_1}} = (\hat{H}_n \vee \hat{P})_{\hat{S}} = (\hat{H}_n)_{\hat{S}}$$

and

$$(\hat{B}_n \vee \hat{H}_n \vee \hat{P})_{\hat{S}_{t_1}} = X.$$

It is then sufficient to verify that $\hat{B}_n$ is $\hat{H}_n \vee \hat{P}_n$-relatively very weak Bernoulli under $\hat{S}_{t_1}$ (see **[4]** for the definition). This is easy, and we omit the details. □

We note that the discrete version of this argument (which does not depend on our work here) answers affirmatively a question posed by Thouvenot in **[5]**, namely whether a transformation must have the weak Pinsker property if one of its powers does.

Proof of theorem 2. Suppose $\{(S^{(n)}, P^{(n)})\}_{n=1}^{\infty}$ converges to (S, P) in $\bar{d}$, and each $(S^{(n)}, (P^{(n)})_{S^{(n)}})$ has the weak Pinsker property. Choose t_0 so that each $S_{t_0}^{(n)}$ and S_{t_0} is ergodic. Choose a partition $Q \subset (P)_S$ such that $(Q)_{S_{t_0}} = (P)_S$. Construct partitions $Q^{(n)} \subset (P^{(n)})_{S^{(n)}}$ so that $\{(S^{(n)}, Q^{(n)})\}_{n=1}^{\infty}$ converges in $\bar{d}$ to (S, Q). It follows that $\{(S_{t_0}^{(n)}, Q^{(n)})\}_{n=1}^{\infty}$ converges to (S_{t_0}, Q) in $\bar{d}$. But proposition 1 of **[5]** implies that each $(S_{t_0}^{(n)}, (Q^{(n)})_{S_{t_0}^{(n)}})$ has a weak Pinsker Property so that, by proposition 2 of **[5]**, $(S_{t_0}, (Q)_{S_{t_0}})$ has the weak Pinsker property. Now since $(Q)_{S_{t_0}} = (P)_S$, theorem 3 implies that $(S, (P)_S)$ has the weak Pinsker property. □

Before proceeding with the proof of theorem 1, we remark that theorem 2 may be strengthened in the following manner.

THEOREM 2′. *Let (S, P) be a process such that for all $\varepsilon > 0$ there exists (S', P') satisfying*

(i) $\bar{d}[(S, P), (S', P')] < \varepsilon$, *and*

(ii) (S', P') *has factors* $(B')_{S'}$ *and* $(H')_{S'}$ *with* $P' \subset_{\varepsilon} (B' \vee H')_{S'}$, $(B')_{S'} \perp (H')_{S'}$, (S', B') *is Bernoulli, and* $h(S', H') < \varepsilon$.

Then (S, P) *has the weak Pinsker property.*

One can prove the corresponding theorem for transformations in the same way that proposition 2 of [5] is proved. One then proves theorem 2′ in the same manner theorem 2 was proved, making use of lemma 4 of [5].

It will be convenient at times to abuse the standard notation $|P - P'|$ by allowing the partitions in question to be indexed by different sets or sets of different cardinalities. If the sets have different cardinalities, the smaller partition is understood to be augmented by a suitable number of copies of the empty set, and we rely on the context to indicate the appropriate correspondence between the elements of the two partitions.

Theorem 1 will follow from propositions 1 and 2 below. Each of these is preceded by two lemmas.

LEMMA 1. *Let (T, P) be finitely determined. Then for all $\varepsilon > 0$ there exist $\delta > 0$, and n such that for all ergodic processes (T', H'), the process $(T, P) \times (T', H')$ is H'-relatively finitely determined to within ε by δ and n. That is, given ergodic $(\bar{T}, \bar{P} \vee \bar{H})$ with*

(1) $(T', H') \approx (\bar{T}, \bar{H})$;

(2) $|\text{dist} \bigvee_{i=0}^{n} \bar{T}^{-i}(\bar{P} \vee \bar{H}) - \text{dist} \bigvee_{i=0}^{n} (T \times T')^{-i}(P \times H')| < \delta$; *and*

(3) $|h(\bar{T}, \bar{P} \vee \bar{H}) - h(T \times T', P \times H')| < \delta$,

we have $\bar{d}_{\bar{H}, H'}[(\bar{T}, \bar{P} \vee \bar{H}), (T \times T', P \times H')] < \varepsilon$.

Proof. Case 1. If (T, P) is an independent process, then this fact is proved in [3].

Case 2. If (T, P) is not independent, then we may choose an independent generator B for (T, P) and apply standard approximation arguments to get the result using case 1. We omit the details. We emphasize the point that δ and n depend only on (T, P) and ε, and not on (T', H'). Furthermore we allow H' here to be countably infinite. □

LEMMA 2. *For all $\varepsilon > 0$, there exists L such that if $(S, (Q)_S)$ has a Rokhlin tower τ of height L such that the partition of τ into Q-columns has finitely or countably many elements and entropy less than La, and τ^c is in a single element of Q, there is a partition $Q' \subset (Q)_S$ such that $|Q' - Q| < \varepsilon$, $(Q')_{S_1} = (Q')_S$, and $h(S, Q') < a + \varepsilon$.*

Proof. The argument is basically that of [2 chapter 12]. We begin by constructing a partition Q_1 which consists of distinct atoms of flow-length 1 at the base of each Q-column, and which agrees with Q elsewhere. We then proceed to make successive modifications as in [2] to obtain Q' with the generating property $(Q')_{S_1} = (Q')_S$. If L is sufficiently large, we can make these modifications on as small a fraction of the space as desired (only needing to add one more atom to Q_1 in doing so), and we will have $|Q' - Q| < \varepsilon$.

