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Representations of Extended Affine Lie Algebras Coordinatized by Certain Quantum Tori

Dedicated to Professor Bruce Allison

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Abstract. An irreducible representation of the extended affine Lie algebra of type A_{n-1} coordinatized by a quantum torus of v variables is constructed by using the Fock space for the principal vertex operator realization of the affine Lie algebra $\widetilde{gl}_n$.

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0. Introduction

Extended affine Lie algebras are a higher-dimensional generalization of affine Kac–Moody Lie algebras introduced by [H-KT] (under the name of irreducible quasi-simple Lie algebras). They can be roughly described as complex Lie algebras which have a nondegenerate invariant form, a self-centralizing finite-dimensional ad-diagonalizable Abelian subalgebra (i.e., a Cartan subalgebra), a discrete irreducible root system, and ad-nilpotency of nonisotropic root spaces (see [AABGP], [BGK] and [ABGP] for more on basic structure theory). Toroidal Lie algebras, which are central extensions of $\dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm 1}, \cdots, t_{v-1}^{\pm 1}]$ ($\dot{\mathfrak{g}}$ is a finite-dimensional simple Lie algebra), are examples of extended affine Lie algebras studied by [F], [W], [MRY], [Y], [EF], [EM] and [BC], among others. There are many extended affine Lie algebras which allow not only the Laurent polynomial algebra $\mathbb{C}[t_0^{\pm 1}, \cdots, t_{v-1}^{\pm 1}]$ as coordinate algebras but also quantum tori, Jordan tori and the octonion torus as coordinate algebras depending on the type of Lie algebra (see [AABGP], [BGK], [BGKN], [AG] and [Yo]). For instance, extended affine Lie algebras of type A_{n-1} are tied up with the Lie algebra $gl_n(\mathbb{C}_Q)$, where $\mathbb{C}_Q$ is a quantum torus $\mathbb{C}_Q[t_0^{\pm 1}, \cdots, t_{v-1}^{\pm 1}]$ associated to a $v \times v$ matrix Q. Quantum tori defined as in [M] are a noncommutative analogue of Laurent polynomial algebras. To get an extended affine Lie algebra, one has to form an appropriate central extension of $\mathrm{gl}_n(\mathbb{C}_Q)$ and add certain outer derivations (just like one obtains an affine Kac–Moody Lie algebra from a loop algebra by forming a one-dimensional central extension and then adding the degree derivation). Representations for Lie algebras coordinatized by certain quantum tori have been studied by [JK], [BS] and [G] in some cases.

In this paper, we will use the underlying Fock space for the principal vertex operator representation of the affine Lie algebra

$$\widetilde{\mathrm{gl}}_{\mathrm{n}} = \mathrm{gl}_{\mathrm{n}}(\mathbb{C}[t_0, t_0^{-1}]) \oplus \mathbb{C}c_0 \oplus \mathbb{C}d_0$$

to construct a family of vertex operators associated with a given pair $(\mathbb{Z}^{\nu-1}, q)$, where q is a $(\nu-1)$-tuple of nonzero complex numbers. These vertex operators together with the Heisenberg algebra form a Lie algebra $\mathcal{V}(\mathbb{Z}^{\nu-1}, q)$. The case $\nu = 1$ is trivial as the resulting Lie algebra represents the affine Lie algebra $\widetilde{\mathrm{gl}}_{\mathrm{n}}$ itself. If $\nu \geqslant 2$ and $(\mathbb{Z}^{\nu-1}, q)$ is generic (see Section 3 for definition), by enlarging the Fock space, we obtain an irreducible representation of an extended affine Lie algebra of type A_{n-1} coordinatized by a quantum torus of ν variables. What it means to say the pair $(\mathbb{Z}^{\nu-1}, q)$ is generic is that one variable in $\mathbb{C}_Q$ has utmost control over the other variables. This assumption makes the lifting of the Lie algebra $\mathcal{V}(\mathbb{Z}^{\nu-1}, q)$ on the enlarged Fock space possible. A representation for such a Lie algebra of type A_1 with a quantum torus of 2 variables was given by [BS] in a different form.

We will consider a more general situation than was done in [G] for the homogeneous construction. The key point is to use the principal gradation on the associative matrix algebra $M_n(\mathbb{C})$ to have a principal realization for our extended affine Lie algebras coordinatized by quantum tori. This is nontrivial if the quantum torus is not commutative. The idea for our construction of vertex operators comes from [KKLW].

