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Compact subgroups in the centralizer of natural factors of an ergodic group extension of a rotation determine all factors

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Abstract For ergodic group extensions of transformations with discrete spectra it is proved that each invariant sub-σ-algebra is determined by a compact subgroup in the centralizer of a natural factor

0 *Introduction*

In [**5**] the set of ergodic measures for compact abelian group extensions of a given transformation was described In the present paper, in a sense, we go further and we study the set of ergodic self-joinings of ergodic group extensions of transformations with discrete spectra These joinings turn out to be natural, namely, every ergodic self-joining of an ergodic compact, abelian group extension $T_\varphi \ (X\times G, \tilde{\mu})\to(X\times G, \tilde{\mu})$ of a transformation with a discrete spectrum $T \ (X,\mu)\to(X,\mu)$ must be the relatively independent extension of an isomorphism between some two natural factors of T_φ (by a natural factor of a G-extension T_φ we mean the action of T_φ on the quotient space $X\times G/H$, for H a closed subgroup of G)

In [**11**] (see also [**3**], [**4**]) Veech proved that for any ergodic transformation U with the 2-fold simplicity property (we use the definition of 2-fold simplicity from [**4**]), there was a one-to-one correspondence between invariant sub-σ-algebras and compact subgroups in the centralizer $C(U)$ of U This Veech correspondence is given by

$$\mathscr{C}\leftrightarrow H(\mathscr{C})=\{S\in C(U) \ (\forall A\in\mathscr{C})S^{-1}A=A\}$$

for each U-invariant sub-σ-algebra $\mathscr{C}$

In this paper, using the structure of self-joinings, we prove that for any T_φ-invariant sub-σ-algebra of an ergodic group extension of a rotation there is a *compact* subgroup in the centralizer of a natural factor giving rise to the Veech correspondence

1 *Ergodic joinings of group extensions of transformations with discrete spectra*

Let T_i ($i=1, \quad, n$) be ergodic automorphisms of Lebesgue spaces $(X_i, \mathscr{B}_i, \mu_i)$, where μ_i is a T_i-invariant probability measure on a σ-algebra $\mathscr{B}_i$ of subsets of X_i

Definition 1 [**9, 4**] By an *n-joining* of $T_1, \ldots, T_n$ we mean any $T_1 \times \ldots \times T_n$-invariant measure λ on $\mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_n$ such that for each $i = 1, \ldots, n$ and each $A_i \in \mathcal{B}_i$

$$\lambda(X_1 \times \ldots \times X_{i-1} \times A_i \times X_{i+1} \times \ldots \times X_n) = \mu_i(A_i)$$

The set of all n-joinings of $T_1, \ldots, T_n$ will be denoted by $J(T_1, \ldots, T_n)$ The subset of $J(T_1, \ldots, T_n)$ consisting of all ergodic measures will be denoted by $J^e(T_1, \ldots, T_n)$ It is clear that if $\lambda \in J(T_1, \ldots, T_n)$ and

$$\lambda = \int_{E(T_1 \ldots T_n)} e \, d\tau(e)$$

is its ergodic decomposition with $E(T_1, \ldots, T_n)$ being the set of all ergodic measures on $\mathcal{B}_1 \otimes \ldots \otimes \mathcal{B}_n$, then

$$\tau(J^e(T_1, \ldots, T_n)) = 1$$

Hence, we can say that the ergodic components of an n-joining are n-joinings In particular, $J^e(T_1, \ldots, T_n)$ is nonempty since $\mu_1 \times \cdot \ldots \times \mu_n \in J(T_1, \ldots, T_n)$

If $n = 2$ we say (for short) *joinings* (instead of 2-joinings) If $T_1 = T_2 = \ldots = T_n = T$ we say *n-self-joinings* of T

Let $T\ (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic automorphism By the *centralizer*, $C(T)$, of T we mean the set of all $S\ (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ commuting with T, i e $ST = TS$ This set is endowed with the weak topology given by $S_n \to S$ iff for each $A \in \mathcal{B}$, $\mu(S_n^{-1} A \,\Delta S^{-1} A) \to 0$ For any $S \in C(T)$ we can define the corresponding *graph joining* μ_S defined on rectangles as

$$\mu_S(A \times B) = \mu(A \cap S^{-1} B) \tag{1}$$

The following characterization of graph joinings can be easily proved

LEMMA 1 *Let $\lambda \in J^e(T, T)$ Then λ is a graph joining iff for any $A \in \mathcal{B}$ there is $\mathcal{B} \in \mathcal{B}$ with $\lambda(A \times X \Delta X \times B) = 0$, i e λ identifies the two marginals sub-σ-algebras of $\mathcal{B} \otimes \mathcal{B}$*

If $S_1, \ldots, S_{n-1} \in C(T)$ then the measure defined as

$$\mu_{S_1 \ldots, S_{n-1}}(A_0 \times A_1 \times \ldots \times A_{n-1}) = \mu(A_0 \cap S_1^{-1} A_1 \cap \ldots \cap S_{n-1}^{-1} A_{n-1})$$

is an element of

$$J^e(\underbrace{T, \ldots, T}_{n \times})$$

Any T-invariant sub-σ-algebra $\ell \subset \mathcal{B}$ is called a *factor* of T (more precisely, the action of T on ℓ is called a factor of T on $\mathcal{B}$) Assume, that two factors ℓ_1, ℓ_2 are isomorphic, i e there exists

$$S\ (T, X_1, \ell_1, \mu) \to (T, X_2, \ell_2, \mu),$$

where X_1, X_2 are the corresponding quotients We can lift this isomorphism to a self-joining λ of T by

$$\lambda(A \times B) = \int_{X_1} E(A \mid \ell_1)(\bar{x}) E(B \mid \ell_2)(S\bar{x}) \, d\mu(\bar{x}) \tag{2}$$

Such a joining is called the *relatively independent extension of the isomorphism S* Note that λ need not be ergodic. In particular, when $\ell = \ell_1 = \ell_2$ and $S = \mathrm{id}$, λ is called the *relatively independent extension of the diagonal measure on* ℓ This joining also need not be ergodic

From now on we assume that $T\ (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an ergodic transformation with discrete spectrum, i e $L^2(X, \mathcal{B}, \mu) = \operatorname{span}\{f_\alpha\ \ \alpha \in \operatorname{Sp}(T),\ f_\alpha \circ T = \alpha f_\alpha\}$, where $Sp(T)$ is the point spectrum of the unitary operator $T\ L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$, $Tf = f \circ T$