If we let R denote the partition that would be obtained by this construction if Q_1 had consolidated the complement of the union of the new atoms in one set, then we would have $(R)_{S_1}=(R)_S$, $(R)_{S_1}=(Q')_S$, and (again for sufficiently large L) we could calculate

$$h(S, Q')=h(S, R')=h(S_1, R')<a+\varepsilon. \qquad \square$$

PROPOSITION 1. *Let (S, X) be a flow and P, B and H partitions of X such that $(B)_S\perp(H)_S$, $(B)_S\vee(H)_S\supset P$ and (S, B) is Bernoulli. Then for all $\varepsilon>0$ there exist partitions $\bar{B}$ and $\bar{H}$ in $(P)_S$ such that $(\bar{B})_S\perp(\bar{H})_S$, $(\bar{B})_S\vee(\bar{H})_S\supset_\varepsilon P$, (S, B) is Bernoulli, and $|h(S, P|_{(\bar{H})_S})-h(S, P|_{(H)_S})|<\varepsilon$.*

Proof of proposition 1. By theorem 3 of [**1**] applied to the factor $(P\vee H)_S$, of $(B\vee H)_S$, we may assume that $(B\vee H)_S=(P\vee H)_S$, so that, in particular, $h(S, B\vee H)=h(S, P\vee H)$. Since (S, B) is an increasing union of (Bernoulli) factors of properly smaller entropy, there is a partition B_1 in $(B)_S$ such that $(B_1)_S\vee(H)_S\supset_{\varepsilon/2} P$ and $u=h(S, B)-h(S, B_1)>0$. We may also assume, without loss of generality, that $(B_1)_{S_1}=(B_1)_S$, $(H)_{S_1}=(H)_S$, and $(P)_{S_1}=(P)_S$.

Choose $n\in\mathbb{N}$ so that $\bigvee_{-n}^{n} S_i(B_1\vee H)\supset_{\varepsilon/2} P$. Choose $\eta_1>0$ so that if partitions R_1 and R_2 satisfy $|R_1-R_2|<\eta_1$, then

$$\left|\operatorname{dist}\bigvee_{-n}^{n} S_iR_1-\operatorname{dist}\bigvee_{-n}^{n} S_iR_2\right|<\varepsilon/2.$$

Let $0<\eta_2<(\eta_1/16)^2$ where $\eta_2^2<\eta_2/1000$, and choose $K\in\mathbb{N}$ so that for all $K'\geq K$

$$\|\psi_{K'}b_1-b_1\|_{X,1/K'}<\tfrac{1}{2}(\eta_2/16)^4.$$

Choose $N\in\mathbb{N}$, a multiple of K, so that $N>100K/\eta_2$. Choose $\delta>0$ and $u\in\mathbb{N}$ as in lemma 1 for the relatively finitely determined condition of $(S_{1/N}, B_1)\times(\tilde{S}_{1/N}, \tilde{H})$ with respect to $\eta_2/3$, (where $(\tilde{S}_{1/N}, \tilde{H})$ is arbitrary). Choose $k\in\mathbb{N}$ and $\delta'>0$ so that if partitions $\bar{B}'$ and $\bar{H}$ satisfy

(1) $|\operatorname{dist}\bigvee_{-k}^{k} S_{i/N}(B_1\vee H\vee P)-\operatorname{dist}\bigvee_{-k}^{k} S_{i/N}(\bar{B}'\vee\bar{H}\vee P)|<\delta'$,

then

(a) $h(S_{1/N}, P|_{(\bar{H})_{S_{1/N}}})<h(S_{1/N}, P|_{(H)_{S_{1/N}}})+\min(\delta/2, \varepsilon/N)$;

(b) $h(S_{1/N}, P|_{(\bar{B}'\vee\bar{H})_{S_{1/N}}})<h(S_{1/N}, P|_{(B_1\vee H)_{S_{1/N}}})+\delta/2$;

(c) $\bigvee_{-n}^{n} S_{i/N}(\bar{B}'\vee\bar{H})\subset_{\varepsilon/2} P$; and

(d) $|\operatorname{dist}\bigvee_0^{u-1} S_{-i/N}(\bar{B}'\vee\bar{H})-\operatorname{dist}\bigvee_0^{u-1} S_{-i/N}(B_1)\times\operatorname{dist}\bigvee_0^{u-1} S_{-i/N}(\bar{H})|<\delta$.

(Note that this last condition may be obtained since $(B_1)_S\perp(H)_S$.)

Choose $\rho>0$ so that if partitions R_1 and R_2 satisfy $|R_1-R_2|<\rho$ then

$$|\operatorname{dist}\bigvee_{-k}^{k} S_{i/N}R_1-\operatorname{dist}\bigvee_{-k}^{k} S_{i/N}R_2|<\delta'/8.$$

Choose $M\in\mathbb{N}$, a multiple of N, so that $\|\psi_Mp-p\|_{X,1/N}<(\rho/100)^2$. Let a, b, and c denote $h(S, P)$, $h(S, H)$ and $h(S, P\vee H)$, respectively. Choose

$$0<\bar{\delta}<\min\left(\frac{\varepsilon}{N}, \frac{\delta}{1000}, \frac{N}{M}\frac{\delta}{20}, \frac{\eta_2}{1000}\right),$$

where $\bar{\delta}$ is subject to further restrictions that will be made explicit in the course of the proof. Choose $L \in \mathbb{N}$, also subject to further restrictions, so that there are sets:

$\mathscr{E}_1 \subset \bigvee_0^{L-1} S_{-i/M}(B_1 \vee H \vee P)$ with $\mu(\bigcup \mathscr{E}_1) > 1-\bar{\delta}^2$, such that for all $\alpha \in \mathscr{E}_1$,

$$\left| \operatorname{dist}_{\alpha} \bigvee_{-k}^{k} S_{i/N}(B_1 \vee H \vee P) - \operatorname{dist}_{X} \bigvee_{-k}^{k} S_{i/N}(B_1 \vee H \vee P) \right| < \delta'/8;$$

$\mathscr{E}_2 \subset \bigvee_0^{L-1} S_{-i/M}(P \vee H)$ with $\mu(\bigcup \mathscr{E}_2) > 1-\bar{\delta}^2$ such that for all $\alpha \in \mathscr{E}_2$,

$$\mu(\alpha) = 2^{-L((c/M)\pm\bar{\delta})};$$

$\mathscr{E}_3 \subset \bigvee_0^{L-1} S_{-i/M}(B_1)$ with $\mu(\bigcup \mathscr{E}_3) > 1-\bar{\delta}^2$ such that for all $\alpha \in \mathscr{E}_3$,

$$\mu(\alpha) = 2^{-L((c-b-u)/M)\pm\bar{\delta})};$$

$\mathscr{E}_4 \subset \bigvee_0^{L-1} S_{-i/M}(H)$ with $\mu(\bigcup \mathscr{E}_4) > 1-\bar{\delta}^2$ such that for all $\alpha \in \mathscr{E}_4$

$$\mu(\alpha) = 2^{-L((b/M)\pm\bar{\delta})};$$

and

$\mathscr{E}_5 \subset \bigvee_0^{L-1} S_{-i/M}(P)$ with $\mu(\bigcup \mathscr{E}_5) > 1-\bar{\delta}^2$ such that for all $\alpha \in \mathscr{E}_5$

$$\mu(\alpha) = 2^{-L((a/M)\pm\bar{\delta})}.$$

We have

$$\begin{aligned}\|\psi_M b_1 - \psi_K(\psi_M b_1)\|_{X,L/M} &= \|\psi_M b_1 - \psi_K b_1\|_{X,L/M} \\ &\le \|\psi_M b_1 - b_1\|_{X,L/M} + \|b_1 - \psi_K b_1\|_{X,L/M} < (\eta_2/16)^4,\end{aligned}$$

so if

$$\mathscr{B} = \left\{ \beta \in \bigvee_0^{L-1} S_{-i/M} B_1 \,\middle|\, \|\psi_M b_1(\beta) - \psi_K(\psi_M b_1(\beta))\|_{L/M} < (\eta_2/16)^2 \right\},$$

then $\mu(\bigcup \mathscr{B}) > 1-(\eta_2/16)^2$. Thus if $\bar{\mathscr{E}}_1 = \{\alpha \in \mathscr{E}_1 | \alpha \subset (\bigcup \mathscr{E}_3) \cap (\bigcup \mathscr{B})\}$, then

$$\mu(\bigcup \bar{\mathscr{E}}_1) > 1-(\eta_2/16)^2 - 2\bar{\delta}^2 > 1-(\eta_2/16)^2,$$

and if $\bar{\bar{\mathscr{E}}}_1 = \{\alpha \in \mathscr{E}_1 | \alpha \subset (\bigcup \mathscr{E}_3)\}$ then

$$\mu(\bigcup \bar{\bar{\mathscr{E}}}_1) > 1-2\bar{\delta}^2.$$

If $\bar{\mathscr{E}}_2 = \{\alpha \in \mathscr{E}_2 | \mu(\alpha \cap (\bigcup \bar{\mathscr{E}}_1) | > 0\}$ then

$$\mu(\bigcup \bar{\mathscr{E}}_2) > 1-(\eta_2/16) - \bar{\delta}^2 > 1-(\eta_2)^2/8,$$

and if $\bar{\bar{\mathscr{E}}}_2 = \{\alpha \in \mathscr{E}_2 | \mu(\alpha \cap (\bigcup \bar{\bar{\mathscr{E}}}_1)) > 0\}$ then

$$\mu(\bigcup \bar{\bar{\mathscr{E}}}_2) > 1-3\bar{\delta}^2.$$

For all $\alpha \in \mathscr{E}_4$ let $\bar{\mathscr{P}}_\alpha$ (resp. $\bar{\bar{\mathscr{P}}}_\alpha$) denote the $\bigvee_0^{L-1} S_{-i/M}\, P$-atoms corresponding to the atoms of $\bar{\mathscr{E}}_2$ (resp. $\bar{\bar{\mathscr{E}}}_2$) covering α. Let $\bar{\mathscr{E}}_6$ be a subset of $\mathscr{E}_4$ and $\bar{\varphi}$ an assignment of a set $\bar{\varphi}(\alpha) \subset \bar{\mathscr{P}}_\alpha \cap \mathscr{E}_5$ to each $\alpha \in \bar{\mathscr{E}}_6$ such that:

for all $\alpha \in \bar{\mathscr{E}}_6$, $|\bar{\varphi}(\alpha)| > 2^{L((c-b/M)-3\bar{\delta})}$;

for all $\alpha \neq \alpha'$, in $\bar{\mathscr{E}}_6$, $\bar{\varphi}(\alpha) \cap \bar{\varphi}(\alpha') = \varnothing$;

for all $\alpha \in \bar{\mathscr{E}}_6$, $\bar{\varphi}(\alpha)$ is maximal with respect to the above properties, and $\bar{\mathscr{E}}_6$ is maximal with respect to the above properties.