Throughout this paper, we denote the field of complex numbers, real numbers and the ring of integers by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$, respectively.

1. Basics

Motivated by the work [KKLW], we shall realize the $n \times n$ matrix algebra $M_n(\mathbb{C})$ as the quotient of a quantum torus. This will provide us with a nice basis for $M_n(\mathbb{C})$ under the principal gradation.

Let ν be a positive integer and $Q = (q_{ij})$ be a $\nu \times \nu$ matrix, where

$$q_{ij} \in \mathbb{C} \setminus \{0\},\ q_{ii} = 1,\ q_{ij} = q_{ji}^{-1}, \quad \text{for} \quad 0 \leqslant \mathrm{i}, \mathrm{j} \leqslant \nu - 1. \tag{1.1}$$

A quantum torus associated to Q (see [M]) is the unital associative $\mathbb{C}$-algebra $\mathbb{C}_Q[t_0^{\pm 1}, \cdots, t_{\nu-1}^{\pm 1}]$ (or, simply $\mathbb{C}_Q$) with generators $t_0^{\pm 1}, \cdots, t_{\nu-1}^{\pm 1}$ and relations

$$t_i t_i^{-1} = t_i^{-1} t_i = 1 \text{ and } t_i t_j = q_{ij} t_j t_i \tag{1.2}$$

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for $0 \leqslant i, j \leqslant \nu - 1$. Write $t^{\boldsymbol{a}} = t_0^{a_0} \cdots t_{\nu-1}^{a_{\nu-1}}$ for $\boldsymbol{a} = (a_0, \cdots, a_{\nu-1}) \in \mathbb{Z}^\nu$. Then

$$t^{\boldsymbol{a}} t^{\boldsymbol{b}} = \left(\prod_{0 \leqslant j \leqslant i \leqslant \nu-1} q_{ij}^{a_i b_j} \right) t^{\boldsymbol{a}+\boldsymbol{b}}, \tag{1.3}$$

where $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^\nu$, and $\mathbb{C}_Q = \sum_{\boldsymbol{a} \in \mathbb{Z}^\nu} \oplus \mathbb{C} t^{\boldsymbol{a}}$.

Note that if Q is a 1×1 matrix, then $\mathbb{C}_Q$ is just the algebra $\mathbb{C}[t_0, t_0^{-1}]$ of Laurent polynomials.

Let n be a positive integer and $n \geqslant 2$. Let $M_n(\mathbb{C})$ be the $n \times n$ matrix algebra and $L = gl_n(\mathbb{C}) = M_n(\mathbb{C})^-$ be the general linear Lie algebra over $\mathbb{C}$.

Consider the Lie algebra $gl_n(\mathbb{C}[t_0, t_0^{-1}])$. Define a central extension as follows:

$$\widehat{L} = \mathrm{gl_n}(\mathbb{C}[t_0, t_0^{-1}]) \oplus \mathbb{C} c_0 \tag{1.4}$$

with the Lie bracket

$$[x_1(t_0^{n_1}), x_2(t_0^{n_2})] = [x_1, x_2](t_0^{n_1+n_2}) + n_1 \delta_{n_1+n_2,0} \mathrm{tr}(x_1 x_2) c_0 \tag{1.5}$$

where $x_1, x_2 \in L, n_1, n_2 \in \mathbb{Z}$, c_0 is a central element of $\widehat{L}$, and tr denotes the matrix trace. We denote

$$\widetilde{L} = \widehat{L} \oplus \mathbb{C} d_0 \tag{1.6}$$

a semi-direct product of $\widehat{L}$ with the degree derivation $d_0 = t_0(\mathrm{d}/\mathrm{d}t_0)$. $\widehat{L}$ (or $\widetilde{L}$) is called the affinization of L.

We shall work with the principal realization of $\widehat{L}$ (or $\widetilde{L}$) based on the $\mathbb{Z}_n$-gradation of L.

Let $^- : \mathbb{Z} \to \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ be the quotient map. Let ε be an nth primitive root of unity. We shall fix ε throughout this paper. Next we shall realize the $n \times n$ matrix algebra as the quotient of a quantum torus. This way will give the motivation for the principal realization of $M_n(\mathbb{C})$.