PROPOSITION 1 *Let* $T \cdot (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ *be an ergodic transformation with discrete spectrum Then*

(i) $C(T)$ *is a group, and*
(ii) $J^e(T, T) = \{\mu_S\ \ S \in C(T)\}$

Proof Obviously (i) follows from (ii) Take $\lambda \in J^e(T, T)$ We will show that λ identifies two marginal sub-σ-algebras $\tilde{\mathcal{B}}_1 = \{A \times X\ \ A \in \mathcal{B}\}$, $\tilde{\tilde{\mathcal{B}}}_2 = \{X \times B\ \ B \in \mathcal{B}\}$ To this end let us look at $L^2(X \times X, \mathcal{B} \otimes \mathcal{B}, \lambda)$ and the corresponding marginal subspaces

$$L^2(\tilde{\mathcal{B}}_1) = \{\tilde{f}\ \ \tilde{f}(x, y) = f(x), f \in L^2(X, \mu)\},$$
$$L^2(\tilde{\tilde{\mathcal{B}}}_2) = \{\tilde{\tilde{f}}\ \ \tilde{\tilde{f}}(x, y) = f(y), f \in L^2(X, \mu)\}$$

Since $\lambda \in J(T, T)$, both $L^2(\tilde{\mathcal{B}}_1)$ and $L^2(\tilde{\tilde{\mathcal{B}}}_2)$ are naturally identified with $L^2(X, \mu)$ Therefore they are spanned by $\{\tilde{f}_\alpha\ \ \alpha \in \operatorname{Sp}(T)\}$, $\{\tilde{\tilde{f}}_\alpha\ \ \alpha \in \operatorname{Sp}(T)\}$ respectively But λ is ergodic, so

$$\tilde{\tilde{f}}_\alpha = a_\alpha\ \ \tilde{f}_\alpha, \quad a_\alpha \in \mathbb{C},$$

and consequently $L^2(\tilde{\mathcal{B}}_1) = L^2(\tilde{\tilde{\mathcal{B}}}_2)$ as two subspaces in $L^2(X \times X, \lambda)$ This is equivalent to saying that $\tilde{\mathcal{B}}_1$ and $\tilde{\tilde{\mathcal{B}}}_2$ are identified by λ An application of Lemma 1 gives the result

Remark The notion of graph joining (1) can be easily transferred to the case $J^e(T_1, T_2)$ where we consider isomorphism between T_1 and T_2. Lemma 1 still works and the proof of Proposition 1 gives rise to a new proof of the well-known result that if T_1 and T_2 are ergodic transformations with discrete spectrum and $\operatorname{Sp}(T_1) = \operatorname{Sp}(T_2)$ then they are isomorphic (actually each ergodic joining between T_1 and T_2 is the graph of an isomorphism)

As an immediate consequence of Proposition 1 we get

COROLLARY 1 *If* $T\ (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ *is an ergodic automorphism with discrete spectrum then*

$$J^e(\underbrace{T, \quad , T}_{n}) = \{\mu_{S_1 \quad S_{n-1}}\ S_1, \quad , S_{n-1} \in C(T)\}.$$

Let G be a compact metric abelian group equipped with a normalized Haar measure ν Let $\varphi\ X \to G$ be a measurable map Define

$$T_\varphi\ \ (X \times G, \mu \times \nu) \to (X \times G, \mu \times \nu),$$
$$T_\varphi(x, g) = (Tx, \varphi(x)g)$$

T_φ is called a *group extension* of T Following [**8**] T_φ is ergodic iff whenever $\alpha \in \hat{G}$ (i e α is a character of G) and a measurable

$$h \quad X \to S^1 = \{z \in \mathbb{C} \quad |z| = 1\} \text{ satisfy } h(Tx)h(x)^{-1} = \alpha(\varphi(x)), \tag{3}$$

then $\alpha = 1$

We will also use the following result

PROPOSITION 2 [7] *Let T_φ be an ergodic G-extension of T Let $\bar{S} \in C(T_\varphi)$ Then there are a continuous group epimorphism $v \quad G \to G$ a measurable map $f \quad X \to G$ and $S \in C(T)$ such that*

$$\bar{S}(x, y) = S_{f,v}(x, y) = (Sx, f(x)v(y)) \tag{4}$$

Let $H \subset G$ be a closed (compact) subgroup Then we can consider the action of T_φ on $X \times G/H$ The factors of this form are called *natural factors* In fact these are the only factors of T_φ that contain the σ-algebra $\{A \times G \quad A \in \mathscr{B}\}$ ([**4**], [**11**])

Our aim is to describe all ergodic self-joinings for an ergodic G-extension of T_φ Without loss of generality we can assume that T is an ergodic rotation on a compact monothetic group X First we will work with the situation where T_φ is not necessarily ergodic Take an ergodic component λ

Let $\Pi \quad X \times G \to X$, $\Pi(x, g) = x$ Then, if λ is an ergodic T_φ-invariant measure on $X \times G$ then $\lambda\Pi^{-1}$ is T-ergodic, hence $\lambda\Pi^{-1} = \mu$ We will also use the following straightforward result

LEMMA 2 *There is a measurable T_φ-invariant subset $Y \subset X \times G$, $\lambda(Y) = 1$ such that for each $(x, g) \in Y$ and for each continuous function f on $X \times G$ (i e $f \in C(X \times G)$)*

$$\lim_{n\to\infty} S_n(f)(x, g) \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ (T_\varphi)^k(x, g) = \int f \, d\lambda$$

Let us denote by H the stabilizer of λ in G, i e

$$H = \{g \in G \quad \lambda g = \lambda\},$$

where $\lambda g(A \times B) = \lambda(A \times Bg^{-1})$ or

$$\int f(x, g) \, d(\lambda g) = \int f(x, hg) \, d\lambda \quad \text{for } f \in C(X \times G)$$

Let us denote $f \circ g(x, h) = f(x, hg)$

LEMMA 3 (i) *H is a closed subgroup of G* (ii) *If $(x, g), (x, h) \in y$ then $hH = gH$*