We see then that $\mu(\bigcup_{\alpha \in \bar{\mathscr{E}}_6} \bar{\varphi}(\alpha))^c < \eta_2/50$. Indeed, if we regard this set as a union of $\bigvee_0^{L-1} S_{-i/M}(P \vee H)$-atoms, then those outside $(\bigcup \bar{\mathscr{E}}_2) \cap (\bigcup \mathscr{E}_4) \cap (\bigcup \mathscr{E}_5)$ form a set of measure less than $(\eta_2^2/8) + 2\bar{\delta}^2 < \eta_2/100$. The rest meet the fewer than $2^{L((b/M)+\bar{\delta})}$ atoms of $\mathscr{E}_4 \backslash \bar{\mathscr{E}}_6$ fewer than $2^{L((c-b/M)-3\bar{\delta})}$ at a time, and each has measure less than $2^{-L((c/M)-\bar{\delta})}$, so that their union has measure less than $2^{-L\bar{\delta}} > \eta_2/100$, provided $L > (1/\bar{\delta}) \log (100/\eta_2)$.

Now let $\bar{\mathscr{E}}_6 \subset \bar{\bar{\mathscr{E}}}_6 \subset \mathscr{E}_4$, and $\bar{\bar{\varphi}}$ be an assignment of a set $\bar{\bar{\varphi}}(\alpha) \subset \bar{\bar{\mathscr{P}}}_\alpha \cap \mathscr{E}_5$ to each $\alpha \in \bar{\bar{\mathscr{E}}}_6$ such that:

for all $\alpha \in \bar{\mathscr{E}}_6$, $\bar{\bar{\varphi}}(\alpha) \supset \bar{\varphi}(\alpha)$;

for all $\alpha \in \bar{\bar{\mathscr{E}}}_6$, $|\bar{\bar{\varphi}}(\alpha)| > 2^{L((c-b/M)-3\bar{\delta})}$;

for all $\alpha \neq \alpha'$ in $\bar{\bar{\mathscr{E}}}_6$, $\bar{\bar{\varphi}}(\alpha) \cap \bar{\bar{\varphi}}(\alpha') = \varnothing$;

for all $\alpha \in \bar{\bar{\mathscr{E}}}_6$, $\bar{\bar{\varphi}}(\alpha)$ is maximal with respect to these properties; and

$\bar{\bar{\mathscr{E}}}_6$ is maximal with respect to these properties.

Arguing as before we see that $\mu(\bigcup_{\alpha \in \bar{\bar{\mathscr{E}}}_6} (\bar{\bar{\varphi}}(\alpha)))^c < 6\bar{\delta}^2$. Indeed if we regard this set as a union of $\bigvee_0^{L-1} S_{-i/M}(P \vee H)$-atoms, then those outside $(\bigcup \mathscr{E}_2) \cap (\bigcup \mathscr{E}_4) \cap (\bigcup \mathscr{E}_5)$ form a set of measure less than $5\bar{\delta}^2$, while the rest meet the fewer than $2^{L((b/M)+\bar{\delta})}$ atoms of $\mathscr{E}_4 \backslash \bar{\bar{\mathscr{E}}}_6$ fewer than $2^{L((c-b/M)-3\bar{\delta})}$ at a time, and each has measure less than $2^{-L((c/M)-\bar{\delta})}$, so that their union has measure less than $2^{-L\bar{\delta}} < \bar{\delta}^2$ provided $L > (2/\bar{\delta}) \log (1/\bar{\delta})$.

Now for each $\alpha \in \bar{\mathscr{E}}_6$, and $\gamma \in \bar{\varphi}(\alpha)$, we let $W(\alpha \cap \gamma) \in \mathscr{E}_3$ be such that $\alpha \cap \gamma \cap W(\alpha \cap \gamma) \in \bar{\mathscr{E}}_1$. For each $\alpha \in \bar{\bar{\mathscr{E}}}_6$ and $\gamma \in \bar{\bar{\varphi}}(\alpha)$, where $W(\alpha \cap \gamma)$ has not yet been defined, let $W(\alpha \cap \gamma) \in \mathscr{E}_3$ be such that $\alpha \cap \gamma \cap W(\alpha \cap \gamma) \in \bar{\bar{\mathscr{E}}}_1$.

Fix $\eta_3 > 0$, whose size will be dictated by the following. Let τ be a Rokhlin tower in (S, P) of height L/M with base F and measure greater than $1 - \eta_3$, such that if

$$G = \{x \mid \|\psi_M p(x) - p(x)\|_{L/M} < \rho/100\},$$

then

$$\left| \operatorname*{dist}_F \left(\bigvee_0^{L-1} S_{-i/M}, P \right) \vee \{G, G^c\} - \operatorname*{dist}_X \left(\bigvee_0^{L-1} S_{-i/M} P \right) \vee \{G, G^c\} \right| < \eta_3.$$

Construct a partition $\tilde{P}$ by setting, for all $x \in F, 0 \le t \le L/M$, $\tilde{p}(S_t x) = \psi_M p(x, t)$, while for all $x \in \tau^c$, $\tilde{p}(x)$ is set equal to a constant symbol. If η_3 is chosen sufficiently small, then $|P - \tilde{P}| < \rho$. Construct $\tilde{H}$ by setting, for all $\alpha \in \bar{\bar{\mathscr{E}}}_6$, and $x \in F \cap (\bigcup \bar{\bar{\varphi}}(\alpha))$, $0 \le t \le L/M$, $\tilde{h}(S_t x) = \psi_M h(\alpha, t)$. This defines $\tilde{H}$ on all but a set of measure less than $6\bar{\delta}^2 + 2\eta_3$, and we put this in a single $\tilde{H}$ set. Finally, for $\tilde{\alpha} \vee \tilde{\gamma} \in \bigvee_0^{L-1} S_{-i/M}(\tilde{H} \vee \tilde{P})$ having the same name as $\alpha \vee \gamma \in \bigvee_0^{L-1} S_{-i/M}(H \vee P)$ where $\alpha \in \bar{\bar{\mathscr{E}}}_6$ and $\gamma \in \bar{\bar{\varphi}}(\alpha)$, and for $x \in F \cap \tilde{\alpha} \vee \tilde{\gamma}$, we let $\tilde{b}(S_t x) = \psi_M \tilde{b}_1(W(\alpha \cap \gamma))$, thus defining a partition $\tilde{B}$ on all but $6\bar{\delta}^2 + 2\eta_3$ of X, and we define $\tilde{b}$ to be constant elsewhere.