Consider the quantum torus $\mathbb{C}_\xi[u_0^{\pm 1}, u_1^{\pm 1}]$, where $\xi = (\begin{smallmatrix} 1 & \varepsilon^{-1} \\ \varepsilon & 1 \end{smallmatrix})$). Define $T\colon \mathbb{C}_\xi \to \mathbb{C}$ to be a $\mathbb{C}$-linear function as

$$T(u_0^{a_0} u_1^{a_1}) = \begin{cases} n, & \text{if both } a_0, a_1 \in n\mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.7}$$

Then the form $(\cdot, \cdot)$ determined by $(x, y) = T(xy)$, for $x, y \in \mathbb{C}_\xi$, is a symmetric invariant form. The radical J of the form is the two-sided ideal of $\mathbb{C}_\xi$ generated by $u_0^n - 1$ and $u_1^n - 1$. Define

$$\mathcal{M}_n = \mathbb{C}_\xi / J \tag{1.8}$$

to be the quotient of $\mathbb{C}_\xi$ by J and identify u_0 and u_1 with their images in $\mathcal{M}_n$.

PROPOSITION 1.9. *$\mathcal{M}_n$ is a simple associative $\mathbb{C}$-algebra of dimension n^2. The induced form $(\cdot, \cdot)$ on $\mathcal{M}_n$ is a symmetric invariant nondegenerate $\mathbb{C}$-bilinear form.*

Proof. Note that $\mathcal{M}_n$ is spanned by $u_0^i u_1^j$, $1 \leqslant i, j \leqslant n$. Let $\mathcal{I}$ be an ideal of $\mathcal{M}_n$ and

$$\sum_{1 \leqslant i,j \leqslant n} a_{ij} u_0^i u_1^j = \sum_{j=1}^n f_j(u_0) u_1^j \in \mathcal{I},$$

where $f_j(u_0) = \sum_{i=1}^n a_{ij} u_0^i$, $a_{ij} \in \mathbb{C}$. We have

$$u_0^{-k} \left(\sum_{j=1}^n f_j(u_0) u_1^j \right) u_0^k = \sum_{j=1}^n \varepsilon^{jk} f_j(u_0) u_1^j \in \mathcal{I}$$

for $1 \leqslant k \leqslant n$. It follows that $f_j(u_0) u_1^j \in \mathcal{I}$ and so $f_j(u_0) \in \mathcal{I}$, for $1 \leqslant j \leqslant n$. Again,

$$u_1^k f_j(u_0) u_1^{-k} = \sum_{i=1}^n \varepsilon^{ik} a_{ij} u_0^i \in \mathcal{I}$$

implies that $a_{ij} u_0^i \in \mathcal{I}$ and so $a_{ij} \in \mathcal{I}$, for $1 \leqslant i, j \leqslant n$. Therefore $\mathcal{I} = \{0\}$ or $\mathcal{M}_n$.

The above procedure also shows that

$$\sum_{1 \leqslant i,j \leqslant n} a_{ij} u_0^i u_1^j = 0 \text{ if and only if } a_{ij} = 0, \text{ for } 1 \leqslant i, j \leqslant n.$$

Hence, $\{u_0^i u_1^j : 1 \leqslant i, j \leqslant n\}$ form a basis for $\mathcal{M}_n$.

The rest of the proof is obvious. □

Let E_{ij} be the $n \times n$ matrix which is 1 in the (i, j)-entry and 0 everywhere else. Let

$$E = E_{12} + \cdots + E_{n-1,n} + E_{n1} \quad \text{and} \quad F = \mathrm{diag}\{\varepsilon, \varepsilon^2, \cdots, \varepsilon^{\mathrm{n}}\}. \tag{1.10}$$

Clearly,

$$E^n = F^n = 1 \quad \text{and} \quad EF = \varepsilon FE. \tag{1.11}$$

We thus have

COROLLARY 1.12. *There is a unique algebra homomorphism* $\phi : \mathcal{M}_n \to M_n(\mathbb{C})$ *such that* $\phi(u_0) = F$ *and* $\phi(u_1) = E$. *Moreover,* ϕ *is an isomorphism with* $\mathrm{tr}(\phi(x)) = T(x)$, *for* $x \in \mathcal{M}_n$. *Therefore,*

$$M_n(\mathbb{C}) = \sum_{1 \leqslant i,j \leqslant n} \oplus \mathbb{C} F^i E^j. \tag{1.13}$$

Remark 1.14. Identifying $M_n(\mathbb{C})$ with $\mathcal{M}_n$ has been implicitly used in [KKLW] (see also [Ma]).