Proof As (i) is obvious, we will prove (ii) Take $f \in C(X \times G)$ Then $(x, g) \in Y$ implies $S_n(f)(x, g) \to_n \int f \, d\lambda$ But

$$S_n(f)(x, g) = S_n(f)(x, hh^{-1}g) = S_n(f \circ h^{-1}g)(x, h)$$

since the action of G on the second coordinate commutes with T_φ But from our assumption, $(x, h) \in Y$ so

$$S_n(f \circ h^{-1}g)(x, h) \xrightarrow[n\to x]{} \int f \circ h^{-1}g \, d\lambda = \int f \, d(\lambda h^{-1}g)$$

Because f is an arbitrary element of $C(X \times G)$, $\lambda h^{-1}g = \lambda$, or, similarly, $h^{-1}g \in H$

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Let us decompose λ over the factor (X, T, μ)

$$\lambda = \int_X \lambda_x \, d\mu(x)$$

LEMMA 4 $\lambda_x = \nu_H g$ μ-a e , *where* ν_H *is Haar measure on* H, $g = g(x)$ *and* $(x, g) \in Y$

Proof Let A be a Borel subset of G, $h \in H$ Let

$$M = \{x \in X \ \ \lambda_x(Ah^{-1}) < \lambda_x(A)\}$$

and suppose that $\mu(M) > 0$ Then

$$\lambda(M \times A) = \lambda h(M \times A) = \lambda(M \times Ah^{-1}) = \int_M \lambda_x(Ah^{-1})\, d\mu(x)$$

$$< \int_M \lambda_x(A)\, d\mu(x) = \lambda(M \times A),$$

a contradiction Similarly we show that $\mu\{x \in X \ \ \lambda_x(Ah^{-1}) > \lambda_x(A)\} = 0$ As a conclusion we have $\lambda_x h = \lambda_x$ μ-a e Let $(x, g) \in Y$ From Lemma 3(ii) it follows that

$$Y \cap (\{x\} \times G) = \{x\} \times gH$$

Hence $\lambda_x(gH) = 1$ This implies $\lambda_x g^{-1}(H) = 1$ But for $h \in H$

$$(\lambda_x g^{-1})h = (\lambda_x h)g^{-1} = \lambda_x g^{-1}$$

Thus $\lambda_x g^{-1}$ is invariant under all translations by elements of H and therefore $\lambda_x g^{-1} = \nu_H$

Remark Lemma 4 implies that if we denote by $\tilde{\lambda} = \int_X \tilde{\lambda}_x \, d\mu(x)$ the image of λ on $X \times G/H$ then $\tilde{\lambda}_x$ is a Dirac measure μ-a e This allows us to define a measurable function f $X \to G/H$ by

$$f(x) = (\tilde{\lambda}_x)^{-1}(1), \tag{5}$$

i e $f(x)$ is the only atom of $\tilde{\lambda}_x$ on G/H (f is measurable since $f^{-1}(A) = E(\chi_{X \times A} | \mathcal{B})^{-1}(1)$, for any Borel subset $A \subset G/H$) Moreover, the T_φ-invariance of λ implies

$$f(Tx) = \varphi(x)f(x) \tag{6}$$

LEMMA 5 *The system* $(X \times G, T_\varphi, \lambda)$ *is isomorphic to* $(X \times H, T_\psi, \mu \times \nu_H)$ *where* ψ $X \to H$ *is measurable* (*i e ergodic* T_φ-*invariant measures induce ergodic group extensions automorphisms*)

Proof Define t $X \to G$ by the formula $t(x) = U(f(x))$, where U is a measurable selector for the natural map $G \to G/H$ (see [**10**], p 5), i e U satisfies $U(gH)H = gH$ Then $t(x) \in f(x)$ μ-a e and, by (6), $\varphi(x)t(x)H = t(Tx)H$ Put

$$\psi(x) = \varphi(x)t(Tx)^{-1}t(x) \in H \tag{7}$$

Therefore from Lemma 4 it follows that

$$j \ \ (X \times H, T_\psi, \mu \times \nu_H) \to (X \times G, T_\varphi, \lambda)$$

acting as $j(x, h) = (x, t(x)h)$ is an isomorphism

Let

$$\Gamma = \{\gamma \in \hat{G} \ \text{ there is a measurable } h \ \ X \to S^1 \text{ such that}$$
$$h(Tx)h(x)^{-1} = \gamma(\varphi(x)) \ \mu\text{-a e}\}$$

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Then Γ is a subgroup of $\hat{G}$ and put

$$F = \operatorname{ann} \Gamma = \{g \in G \text{ for each } \gamma \in \Gamma,\ \gamma(g) = 1\}$$

LEMMA 6 $F = H$

Proof Let $g_0 \in H$, $\gamma \in \Gamma$ Then $\gamma(\varphi(x)) = h(Tx)h(x)^{-1}$ and let us define a function w $X \times G \to S^1$ setting

$$w(x, g) = h(x)^{-1}\gamma(g)$$

Then for μ-a e x and for *all* g, w is T_φ-invariant The ergodicity of λ forces w to be constant λ-a.e , i e $h(x)^{-1}\gamma(g) = c \neq 0$ Moreover

$$c = \int w(x, g)\, d\ell = \int h(x)^{-1}\gamma(g)\, d\lambda = \int h(x)^{-1}\gamma(g)\, d(\lambda g_0)$$

$$= \int h(x)^{-1}\gamma(gg_0)\, d\lambda = \gamma(g_0)c$$

Hence $\gamma(g_0) = 1$ and therefore $g_0 \in F$ Now, let $g \in F$ If $g \notin H$ then there is a character γ such that

$$\gamma(g) \neq 1 \quad \text{and} \quad \gamma(H) = 1$$

From (7) it follows that

$$\gamma(\varphi(x)) = \gamma(t(Tx)\psi(x)t(x)^{-1}) = \gamma(t(Tx))\gamma(t(x))^{-1},$$

since $\psi(x) \in H$ This implies $\gamma \in \Gamma$ and consequently $\gamma(g) = 1$ which is a contradiction

Remark. The results contained in Lemmata 3-6 can be deduced from [**5**] We include these results for completeness as well as for new and simple proofs

Now, we are in a position to pass to our main problem, namely, to describe all ergodic self-joinings of T_φ We assume that