Now by the construction of W, each atom of the trace of $\bigvee_0^{L-1} S_{-i/M}(\tilde{B} \vee \tilde{H} \vee \tilde{P})$ on F outside a set of relative measure $\bar{\delta}^2 + \eta_3$ in F has a name from $\mathscr{E}_1$, so if $\bar{\delta}$ and η_3 are sufficiently small (with respect to ρ), we have

$$\left| \operatorname{dist} \bigvee_{-k}^{k} S_{i/N} \left(\tilde{B} \vee \tilde{H} \vee \tilde{P} \right) - \operatorname{dist} \bigvee_{-k}^{k} S_{i/N}(B_1 \vee H \vee P) \right| < \delta'/4.$$

Since $|P - \tilde{P}| < \rho$, we have

$$\left| \operatorname{dist} \bigvee_{-k}^{k} S_{i/N}(\tilde{B} \vee \tilde{H} \vee P) - \operatorname{dist} \bigvee_{-k}^{k} S_{i/N}(B_1 \vee H \vee P) \right| < 3\delta'/8.$$

Also by the construction of W, each atom of the trace of $\bigvee_0^{L-1} S_{-i/M}(\tilde{B})$ on F outside a set of relative measure $(\eta_2/50)+\eta_3$ in F has a name from $\mathscr{B}$, so that, arguing as in [**1**], we see that $\|\psi_N\tilde{b}-\tilde{b}\|_{X,1/N}<\eta_2/10$. (Again assuming η_3 sufficiently small.)

We observe that $\tilde{P}$ subdivides τ into fewer than $2^{L((a/M)+\bar{\delta})}$ columns, partitioned by the $\tilde{H}$-columns into groups of at least $2^{L((c-b/M)-3\bar{\delta})}$ columns, so that $\tilde{H}$ has at most $2^{L((a+b-c)/M)+4\bar{\delta})}$ columns on τ. Furthermore, since the columns of $\tilde{B}$ on τ have names from $\mathscr{E}_3$, there are fewer than $2^{L((c-b-u)/M)+\bar{\delta})}$ of them.

Thus if L was chosen sufficiently large, we may apply lemma 2 to obtain $\bar{B}'$ and $\bar{H}$ in $(P)_S$ such that $(\bar{B}')_{S_1}=(\bar{B}')_S$, $(\bar{H})_{S_1}=(\bar{H})_S$, $h(S_{1/M}, \bar{B}')<(c-b-u)/M+\bar{\delta}$, $h(S_{1/M}, \bar{H})<(a+b-c/M)+4\bar{\delta}$, $\|\psi_N\bar{b}'-\bar{b}'\|_{X,1/N}<\eta_2/3$, and condition (1) holds. Condition (1a) then gives us

(2) $|h(S_{1/N}, \bar{H})-((a-(c-b))/N)|<\min(\delta/2, \varepsilon/N)$

so that $|h(S, P|_{(\bar{H})_S})-h(S, P|_{(H)_S})|<\varepsilon$ as desired. Furthermore, we have

$$h(S_{1/N}, \bar{B}'\vee\bar{H})\le h(S_{1/N}, \bar{B}')+h(S_{1/N}, \bar{H})<\frac{a-u}{N}+\frac{5\bar{\delta}M}{N}<\frac{a-u}{N}+\frac{\delta}{2},$$

which with conditions (2) and (1b) gives

$$|h(S_{1/N}, \bar{B}'\vee\bar{H})-h((S_{1/N}, B_1)\times(S_{1/N}, \bar{H}))|<\delta$$

as desired. Since (1d) holds as well, we have by lemma 3 of [**1**], that

$$\bar{d}_{\bar{H}}[(S, \bar{B}'\vee\bar{H}), (S, B_1)\times(S, \bar{H})]<\eta_2<\left(\frac{\eta_1}{16}\right)^2,$$

so by corollary 1 of [**1**] there is a partition $\bar{B}$ in $(P)_S$ with $|\bar{B}-\bar{B}'|<\eta_1$, $(S, \bar{B})\approx(S, B_1)$ and $(\bar{B})_S\perp(\bar{H})_S$. By condition (1c) and the choice of η_1, we have $\bigvee_{-n}^{n} S_i(\bar{B}\vee\bar{H})\supset_\varepsilon P$ and we are done. □

LEMMA 3. *If (S, X) is a flow with partitions H and Q such that $(H)_S$ has a Bernoulli complement in (S, X) and $(Q)_S\supset(H)_S$, then for all $\varepsilon>0$ there exists $\bar{Q}$ with $|\bar{Q}-Q|<\varepsilon$, $(S, \bar{Q}\vee H)\approx(S, Q\vee H)$, and $(\bar{Q})_S$ has a Bernoulli complement in (S, X). (If $h(S, Q)=h(S)$, then this Bernoulli complement is trivial; that is $(\bar{Q})_S=X$.)*