LEMMA 1.15 $M_n(\mathbb{C})$ *has the following* $\mathbb{Z}_n$*-gradation:*

$$M_n(\mathbb{C}) = \oplus_{\bar{j} \in \mathbb{Z}_n} M_n(\mathbb{C})_{(\bar{j})},$$

where $M_n(\mathbb{C})_{(\bar{j})} = \sum_{i=1}^n \oplus \mathbb{C} F^i E^j$, *for* $\bar{j} \in \mathbb{Z}_n$.

It is easy to see that the above gradation coincides with the 'principal gradation' given by $\deg E_{ij} = \bar{j} - \bar{i}$. This gradation on $M_n(\mathbb{C})$ is really needed later when we deal with the matrix Lie algebra with entries in a non-commutative quantum torus $\mathbb{C}_Q$.

Clearly, $L = gl_n(\mathbb{C}) = \oplus_{\bar{j}\in\mathbb{Z}_n} L_{(\bar{j})}$ is $\mathbb{Z}_n$-graded as well, where $L_{(\bar{j})} = \sum_{i=1}^{n} \oplus \mathbb{C}F^i E^j$. Note that the matrix A_i in Example 1 of [KKLW] is exactly $\sum_{j=1}^{n} F^i E^j$, for $1 \leqslant i \leqslant n-1$.

Set

$$L_p = \sum_{i=1}^{n} \sum_{j\in\mathbb{Z}} \oplus \mathbb{C}F^i E^j(t_0^j) \tag{1.16}$$

and form the one-dimensional central extension

$$\widehat{L}_p = L_p \oplus \mathbb{C}c_0 \tag{1.17}$$

with the Lie bracket

$$[x_1(t_0^{n_1}), x_2(t_0^{n_2})] = [x_1, x_2](t_0^{n_1+n_2}) + \frac{n_1}{n}\delta_{n_1+n_2,0}\mathrm{tr}(x_1x_2)c_0, \tag{1.18}$$

where $x_1, x_2 \in L, n_1, n_2 \in \mathbb{Z}$, c_0 is a central element of $\widehat{L}_p$. We denote

$$\widetilde{L}_p = \widehat{L} \oplus \mathbb{C}d_0. \tag{1.19}$$

Note that $E_{ij} \in L_{(\bar{j}-\bar{i})}$. The following result can be easily verified. Later in Proposition 3.10 we shall prove a more general result.

LEMMA 1.20. *The Lie algebra $\widetilde{L}$ is isomorphic to $\widetilde{L}_p$ and the isomorphism is given by*

$$E_{ij}(t_0^k) \longmapsto E_{ij}(t_0^{j-i+kn}) - \frac{i}{n}\delta_{ij}\delta_{k,0}c_0,$$
$$c_0 \longmapsto c_0, \quad d_0 \longmapsto \frac{1}{n}\left(d_0 + \sum_{i=1}^{n} iE_{ii}\right),$$

where $1 \leqslant i, j \leqslant n$, $k \in \mathbb{Z}$.

$\widehat{L}_p$ (or $\widetilde{L}_p$) is called the principal realization of $\widehat{L}$ (or $\widetilde{L}$). It has a principal subalgebra

$$\widehat{H} = \mathbb{C}c_0 \oplus \sum_{i\in\mathbb{Z}} \oplus \mathbb{C}E^i(t_0^i). \tag{1.21}$$

Define

$$\widehat{H}^{\pm} = \sum_{i\in\pm\mathbb{Z}_+} \oplus \mathbb{C}E^i(t_0^i), \tag{1.22}$$

where $\mathbb{Z}_+ = \{i \in \mathbb{Z} : i > 0\}$, and write $E(i) = E^i(t_0^i)$, for $i \in \mathbb{Z}$. Then

$$\widehat{H} = \widehat{H}^+ \oplus (\mathbb{C}c_0 \oplus \mathbb{C}E(0)) \oplus \widehat{H}^-$$

and

$$\mathfrak{s} = \widehat{H}^+ \oplus \mathbb{C}c_0 \oplus \widehat{H}^- \tag{1.23}$$