T_φ is an ergodic G-extension

Let Π $X \times G \times X \times G \to X \times X$ be defined as

$$\Pi(x, g, y, h) = (x, y)$$

Assume that $\bar{\lambda} \in J^e(T_\varphi, T_\varphi)$ Then by Proposition 1,

$$\bar{\lambda}\Pi^{-1} = \mu_S$$

for some $S \in C(T)$ Hence

LEMMA 7

$$\bar{\lambda}\left(\bigcup_{x \in X} \{x\} \times G \times \{Sx\} \times G\right) = 1$$

We define a measure λ on $X \times G \times G$ as follows

$$\lambda(A \times B \times C) = \bar{\lambda}(A \times B \times SA \times C)$$

Put

$$\alpha \quad \bigcup_{x \in X} \{x\} \times G \times \{Sx\} \times G \to X \times G \times G,$$

$$\alpha(x, g, Sx, h) = (x, g, h)$$

Then we see that λ is just the image of $\bar{\lambda}$ via α Also $T_{\varphi\times\varphi\circ S}\circ\alpha=\alpha\circ(T_\varphi\times T_\varphi)$ Therefore the Lemma below is clear

LEMMA 8 *The function α is an isomorphism of $(X\times G\times X\times G,\ T_\varphi\times T_\varphi,\ \bar{\lambda})$ and $(X\times G\times G,\ T_{\varphi\times\varphi\circ S},\lambda)$*

In what follows we will consider $T_{\varphi\times\varphi\circ S}$ and the measure λ on $X\times G\times G$ Let $H\subset G\times G$, $H=\{(g_1,g_2)\in G\times G\ \ \lambda(g_1,g_2)=\lambda\}$,

$$H_1,H_2\subset G,\quad H_1=\{g\in G\ \ (g,e)\in H\},\quad H_2=\{g\in G\ \ (e,g)\in H\},$$

where e is the unit element of G

Then, obviously, H_1,H_2 are closed subgroups of H If we put $\Gamma=\{(\gamma_1,\gamma_2)\in\hat{G}\times\hat{G}$ there is a measurable function $h\ \ X\to S^1$ such that $\gamma_1(\varphi(x))\gamma_2(\varphi(Sx))=h(Tx)h(x)^{-1}$ then from Lemma 6, $H=\operatorname{ann}\Gamma$ and therefore

$$H_i=\operatorname{ann}\Gamma_i,\quad i=1,2,\tag{8}$$

where $\Gamma_i=\Pi_i(\Gamma)$, $\Pi_i\ \ \hat{G}\times\hat{G}\to\hat{G}$, $\Pi_i(\gamma_1,\gamma_2)=\gamma_i$

LEMMA 9 *Γ is a 'diagonal' subgroup of $\hat{G}\times\hat{G}$, i e ,*

$$(\gamma_1,\gamma_2)\in\Gamma,\ (\gamma_1,\gamma_2')\in\Gamma\quad\text{imply } \gamma_2=\gamma_2',$$
$$(\gamma_1,\gamma_2)\in\Gamma,\ (\gamma_1',\gamma_2)\in\Gamma\quad\text{imply } \gamma_1=\gamma_1'$$

Proof This is an obvious consequence of ergodicity of T_φ and (3)

LEMMA 10 *There is a group isomorphism $\hat{w}\ \ \Gamma_2\to\Gamma_1$*

Proof This follows from Lemma 9 (as Γ is a subgroup of $\hat{G}\times\hat{G}$) that

$$\hat{w}(\gamma_2)=\gamma_1\quad\text{iff}\quad(\gamma_1,\gamma_2)\in\Gamma$$

is a well-defined group isomorphism of Γ_1 and Γ_2

Let $w\ \ G/H_1\to G/H_2$ be the group isomorphism determined by

$$\hat{w}\ \ (G/H_2)^\wedge\to(G/H_1)^\wedge,$$
$$\hat{w}(\gamma_2)=\gamma_2 w,$$

where $(G/H_i)^\wedge$ is naturally identified with Γ_i as $\operatorname{ann}\Gamma_i=H_i$, $i=1,2$

LEMMA 11

$$H=\bigcup_{g\in G} gH_1\times w(g^{-1}H_1)$$

Proof Let $g\in G$, $(\hat{w}(\gamma_2),\gamma_2)\in\Gamma$ Then

$$\begin{aligned}(\hat{w}(\gamma_2),\gamma_2)(gH_1\times w(g^{-1}H_1))&=\hat{w}(\gamma_2)(gH_1)\ \ \gamma_2(w(g^{-1}H_1))\\&=\hat{w}(\gamma_2)(gH_1)\ \ \hat{w}(\gamma_2)(g^{-1}H_1)\\&=\hat{w}(\gamma_2)(H_1)=1,\end{aligned}$$

since (8) holds Therefore $gH_1\times w(g^{-1}H_1)\subset H$

Now, let $(g,h)\in H$ We wish to show that

$$hH_2\ \ w(gH_1)=H_2\tag{9}$$

Indeed, let $\gamma_2 \in \Gamma_2$,

$$\gamma_2(hH_2 \; w(gH_1)) = \gamma_2(w(gH_1)) \; \gamma_2(hH_2) = \hat{w}(\gamma_2)(gH_1) \; \gamma_2(hH_2)$$
$$= (\hat{w}(\gamma_2), \gamma_2)(gH_1 \times hH_2) = 1$$

Now, from (9), $w(gH_1)^{-1} = hH_2$, so $h \in w(g^{-1}H_1)$

Let $\hat{p} \; \Gamma_2 \to \Gamma$ be defined by

$$\hat{p}(\gamma_2) = (\hat{w}(\gamma_2), \gamma_2) \tag{10}$$

and let $p \; (G \times G)/H \to G/H_2$ be determined by

$$\hat{p}(\gamma_2) = \gamma_2 \circ p \tag{11}$$

Put $\bar{f} \; X \to G/H_2$

$$\bar{f} = p \circ f, \tag{12}$$

where f is defined by (5)

Let $v \; G/H_1 \to G/H_2$ be the topological group isomorphism defined by

$$v(gH_1) = w(g^{-1}H_1) \tag{13}$$

Finally, let us define $S_{\bar{f} v} \; X \times G/H_1 \to X \times G/H_2$ setting

$$S_{\bar{f},v}(x, gH_1) = (Sx, \bar{f}(x)v(gH_1)) \tag{14}$$

LEMMA 12 *The map* $S_{\bar{f},v}$ *establishes an isomorphism of the natural factors* $(X \times G/H_1, T_\varphi, \mu \times \nu)$ *and* $(X \times G/H_2, T_\varphi, \mu \times \nu)$