Proof. Choose $\tilde{B}$ so that $(S, \tilde{B})$ is Bernoulli, $(\tilde{B})_S\perp(Q)_S$, and $h(S, \tilde{B}\vee Q)=h(S)$. This is possible by corollary 1 of [**1**]. By theorem 3 of [**1**], $(H)_S$ has a Bernoulli complement in $(\tilde{B}\vee Q)_S$. Now repeated application of lemma 5 of [**1**] yields $\bar{B}\vee\bar{Q}$ such that $|\bar{B}\vee\bar{Q}-B\vee Q|<\varepsilon$, $(S, \bar{B}\vee\bar{Q})\approx(S, B\vee Q)$ and $(\bar{B}\vee\bar{Q})_S=X$, so that $\bar{Q}$ has the desired properties. □

LEMMA 4. *If (S, X) is a flow with partitions B, Q and P and $\varepsilon>0$ such that $(B\vee Q)_{S_1}=(B\vee Q)_S$, $(Q)_{S_1}=(Q)_S$, $(B)_S\perp(Q)_S$, (S, B) is Bernoulli, and $(B\vee Q)_S\supset_\varepsilon P$, then for all $\varepsilon_1>0$ there exists $\delta>0$ such that if $\bar{Q}$ is a partition satisfying*

(1) $(\bar{Q})_{S_1}=(\bar{Q})_S$;

(2) $|\bar{Q}-Q|<\delta$; *and*

(3) $|h(S, \bar{Q})-h(S, Q)|<\delta$;

then there exists a partition $\bar{B}$ such that $(S, \bar{B})$ is Bernoulli, $(\bar{B})_S\perp(\bar{Q})_S$, and $(\bar{B}\vee\bar{Q})_S\supset_{\varepsilon+\varepsilon_1} P$. We allow $\bar{Q}$ here to be countably infinite.

Proof. Choose a partition $B_1 \subset (B)_S$ so that $(B_1)_{S_1} = (B_1)_S$, $h(S, B) - h(S, B_1) = u > 0$, and $(B_1 \vee Q)_S \supset_{\varepsilon+(\varepsilon_1/10)} P$. Choose N so that $\bigvee_{i=-N}^{N} S_i(B_1 \vee Q) \supset_{\varepsilon+(\varepsilon_1/10)} P$. Choose $\xi > 0$ so that $|R_1 - R_2| < \xi$ implies

$$\left| \text{dist} \bigvee_{-N}^{N} S_i R_1 - \text{dist} \bigvee_{-N}^{N} S_i R_2 \right| < \frac{\varepsilon_1}{10}.$$

Let $\xi' < (\xi/16)^2$ and choose M so that

(4) $\|\psi_M b_1 - b_1\|_{X, 1/M} < \xi'/3$.

Choose (δ', n) by lemma 1 so that for arbitrary $(\hat{T}, \hat{Q})$, $(S_{1/M}, B_1) \times (\hat{T}, \hat{Q})$ is $\hat{Q}$-relatively finitely determined to $\xi'/3$. Now choose $\delta < \min(\delta', \xi)$ so that conditions (2) and (3) imply that

(5) $|\text{dist} \bigvee_1^n S_{-i/M}(B_1 \vee \bar{Q}) - \text{dist} \bigvee_1^n S_{-i/M} B_1 \times \bigvee_1^n S_{-i/M} \bar{Q}| < \delta'$, and

(6) $|h(S_{1/M}, B_1 \vee \bar{Q}) - h[(S_{1/M}, B_1) \times (S_{1/M}, \bar{Q})]| < \delta'$, and

(7) $|h(S_{1/M}, \bar{Q}) - h(S_{1/M}, Q)| < u/M$.

Thus if (2) and (3) hold for δ, (5) and (6) imply

$$\bar{d}_{\bar{Q}}[(S_{1/M}, B_1 \vee \bar{Q}), [(S_{1/M}, B_1) \times (S_{1/M}, \bar{Q})]] < \xi'/3,$$

so that (4) gives $\bar{d}_{\bar{Q}}[(S, B_1 \vee \bar{Q}), [(S, B_1) \times (S, \bar{Q})]] < \xi'$. With conditions (1) and (2), we have

$$h(S, B_1 \vee \bar{Q}) \leq h(S, B_1) + h(S, \bar{Q}) < h(S, B_1) + h(S, Q) + u \leq h(S).$$

Hence by corollary 1 of [1], there exists $\bar{B}$ with $|\bar{B} - B_1| < \xi$, $(\bar{B})_S \perp (\bar{Q})_S$, and $(S, \bar{B}) \approx (S, B_1)$, and by the choice of ξ we have

$$\left| \text{dist} \bigvee_{i=-N}^{N} S_i(\bar{B} \vee \bar{Q} \vee P) - \text{dist} \bigvee_{i=-N}^{N} S_i(B_1 \vee Q \vee P) \right| < \varepsilon_1/5,$$

so that $\bigvee_{i=-N}^{N} S_i(\bar{B} \vee \bar{Q} \vee P) \supset_{\varepsilon+\varepsilon_1} P$. □

PROPOSITION 2. *Let (S, X) be a flow with a generator P and factors $(B_n)_S$ and $(H_n)_S$, $n = 1, 2, 3, \ldots$, such that $h(S, H_n) < \varepsilon_n$, $(B_n)_S \perp (H_n)_S$, $(B_n \vee H_n)_S \supset_{\varepsilon_n} P$, and (S, B_n) is Bernoulli, where $\lim_{n\to\infty} \varepsilon_n = 0$. Then for all partitions Q satisfying $(Q)_{S_1} = (Q)_S$ and $\varepsilon > 0$, there is a partition Q' such that $|Q - Q'| < \varepsilon$, $(Q')_{S_1} = (Q')_{S'}$, $|h(S, Q') - h(S, Q)| < \varepsilon$, and $(Q')_S$ has a Bernoulli complement in (S, X).*