is a Heisenberg algebra. Let

$$S(\widehat{H}^-) = \mathbb{C}[E(i) : i \in -\mathbb{Z}_+] \tag{1.24}$$

denote the symmetric algebra of $\widehat{H}^-$, which is the algebra of polynomials in infinitely many variables $E(i), i \in -\mathbb{Z}_+$. Let $\widetilde{H} = \widehat{H} \oplus \mathbb{C}d_0$. $S(\widehat{H}^-)$ is an $\widetilde{H}$-module in which c_0 acts as 1, d_0 acts as the degree operator (i.e. $d_0E(i) = iE(i)$), $E(0)$ acts as a scalar. Then

$$[E(i), E(j)] = i\delta_{i+j,0}, \qquad [d_0, E(i)] = iE(i) \tag{1.25}$$

for $i, j \in \mathbb{Z}$.

We define

$$E(z) = \sum_{j\in\mathbb{Z}} E(j)z^{-j} \in (\mathrm{End}S(\widehat{H}^-))[[z, z^{-1}]]. \tag{1.26}$$

Finally, we set

$$\delta(z) = \sum_{j\in\mathbb{Z}} z^j \in \mathbb{C}[[z, z^{-1}]], \tag{1.27}$$

formally the Fourier expansion of the δ-function, and

$$(D\delta)(z) = D\delta(z) = \sum_{j\in\mathbb{Z}} jz^j, \tag{1.28}$$

where $D = z(\mathrm{d}/\mathrm{d}z)$.

2. Construction of Vertex Operators

Let (Λ, q) be a pair, where $q = (q_1, \cdots, q_N)$ is a fixed N-tuple of nonzero complex numbers and Λ is a sub-semigroup of $\mathbb{R}^N$ (i.e., a subset of $\mathbb{R}^N$ containing 0 and closed under addition). Write $q^{\boldsymbol{r}} = q_1^{r_1} \cdots q_N^{r_N}$ for $\mathbb{R} = (r_1, \cdots, r_N) \in^N$. We shall fix one choice for $\ln q_i$ such that $q^{\boldsymbol{r}} =^{\sum_{i=1}^N r_i \ln q_i}$ for all $\boldsymbol{r} \in \Lambda$.

Set $\Lambda_0 = \{\boldsymbol{r} \in \Lambda : q^{\boldsymbol{r}} = 1\}$.

ASSUMPTION 2.1. Given a pair (Λ, q), we always assume that

$$\{q^{\boldsymbol{r}} : \boldsymbol{r} \in \Lambda\} \cap \{\varepsilon^i : 1 \leqslant i \leqslant n\} = \{1\}.$$

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Remark 2.2. The above assumption is equivalent to saying that $q^{r} = \varepsilon^{i}$ if and only if $q^{r} = \varepsilon^{i} = 1$. Namely,
$q^{r} \neq \varepsilon^{i}$ if and only if $\bar{i} \neq 0$ and $\boldsymbol{r} \in \Lambda$, or $\bar{i} = 0$ but $\boldsymbol{r} \in \Lambda \setminus \Lambda_0$;
$q^{r} = \varepsilon^{i}$ if and only if $\boldsymbol{r} \in \Lambda_0$ and $\bar{i} = 0$.

For $\boldsymbol{r} \in \Lambda$, $1 \leqslant i \leqslant n$, we define the vertex operator $X^{(\bar{i})}(\boldsymbol{r}, z)$ as follows.

$$X^{(\bar{i})}(\boldsymbol{r}, z) = \exp\left(-\sum_{j\in -\mathbb{Z}_+} \frac{\varepsilon^{-ij} - q^{-\boldsymbol{r}j}}{j} E(j) z^{-j}\right) \exp\left(-\sum_{j\in \mathbb{Z}_+} \frac{\varepsilon^{-ij} - q^{-\boldsymbol{r}j}}{j} E(j) z^{-j}\right). \tag{2.3}$$

Clearly, we have $X^{(\bar{i})}(\boldsymbol{r}, z) \in (\mathrm{End}\mathrm{S}(\widehat{\mathrm{H}}^{-}))[[\mathrm{z}, \mathrm{z}^{-1}]]$ and so we have

$$X^{(\bar{i})}(\boldsymbol{r}, z) = \sum_{j\in\mathbb{Z}} x^{(\bar{i})}(\boldsymbol{r}, j) z^{-j}, \tag{2.4}$$

where $x^{(\bar{i})}(\boldsymbol{r}, j) \in \mathrm{End} S(\widehat{H}^{-})$, for $1 \leqslant i \leqslant n$, $j \in \mathbb{Z}$ and $\boldsymbol{r} \in \Lambda$.