Proof It is sufficient to show that $S_{\bar{f},v} \circ T_\varphi = T_\varphi \circ S_{\bar{f},v}$ This is equivalent to proving the equality

$$\bar{f}(Tx)\bar{f}(x)^{-1}v(\varphi(x)H_1)\varphi(Sx)^{-1}H_2 = H_2 \quad \mu\text{-a e} \tag{15}$$

Using (6) and (12), (15) can be reduced to showing that

$$p((\varphi(x), \varphi(Sx))H)v(\varphi(x)H_1)\varphi(Sx)^{-1}H_2 = H_2$$

Take $\gamma \in \Gamma_2$ Then by (12), (10) and (13)

$$\gamma[p((\varphi(x), \varphi(Sx))H) \; v(\varphi(x)H_1)\varphi(Sx)^{-1}H_2]$$
$$= \gamma \circ p((\varphi(x), \varphi(Sx))H) \; \gamma(v(\varphi(x)H_1)) \; \gamma((\varphi(Sx)^{-1})H_2)$$
$$= \hat{p}(\gamma)((\varphi(x), \varphi(Sx))H) \; \gamma(v(\varphi(x)H_1)) \; \gamma((\varphi(Sx)H_2)^{-1})$$
$$= \hat{w}(\gamma)(\varphi(x)H_1) \; \gamma(\varphi(Sx)H_2) \; \gamma(v(\varphi(x)H_1)) \; \gamma(\varphi(Sx)H_2)^{-1} = 1$$

THEOREM 1 *If* $T_\varphi \; (X \times G, \mu \times \nu) \to (X \times G, \mu \times \nu)$ *is an ergodic group extension of a transformation with discrete spectrum and* $\bar{\lambda} \in J^e(T_\varphi, T_\varphi)$ *then there exist closed subgroups* $H_1 \subset G$, $H_2 \subset G$ *and an isomorphism of the corresponding natural factors* $\bar{S} \; (X \times G/H_1, T_\varphi) \to (X \times G/H_2, T_\varphi)$ *such that for any Borel sets* $A \subset X \times G$, $B \subset X \times G$

$$\bar{\lambda}(A \times B) = \int_{X \times G/H_1} E(A|H_1)(x, gH_1) \quad E(B|H_2)(\bar{S}(x, gH_1)) \, d(\mu \times \nu)(x, gH_1),$$

where $E(A|H_i)$ *denotes the conditional expectation with respect to the natural factor* $(X \times G/H_i, \mu \times \nu)$, $i = 1, 2$ *(i e* $\bar{\lambda}$ *is the relatively independent extension of an isomorphism of two natural factors)*

Proof It follows from Lemmata 4, 11, 12, since

$$\nu_H = \int_{G/H_1} \nu_{H_1} \quad gH_1 \times \nu_{H_2} \quad v(gH_1)\, d\nu(gH_1)$$

Although we have dealt with the case $\bar{\lambda} \in J^e(T_\varphi, T_\varphi)$, all Lemmata 7–12 go through when $\bar{\lambda} \in J^e(T_{\varphi_1}, T_{\varphi_2})$ where $\varphi_i \quad X \to G_1, \varphi_2 \quad X \to G_2, T_{\varphi_1}, T_{\varphi_2}$ are ergodic This proves the following

THEOREM 2 *Let $T_{\varphi_i} \quad (X \times G_i, \mu \times \nu_i) \to (X \times G_i, \mu \times \nu_i)$ be an ergodic group extension of T, $i = 1, 2$ If $\bar{\lambda} \in J^e(T_{\varphi_1}, T_{\varphi_2})$ then there exist two closed subgroups $H_i \subset G_i$ and an isomorphism of the natural factors*

$$\bar{S} \quad (X \times G_1/H_1, T_{\varphi_1}, \mu \times \nu_1) \to (X \times G_2/H_2, T_{\varphi_2}, \mu \times \nu_2)$$

such that for any Borel sets $A \subset X \times G_1$, $B \subset X \times G_2$

$$\bar{\lambda}(A \times B) = \int_{X \times G_1/H_1} E(A|H_1)(x, gH_1) \quad E(B|H_2)(\bar{S}(x, gH_1))\, d(\mu \times \nu_1)(x, gH_1)$$

Remark. A combination of Lemma 5 and Theorem 2 allows us to describe all ergodic n-joinings of ergodic extensions $T_{\varphi_1}, \quad, T_{\varphi_n}$ Indeed, let $\lambda \in J^e(T_{\varphi_1}, \quad, T_{\varphi_n})$ Then the measure $\bar{\lambda}$ given by

$$\bar{\lambda}(A_1 \times A_2 \times \quad \times A_{n-1}) = \lambda(A_1 \times \quad \times A_{n-1} \times (X \times G_n))$$

is an ergodic $n-1$-joining of $T_{\varphi_1}, \quad, T_{\varphi_{n-1}}$ From Lemma 5 it follows that $(T_{\varphi_1} \times \quad \times T_{\varphi_{n-1}}, \bar{\lambda})$ is isomorphic to some H-extension of T Therefore λ is an ergodic joining of this H-extension of T and T_{φ_n} Then we apply Theorem 2

Remark The result of Theorem 2 can be generalized as follows Let $T \quad (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be an ergodic (not necessarily with a discrete spectrum) transformation of a Lebesgue space, $\varphi \quad X \to G$ an ergodic cocycle If $\bar{\lambda} \in J^e(T_\varphi, T_\varphi)$ projected on $J^e(T, T)$ is the graph joining of an $S \in C(T)$ then $\bar{\lambda}$ must satisfy the conclusion of Theorem 1

2 *Structure of factors of group extensions of transformations with discrete spectra (Veech theorem)*