Proof. Write the rationals as $\bigcup_{i=0}^{\infty} \alpha_i$, where each α_i is finite, and $\alpha_i \subset \alpha_{i+1}$. Choose m_0 so that

$$\bigvee_{i=-m_0}^{m_0} S_i Q \supset_{\frac{1}{4}} \bigvee_{t \in \alpha_0} S_t Q,$$

and choose $\hat{\varepsilon}$ so that $|Q - Q'| < \hat{\varepsilon}$ implies

$$\bigvee_{i=-m_0}^{m_0} S_i Q' \supset_{\frac{1}{2}} \bigvee_{t \in \alpha_0} S_t Q'.$$

Without loss of generality, we may assume that $\varepsilon < \hat{\varepsilon}$. We may also assume that for all n, $(B_n)_{S_1} = (B_n)_S$, $(H_n)_{S_1} = (H_n)_S$, $H_n = \{H_n^1, H_n^2\}$ with $\mu(H_n^2) < \varepsilon_n$, and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Choose n_1 so that $\varepsilon_{n_1} < \varepsilon/10$ and so that for partitions $\hat{Q}_i$ labelled like Q, $|\hat{Q}_1 - \hat{Q}_2| < 3\varepsilon_{n_1}$ implies $|h(S_1, \hat{Q}_1) - h(S_1, \hat{Q}_2)| < \varepsilon/4$.

Choose N so that $\bigvee_{i=-N}^{N} S_i(B_{n_1} \vee H_{n_1}) \supset_{\varepsilon_{n_1}} Q$. Choose $\eta > 0$ so that $|R_1 - R_2| < \eta$ implies

$$\left| \operatorname{dist} \bigvee_{i=-N}^{N} S_i R_1 - \operatorname{dist} \bigvee_{i=-N}^{N} S_i R_2 \right| < \varepsilon_{n_1}.$$

Choose M so that

$$\|\psi_M(b_{n_1} \vee h_{n_1}) - (b_{n_1} \vee h_{n_1})\|_{X, 1/M} < \left(\frac{\eta}{100}\right)^2.$$

Fix $\delta' > 0$ whose size will be dictated by the following. Choose L so that a collection

$$\mathscr{C}_1 \subset \bigvee_{i=0}^{L-1} S_{-i/M}(B_{n_1} \vee H_{n_1} \vee Q)$$

satisfies $\mu(\bigcup \mathscr{C}_1) > 1 - \delta'$, and the distribution of $\bigvee_{i=-N}^{N} S_i(B_{n_1} \vee H_{n_1} \vee Q)$ on the name of each $\beta \in \mathscr{C}_1$ is within δ' of the actual distribution, and the number of $\bigvee_{i=0}^{L-1} S_{-i/M} Q$ atoms that contain atoms of $\mathscr{C}_1$ is less than $2^{L(h(S_{1/M}, Q) + (\delta'/M))}$.

Let $G = \{x \mid \|\psi_M(b_{n_1} \vee h_{n_1})(x) - (b_{n_1} \vee h_{b_1})(x)\|_{L/M} < \eta/100\}$ so that $\mu(G) > 1 - (\eta/100)$. Build a Rokhlin tower in $(S, (B_{n_1} \vee H_{n_1})_S)$ of measure greater than $1 - \delta'$ such that

$$\left| \operatorname*{dist}_{(F, \bar{\mu})} \{G, G^c\} \vee \bigvee_{i=0}^{L-1} S_{-i/M}(B_{n_1} \vee H_{n_1}) - \operatorname*{dist}_{X} \{G, G^c\} \vee \bigvee_{i=0}^{L-1} S_{-i/M}(B_{n_1} \vee H_{n_1}) \right| < \delta',$$

where $(F, \bar{\mu})$ is the base of the tower with its normalized cross-sectional measure.

Let $\mathscr{C}_2$ denote the set of $\bigvee_{i=0}^{L-1} S_{-i/M}(B_{n_1} \vee H_{n_1})$ atoms which contain atoms of $\mathscr{C}_1$. Construct partitions $\tilde{B}_{n_1}$ and $\tilde{H}_{n_1}$ by setting, for all $x \in F \cap (\bigcup \mathscr{C}_2)$ and $0 \le t \le L/M$,

$$(\tilde{b}_{n_1} \vee \tilde{h}_{n_1})(S_t x) = \psi_M(b_{n_1} \vee h_{n_1})(x, t)$$

and by putting the rest of the space in single $\tilde{B}_{n_1}$ and $\tilde{H}_{n_1}$ atoms. Now construct $\tilde{Q}$ so that for all $x \in F \cap (\bigcup \mathscr{C}_2)$, $(\tilde{b}_{n_1} \vee \tilde{h}_{n_1} \vee \tilde{q})(S_{i/M} x)$, $0 \le i \le L-1$, gives a name from $\mathscr{C}_1$, and for $0 \le t \le L/M$, $\tilde{q}(x, t) = \psi_M \tilde{q}(x, t)$, and $\tilde{q}$ is constant elsewhere.