Remark 2.5. In the definition of the vertex operators (2.3), $X^{(\bar{i})}(\boldsymbol{r} + \boldsymbol{r}', z) = X^{(\bar{i})}(\boldsymbol{r}, z)$ whenever $\boldsymbol{r}' \in \Lambda_0$, where $\boldsymbol{r} \in \Lambda$, $1 \leqslant i \leqslant n$. Also, $X^{(\bar{i})}(\boldsymbol{r}, z) = 1$ when $q^{r} = \varepsilon^{i} (= 1)$.

Next we shall derive the commutator relations for our vertex operators. The technique follows from [LW], [KKLW], [FK], [S] and [FLM].

PROPOSITION 2.6. *For any* $1 \leqslant i \leqslant n$, $k \in \mathbb{Z}$, $\boldsymbol{r} \in \Lambda$, *we have*

$$[E(k), X^{(\bar{i})}(\boldsymbol{r}, z)] = (\varepsilon^{ik} - q^{\boldsymbol{r}k}) z^{k} X^{(\bar{i})}(\boldsymbol{r}, z), \tag{2.7}$$

$$[d_0, X^{(\bar{i})}(\boldsymbol{r}, z)] = -D X^{(\bar{i})}(\boldsymbol{r}, z). \tag{2.8}$$

The normal ordering can be defined as usual, see for example [FLM] in the twisted case. Thus

$$: X^{(\bar{i})}(\boldsymbol{r}, z) := X^{(\bar{i})}(\boldsymbol{r}, z). \tag{2.9}$$

Remark 2.10. d_0 can be rewritten as

$$d_0 = -\frac{1}{2}\sum_{j\in\mathbb{Z}} : E(-j)E(j) := -\sum_{j\in\mathbb{Z}_+} E(-j)E(j) \in \mathrm{End} S(\widehat{H}^{-}).$$

We define

$$
\begin{aligned}
&: X^{(\bar{i})}(\boldsymbol{r}_1, z_1) X^{(\bar{k})}(\boldsymbol{r}_2, z_2) : \\
&\quad = \exp\left(- \sum_{j \in -\mathbb{Z}_+} \frac{(\varepsilon^{-ij} - q^{-\boldsymbol{r}_1 j}) E(j) z_1^{-j} + (\varepsilon^{-kj} - q^{-\boldsymbol{r}_2 j}) E(j) z_2^{-j}}{j} \right) \times \\
&\quad\quad \times \exp\left(- \sum_{j \in \mathbb{Z}_+} \frac{(\varepsilon^{-ij} - q^{-\boldsymbol{r}_1 j}) E(j) z_1^{-j} + (\varepsilon^{-kj} - q^{-\boldsymbol{r}_2 j}) E(j) z_2^{-j}}{j} \right)
\end{aligned}
$$

for $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \Lambda$, $1 \leqslant i, k \leqslant n$. Then one has

$$
: X^{(\bar{i})}(\boldsymbol{r}_1, z_1) X^{(\bar{k})}(\boldsymbol{r}_2, z_2) :=: X^{(\bar{k})}(\boldsymbol{r}_2, z_2) X^{(\bar{i})}(\boldsymbol{r}_1, z_1) : . \tag{2.12}
$$

We have the following basic result.

LEMMA 2.13. *For* $1 \leqslant i, k \leqslant n$, $\boldsymbol{r}_1, \boldsymbol{r}_2 \in \Lambda$,

$$
\begin{aligned}
&\exp\left(- \sum_{j \in \mathbb{Z}_+} \frac{\varepsilon^{-ij} - q^{-\boldsymbol{r}_1 j}}{j} z_1^{-j} \right) \exp\left(- \sum_{j \in -\mathbb{Z}_+} \frac{\varepsilon^{-kj} - q^{-\boldsymbol{r}_2 j}}{j} z_2^{-j} \right) \\
&\quad = \exp\left(- \sum_{j \in -\mathbb{Z}_+} \frac{\varepsilon^{-kj} - q^{-\boldsymbol{r}_2 j}}{j} z_2^{-j} \right) \exp\left(- \sum_{j \in \mathbb{Z}_+} \frac{\varepsilon^{-ij} - q^{-\boldsymbol{r}_1 j}}{j} z_1^{-j} \right) \times \\
&\quad\quad \times \left(1 - \frac{\varepsilon^k z_2}{\varepsilon^i z_1}\right) \left(1 - \frac{q^{\boldsymbol{r}_2} z_2}{q^{\boldsymbol{r}_1} z_1}\right) \left(1 - \frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1} z_1}\right)^{-1} \left(1 - \frac{q^{\boldsymbol{r}_2} z_2}{\varepsilon^i z_1}\right)^{-1}
\end{aligned} \tag{2.14}
$$