Let $T_\varphi \quad (X \times G, \tilde{\mathcal{B}}, \tilde{\mu}) \to (X \times G, \tilde{\mathcal{B}}, \tilde{\mu})$ be an ergodic group extension of a transformation with discrete spectrum $T \quad (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$, $\tilde{\mu} = \mu \times \nu_G$ and $\tilde{\mathcal{B}}$ the corresponding product σ-algebra For each closed subgroup $H \subset G$ we have a natural factor $T_\varphi \quad (X \times G/H, \tilde{\mu}) \to (X \times G/H, \tilde{\mu})$. Let $C_1(T_\varphi, H)$ denote the group of all invertible elements of the centralizer of T_φ on $(X \times G/H, \tilde{\mu})$ Assume that $\mathcal{H} \subset C_1(T_\varphi, H)$ is a subgroup Then this $\mathcal{H}$ determines a factor of T_φ on $X \times G/H$ (and hence a factor of T_φ on $X \times G$) by

$$\ell(\mathcal{H}) = \{A \in \tilde{\mathcal{B}} \quad \text{for each } \bar{S} \in \mathcal{H}, \bar{S}A = A\}$$

The point is that when we pass through all *compact* $\mathcal{H}$ for all possible closed $H \subset G$ we get all factors (Theorem 3)

Let $\ell \subset \tilde{\mathscr{B}}$ be a T_φ-invariant sub-σ-algebra Following (2) this ℓ gives rise to a self-joining of T_φ by

$$\tilde{\mu} \times_\ell \tilde{\mu}(A \times B) = \int_{\bar{X}} E(A|\ell)(\bar{x}) \quad E(B|\ell)(\bar{x})\, d\tilde{\mu}(\bar{x}),$$

where $\bar{X}$ is the quotient corresponding to ℓ

Put $\lambda = \tilde{\mu} \times_\ell \tilde{\mu}$ Since λ is not necessarily ergodic, let

$$\lambda = \int_{J^e(T_\varphi, T_\varphi)} e\, d\gamma(e) \tag{16}$$

be its ergodic decomposition, γ a probability measure on $J^e(T_\varphi, T_\varphi)$

The following lemma is well-known ([4], [9])

LEMMA 13 *Let A be a Borel subset of $X \times G$ Then $A \in \ell$ iff* $\lambda(A \times A^c \cup A^c \times A) = 0$

Let $E = \{e \in J^e(T_\varphi, T_\varphi)$ for each $A \in \ell$, $e(A \times A^c \cup A^c \times A) = 0\}$

LEMMA 14 $\gamma(E) = 1$

The proof of this is easy and is therefore omitted

LEMMA 15 *Let $e \in J^e(T_\varphi, T_\varphi)$ By Theorem* 1,

$$e = \int_{X \times G/H_1} E(\quad|H_1)(x, gH_1) \cdot E(\quad|H_2)(S_{f,v}(x, gH_1))\, d\tilde{\mu}(x, gH_1)$$

Then $e \in E$ iff $\ell \subset \tilde{\mathscr{B}}_{H_1 H_2}$ and for each $A \in \ell$, $S_{f,v}^{-1}(A) = A$

Proof We start with the following observation

$$\tilde{\mathscr{B}}_{H_1} \cap \tilde{\mathscr{B}}_{H_2} = \tilde{\mathscr{B}}_{H_1 H_2} \tag{17}$$

since $\tilde{\mathscr{B}}_J = \{A \in \tilde{\mathscr{B}}$ for each $g \in J, Ag = A\}$, $J \subset G$ closed Hence, the sufficiency easily follows Denote $\bar{S} = S_{f,v}$ Let $A \in \ell$, $e \in E$ Then

$$e(A \times A^c) = 0 \quad \text{and} \quad e(A^c \times A) = 0$$

The definition of e implies

$$\int_{X \times G/H_1} E(A|H_1) \quad E(A^c|H_2)\bar{S}\, d\tilde{\mu}_{H_1} = 0, \tag{18}$$

$$\int_{X \times G/H_1} E(A^c|H_1) \cdot E(A|H_2)\bar{S}\, d\tilde{\mu}_{H_1} = 0 \tag{19}$$

Assume that $E(A|H_1)(x, gH_1) \neq 0, 1$ Hence, by (18) $E(A^c|H_2) \circ \bar{S}(x, gH_1) = 0$, i.e $E(A|H_2) \circ \bar{S}(x, gH_1) = 1$ It follows that $\bar{S}(x, gH_1) \subset A$ But, from (19) $E(A|H_2) \circ \bar{S}(x, gH_1) = 0$ (since $E(A^c|H_1)(x, gH_1) \neq 0, 1$), a contradiction We conclude that for $\tilde{\mu}_{H_1}$-a.e (x, gH_1), $E(A|H_1)(x, gH_1) = 0$ or 1, i e $A \in \tilde{\mathscr{B}}_{H_1}$ Suppose that $A \notin \tilde{\mathscr{B}}_{H_2}$ Then for a set of positive $\tilde{\mu}_{H_1}$ measure

$$E(A|H_2) \circ \bar{S}(x, gH_1) \neq 0, 1,$$

and

$$E(A^c|H_2) \circ \bar{S}(x, gH_2) \neq 0, 1$$

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However this implies (see (18), (19)) that either $(x, gH_1) \subset A$ or $(x, gH_1) \subset A^c$ which is a contradiction Therefore, from (17), $A \in \tilde{\mathscr{B}}_{H_1 H_2}$ Moreover

$$0 = \int_{X \times G/H_1} \chi_A \ \chi_{A^c} \circ \bar{S}\, d\tilde{\mu} = \int_{X \times G/H_1} \chi_{A \cap \bar{S}^{-1} A^c}\, d\tilde{\mu}$$

forces $\bar{S}^{-1}A = A$ to hold This completes the proof

Let H be the largest closed subgroup of G such that

$$\ell \subset \tilde{\mathscr{B}}_H$$

Such a group exists, as we can take H as being the closure of the group generated by $\{H_1 \ \ell \subset \tilde{\mathscr{B}}_{H_1}\}$ Since the map $\tilde{f}\ G \to L^2(X \times G, \tilde{\mu})$ given by $\tilde{f}(g) = f \circ g$ where $f \circ g(x, h) = f(x, hg)$, $f \in L^2(X \times G, \tilde{\mu})$ is continuous, $\ell \subset \tilde{\mathscr{B}}_H$ In other words, there exists a smallest natural factor of T_φ, containing ℓ We will consider *this* factor as a group extension for which ℓ is a factor