If δ' is chosen sufficiently small, we have

$$\left| \operatorname{dist} \bigvee_{i=-N}^{N} S_i(\tilde{B}_{n_1} \vee \tilde{H}_{n_1} \vee \tilde{Q}) - \operatorname{dist} \bigvee_{i=-N}^{N} S_i(B_{n_1} \vee H_{n_1} \vee Q) \right| < \varepsilon_{n_1}/2$$

and

$$|\tilde{B}_{n_1} \vee \tilde{H}_{n_1} \vee \tilde{Q} - B_{n_1} \vee H_{n_1} \vee Q| < \eta$$

so that, (by the choice of η),

$$\left| \operatorname{dist} \bigvee_{i=-N}^{N} S_i(B_{n_1} \vee H_{n_1} \vee \tilde{Q}) - \operatorname{dist} \bigvee_{i=-N}^{N} S_i(B_{n_1} \vee H_{n_1} \vee Q) \right| < \varepsilon_{n_1}.$$

Hence $|\tilde{Q} - Q| < 2\varepsilon_{n_1}$. Now by lemma 2, (if L was chosen sufficiently large), we can construct $\tilde{Q}_1 \subset (B_{n_1} \vee H_{n_1})_S$ such that $|\tilde{Q}_1 - \tilde{Q}| < \varepsilon_{n_1}$, $(\tilde{Q}_1)_{S_1} = (\tilde{Q}_1)_S$, and $h(S, \tilde{Q}_1) < h(S, Q) + (\varepsilon/4)$. But by the choice of ε_{n_1}, we have in fact $|h(S, \tilde{Q}_1) - h(S, Q)| < \varepsilon/4$.

Let $\bar{Q}_1 = \tilde{Q}_1 \vee H_{n_1}$. Applying lemma 3 to $(S, (B_{n_1} \vee H_{n_1})_S)$, we obtain a partition $Q_1 \subset (B_{n_1} \vee H_{n_1})_S$ such that $|Q_1 - \bar{Q}_1| < \varepsilon_{n_1}$, $(S, Q_1 \vee H_{n_1}) \approx (S, \bar{Q}_1 \vee H_{n_1})$, and $(Q_1)_S$ has a Bernoulli complement in $(S, (B_{n_1} \vee H_{n_1})_S)$. Note that $|Q_1 - Q| < \varepsilon/2$ and $|h(S, Q_1) - h(S, Q)| < \varepsilon/2$.

We now iterate the above construction to obtain $n_i\uparrow\infty$, $\delta_i\downarrow 0$, and partitions Q_i and $\hat{B}_i$ satisfying $(Q_i)_{S_1}=(Q_i)_S$, $|Q_{i+1}-Q_i|<\delta_i/2$, $|h(S, Q_{i+1})-h(S, Q_i)|<\delta_i/2$, $(S, \hat{B}_i)$ is Bernoulli, $(\hat{B}_i)_S\perp(Q_i)_S$, $(B_i\vee Q_i)_S \supset_{\varepsilon_{n_i}} P$ and δ_i is chosen so that $\delta_i<\delta_{i-1}/2$, $(\delta_1<\varepsilon/2)$, and if a partition Q' satisfies

(1) $|Q'-Q_i|<\delta_i$,

then there exist m_i such that

$$\bigvee_{i=-m_i}^{m_i} S_iQ' \underset{1/2^{i+1}}{\supset} \bigvee_{t\in\alpha_i} S_tQ'.$$

Further, if Q' satisfies in addition

(2) $(Q')_{S_1}=(Q')_S$, and

(3) $|h(S, Q')-h(S, Q_i)|<\delta_i$,

then there is a partition $\bar{B}_i$ such that $(S, \bar{B}_i)$ is Bernoulli, $(\bar{B}_i)_S\perp(Q')_S$, and $(\bar{B}_i\vee Q')_S \supset_{2\varepsilon_{n_i}} P$. (We are using lemma 4 here.)

We then let $Q'=\lim_{i\to\infty} Q_i$ and conclude from the preceding conditions that $(Q')_{S_1}=(Q')_S$, $|Q'-Q|<\varepsilon$, and $|h(S, Q')-h(S, Q)|<\varepsilon$. We also conclude that Q' satisfies conditions (1), (2) and (3). Using the sequence of processes $(S, \bar{B}_i\vee Q')$ as described above, we apply theorems 3 and 4 of [**1**] to conclude that $(Q')_S$ has Bernoulli complement in $(S, (P)_S)=(S, X)$. □

Proof of theorem 1. Suppose that (S, X) has the weak Pinsker property. Let $(Q)_S$ be a factor of (S, X). Proposition 1 applied to the direct factors $(B_n)_S$ and $(H_n)_S$ given by the hypothesis yield factors $(\bar{B}_n)_S$ and $(\bar{H}_n)_S$ of $(Q)_S$ satisfying the hypotheses of proposition 2. Consequently, proposition 2 yields a factor $(H'_1)_S\subset(Q)_S$ of arbitrarily small entropy (say $h(S, H'_1)<\frac{1}{2}$) with a Bernoulli complement (S, B'_1) in (S, Q). Repeating this argument with (S, H'_1) in place of (S, Q) yields a factor (S, H'_2) of (S, H'_1) with $h(S, H'_2)<\frac{1}{4}$, having a Bernoulli complement (S, B'_2) in (S, H'_1) and hence a Bernoulli complement $(S, B'_1\vee B'_2)$ in (S, Q). Continuing in this manner we exhibit the weak Pinsker property for $(S, (Q)_S)$. □

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REFERENCES

[1] A. Fieldsteel. The relative isomorphism theorem for Bernoulli flows. *Israel J. Math.*, **40** (1981), 197–216.

[2] D. S. Ornstein. *Ergodic Theory, Randomness, and Dynamical Systems.* Yale University Press: New Haven and London, 1974.

[3] J.-P. Thouvenot. Quelques propriétés des systèmes dynamique qui se décomposent en un produit de deux systèmes dout l'un est un schème de Bernoulli. *Israel J. Math.*, **21** (1975), 177–203.

[4] J.-P. Thouvenot. Remarques sur les systèmes dynamiques donnes avec plusieurs facteurs. *Israel J. Math.*, **21** (1975), 215–230.

[5] J.-P. Thouvenot. On the stability of the weaker Pinsker property. *Israel J. Math.*, **27** (1977), 150–162.