LEMMA 16 *If* $H(\ell) = \{\bar{S} \in C(T_\varphi, X \times G/H)$ *for each* $A \in \ell\, \bar{S}^{-1}A = A\}$ *then* $H(\ell) \subset C_1(T_\varphi, X \times G/H)$ (*i e all elements from the centralizer of* $T_\varphi\ (X \times G/H, \tilde{\mu}) \to (X \times G/H, \tilde{\mu})$ *which do not move any* $A \in \ell$ *are invertible*)

Proof If $\bar{S} \in H(\ell)$ is not invertible, so $\bar{S}^{-1}$ carries the whole σ-algebra $\tilde{\mathscr{B}}_H$ to a smaller sub-σ-algebra which is a natural factor of $T_\varphi\ (X \times G/H, \tilde{\mu}) \to (X \times G/H, \tilde{\mu})$ Hence, this factor is determined by a closed (nontrivial) subgroup F of G/H Then it is clear that $\ell \subset \tilde{\mathscr{B}}_{F_1}$ where F_1 is the inverse image of F under the natural map $G \to G/H$ If F is not trivial, $F_1 \not\supseteq H$ and we get a contradiction

By exactly the same arguments we can prove the following

LEMMA 17 *For each* $e \in E$ (*E considered for* T_φ, $T_\varphi\ (X \times G/H, \tilde{\mu}) \to (X \times G/H, \tilde{\mu})$), $e = (\tilde{\mu})_{\bar{S}}$ *and* $\bar{S}$ *is an invertible element of the centralizer of* T_φ, $T_\varphi\ (X \times G/H, \tilde{\mu}) \to (X \times G/H, \tilde{\mu})$

THEOREM 3 (Veech Theorem) *If* ℓ *is a* T_φ-*invariant sub-*σ-*algebra for an ergodic group extension* $T_\varphi\ (X \times G, \tilde{\mu}) \to (X \times G, \tilde{\mu})$ *of a transformation with discrete spectrum, then there exists a natural factor of* T_φ, $T_\varphi\ (X \times G/H, \tilde{\mu}_H) \to (X \times G/H, \tilde{\mu})$ *such that* $\ell = \{A \in \tilde{\mathscr{B}}_H$ *for each* $\bar{S} \in H(\ell), \bar{S}A = A\}$ *and* $H(\ell)$ *is a compact subgroup of the centralizer of* $T_\varphi\ (X \times G/H, \tilde{\mu}) \to (X \times G/H, \tilde{\mu})$

Proof This natural factor is taken as the smallest natural factor of T_φ which contains ℓ Then Lemmata 16 and 17 reduce our problem to the following for *this* natural factor the relatively independent extension of the diagonal measure on ℓ has the ergodic decomposition which consists of some invertible $\bar{S}$'s belonging to the centralizer of $T_\varphi\ (X \times G/H, \tilde{\mu}) \to (X \times G/H, \tilde{\mu})$ We are now in the situation of Theorem 1 8 2 from [4]

Remark. From Theorem 3 it follows that for each factor ℓ of an ergodic group extension $T_\varphi\ (X \times G, \tilde{\mathscr{B}}, \tilde{\mu}) \to (X \times G, \tilde{\mathscr{B}}, \tilde{\mu})$ we can pass from ℓ to $\tilde{\mathscr{B}}$ in two steps, each one of which is a group extension operation (the first not necessarily abelian)

Remark. Although, throughout the paper we have dealt with a discrete spectrum rotation $T\ (X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$, Theorem 3 is still valid if we replace T by a 2-fold

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simple transformation (see [4]), i e a transformation where besides graph joinings μ_S, $S \in C(T)$ we admit only $\mu \times \mu$ as a new ergodic self-joining of T

Example 1 Let T $(X, \mathscr{B}, \mu) \to (X, \mathscr{B}, \mu)$ be defined as $Tx = x + \alpha$, where $X = [0, 1)$ (mod 1), μ is the Lebesgue measure and α is irrational Let φ $X \to X$, $\varphi(x) = x$ Using the following classical result ([1])

For $m \in \mathbb{Z}$, $m \neq 0$, $b \in [0, 1)$ the cocycle

$$\psi(x) = mx + b \quad \text{is ergodic,} \tag{20}$$

one can easily compute the centralizer of T_φ as well as its natural factors $T_{m\varphi}$ $(m \in \mathbb{N})$

$$C(T_{m\varphi}) = \{S_{f,v} \qquad \},$$

$$f(Tx) - f(x) = m\varphi(Sx) - v\varphi(x),$$

$$f(x+\alpha) - f(x) = m(x+\beta) - smx = m(1-s)x + m\beta$$

Hence, from (20) $s = 1$ and using Anzai's result [1], $m\beta = m'\alpha$ for an integer m' Therefore the centralizer T_φ does not contain nontrivial compact subgroups, i e subgroups for which the projection on the first coordinate is different from {id} Consequently from $C(T_\varphi)$ we can read merely all natural factors, while for instance the transformation

$$U(x, y) = (x + 2\alpha, x + y)$$

is a factor of T_φ (via the map $(x, y) \mapsto (2x, 2y)$) However this factor can be read from the centralizer of $T_{2\varphi}$ as the group $\{0, \frac{1}{2}\}$ can be lifted to the centralizer of $T_{2\varphi}$

Remark. These circle extensions of some rotations are well-known to be coalescent (i e their centralizers are groups) However in [6] some new examples of ergodic circle extensions of rotations are constructed with the coalescence property being lost

Example 2 It would be interesting to know whether for ergodic group extensions the following formula holds

$$C((T_\varphi)^n) = C(T_\varphi), \quad n \geq 2 \tag{21}$$

It is not difficult to see that total ergodicity (i e

$$\lambda \in J^e((T_\varphi)^n, (T_\varphi)^n) \quad \text{for each natural } n) \tag{22}$$

of all $\lambda \in J^e(T_\varphi, T_\varphi)$ forces (21) to be true Indeed, let $\bar{S} \in C((T_\varphi)^n)$ Then take

$$\lambda = \frac{1}{n}(\tilde{\mu}_{\bar{S}} + \tilde{\mu}_{\bar{S}} \circ T_\varphi + \qquad + \tilde{\mu}_{\bar{S}} \circ (T_\varphi)^{n-1})$$

It is not hard to see that $\lambda \in J(T_\varphi, T_\varphi)$ is in fact ergodic Then from (22) it follows that $\lambda \in J^e((T_\varphi)^n, (T_\varphi)^n)$ and consequently $\lambda = \tilde{\mu}_{\bar{S}}$, i e $\bar{S} \in C(T_\varphi)$ Nevertheless (21) does not hold in general For instance for the examples from Example 1, $T_\varphi(x, y) = (x + \alpha, x + y)$, $C(T_\varphi) \neq C((T_\varphi)^2)$ as $\frac{1}{2}$ can be lifted to the centralizer of $(T_\varphi)^2(x, y) = (x + 2\alpha, 2x + \alpha + y)$

It is also interesting to ask whether there is any relation between two isomorphic sub-σ-algebras ℓ_1, ℓ_2 of a G-extension T_φ and the subgroups $H(\ell_1)$, $H(\ell_2)$ in the centralizers of the smallest natural factors containing these two sub-σ-algebras It will follow from Theorem 4 that the answer is positive and the corollary after this theorem says what this relation is

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Assume that U is an isomorphism of two invariant sub-σ-algebras ℓ_1, ℓ_2 of T_φ. Let $X\times G/H_1$ and $X\times G/H_2$ be the smallest natural factors of T_φ containing algebras ℓ_1 and ℓ_2, respectively, as factors.

THEOREM 4. *There exists an isomorphism* $\bar{S}: X\times G/H_1\to X\times G/H_2$ *satisfying* $\bar{S}|_{\ell_1}=U$.

Proof. The proof consists of two steps. First we will establish the following property of ergodic joinings of ℓ_1 and ℓ_2.

If ν is an ergodic joining of ℓ_1 and ℓ_2 then ν

$$\text{is the projection of some ergodic joining of } X\times G/H_1 \text{ and } X\times G/H_2. \tag{23}$$

Indeed, set $\hat{\nu}$ to be the relatively independent extension of ν to $(X\times G/H_1)\times(X\times G/H_2)$, i.e.

$$\hat{\nu}=\int_{\ell_1\otimes\ell_2} E(\cdot\,|\ell_1)(\bar{x})E(\cdot\,|\ell_2)(\bar{y})\,d\nu.$$

Obviously, $\hat{\nu}$ need not be ergodic. Let

$$\hat{\nu}=\int_{J^e(H_1,H_2)}\tau\,d\gamma(\tau)$$

be the ergodic decomposition of $\hat{\nu}$.

If Π_i is the projection of $X\times G/H_i$ onto the (quotient) Lebesgue space corresponding to ℓ_i, $i=1,2$, then

$$\nu=\hat{\nu}\circ(\Pi_1\times\Pi_2)=\int_{J^e(H_1,H_2)}\tau\circ(\Pi_1\times\Pi_2)\,d\gamma(\tau).$$

Ergodicity of ν yields that for γ-a.e. τ, $\tau\circ(\Pi_1\times\Pi_2)=\nu$. In particular, there exists an ergodic joining τ such that $\tau\circ(\Pi_1\times\Pi_2)=\nu$, and property (23) is proved.

To end the proof of Theorem 4, denote by $\tilde{\mu}_U$ the graph joining on $\ell_1\otimes\ell_2$, corresponding to the isomorphism U. By virtue of (23) there is a measure $\tau\in J^e(H_1,H_2)$ such that

$$\tilde{\mu}_U=\tau\circ(\Pi_1\times\Pi_2). \tag{24}$$

Take $A\in\ell_1$. Then, by definition of $\tilde{\mu}_U$,

$$\tilde{\mu}_U(A\times U(A^c))=\tilde{\mu}(A\cap U^{-1}U(A^c))=0.$$

On the other hand, using (24) we have

$$\tilde{\mu}_U(A\times U(A^c))=\tau\circ(\Pi_1\times\Pi_2)(A\times U(A^c))=\tau(A\times U(A^c)) \tag{25}$$

since $A\times U(A^c)\in\ell_1\otimes\ell_2$.

There are subgroups $\tilde{F}_i\subset G/H_i$, $i=1,2$, and isomorphism $\bar{S}: X\times G/F_1\to X\times G/F_2$ (where F_i is the subgroup of G, for which G/F_i is naturally isomorphic to $(G/H_i)/\tilde{F}_i$, $i=1,2$) satisfying

$$\tau=\int_{X\times G/F_1} E(\cdot\,|F_1)\cdot E(\cdot\,|F_2)\circ\bar{S}\,d\tilde{\mu}.$$

Therefore, by (24) and (25)

$$0=\int_{X\times G/F_1} E(A|F_1)(\bar{x})E(U(A^c)|F_2)\circ \bar{S}\,d\tilde{\mu}$$
$$=\int_{X\times G/F_1} E(A|F_1)(\bar{x})E(\bar{S}^{-1}U(A^c)|F_1)(\bar{x})\,d\tilde{\mu} \tag{26}$$

Similarly

$$0=\tilde{\mu}_U(A^c\times U(A))$$
$$=\int_{X\times G/F_1} E(A^c|F_1)(\bar{x})\quad E(\bar{S}^{-1}U(A)|F_1)(\bar{x})\,d\tilde{\mu} \tag{27}$$

Now, if $\tilde{F}_1$ is a nontrivial subgroup of X/H_1, i e $F_1 \not\supseteq H_1$, then for some set $A\in \ell_1$ the function $E(A|F_1)$ is not a characteristic function In other words, for a set of positive measure, $E(A|F_1)(\bar{x})\neq 0, 1$ By (26) and (27), for such an $\bar{x}$,

$$0=E(A|F_1)(\bar{x})\quad E(\bar{S}^{-1}U(A^c)|F_2)(\bar{x})$$

and

$$0=E(A^c|F_1)(\bar{x})\quad E(\bar{S}^{-1}U(A)|F_2)(\bar{x})$$

Using the same arguments as in the proof of Lemma 15, we obtain a contradiction Therefore $F_1=H_1$ Since U is an isomorphism, $F_2=H_2$ Thus $\bar{S}$ is an isomorphism of $X\times G/H_1$ and $X\times G/H_2$ and $\bar{S}|_{\ell_1}=U$

Corollary 2 *If ℓ_1, ℓ_2 are two isomorphic invariant sub-σ-algebras then there is an isomorphism $\bar{S}$ of the smallest natural factors $X\times G/H_1$ and $X\times G/H_2$ of T_φ containing ℓ_1, ℓ_2 respectively, such that $H(\ell_2)=\bar{S}H(\ell_1)\bar{S}^{-1}$* □

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