in the formal power series algebra $(\mathrm{End} S(\widehat{H}^-))[[z_1^{-1}, z_2]] \subseteq (\mathrm{End} S(\widehat{H}^-))\{z_1, z_2\}$ *(for notation see* [FLM]*). So*

$$
\begin{aligned}
&X^{(\bar{i})}(\boldsymbol{r}_1, z_1) X^{(\bar{k})}(\boldsymbol{r}_2, z_2) \\
&\quad = : X^{(\bar{i})}(\boldsymbol{r}_1, z_1) X^{(\bar{k})}(\boldsymbol{r}_2, z_2) : \times \\
&\quad\quad \times \left(1 - \frac{\varepsilon^k z_2}{\varepsilon^i z_1}\right) \left(1 - \frac{q^{\boldsymbol{r}_2} z_2}{q^{\boldsymbol{r}_1} z_1}\right) \left(1 - \frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1} z_1}\right)^{-1} \left(1 - \frac{q^{\boldsymbol{r}_2} z_2}{\varepsilon^i z_1}\right)^{-1} .
\end{aligned} \tag{2.15}
$$

Proof.

$$
\begin{aligned}
&\left[- \sum_{j \in \mathbb{Z}_+} \frac{\varepsilon^{ij} - q^{-\boldsymbol{r}_1 j}}{j} z_1^{-j}, - \sum_{j \in -\mathbb{Z}_+} \frac{\varepsilon^{-kj} - q^{-\boldsymbol{r}_2 j}}{j} z_2^{-j} \right] \\
&\quad = - \sum_{j \in \mathbb{Z}_+} \frac{(\varepsilon^{-ij} - q^{-\boldsymbol{r}_1 j})(\varepsilon^{kj} - q^{\boldsymbol{r}_2 j})}{j} \left(\frac{z_2}{z_1}\right)^j
\end{aligned}
$$

$$= -\sum_{j\in\mathbb{Z}_+}\frac{1}{j}\left(\left(\frac{\varepsilon^k z_2}{\varepsilon_i z_1}\right)^j+\left(\frac{q^{\mathbf{r}_2}z_2}{q^{\mathbf{r}_1}z_1}\right)^j-\left(\frac{\varepsilon^k z_2}{q^{\mathbf{r}_1}z_1}\right)^j-\left(\frac{q^{\mathbf{r}_2}z_2}{\varepsilon^i z_1}\right)^j\right)$$
$$= \ln\left(1-\frac{\varepsilon^k z_2}{\varepsilon^i z_1}\right)+\ln\left(1-\frac{q^{\mathbf{r}_2}z_2}{q^{\mathbf{r}_1}z_1}\right)-\ln\left(1-\frac{\varepsilon^k z_2}{q^{\mathbf{r}_1}z_1}\right)-\ln\left(1-\frac{q^{\mathbf{r}_2}z_2}{\varepsilon^i z_1}\right),$$

which immediately implies the lemma. □

To calculate the commutators of vertex operators, we need some more notation and identities.

Set

$$R^{(\bar{i})}_{(\bar{k})}(\mathbf{r}_1,\mathbf{r}_2,z_1,z_2) = :X^{(\bar{i})}(\mathbf{r}_1,z_1)X^{(\bar{k})}(\mathbf{r}_2,z_2):\left(1-\frac{\varepsilon^k z_2}{\varepsilon^i z_1}\right)\left(1-\frac{q^{\mathbf{r}_2}z_2}{q^{\mathbf{r}_1}z_1}\right)\frac{q^{\mathbf{r}_1}z_1}{\varepsilon^k z_2}. \tag{2.16}$$

Then

$$X^{(\bar{i})}(\mathbf{r}_1,z_1)X^{(\bar{k})}(\mathbf{r}_2,z_2) = R^{(\bar{i})}_{(\bar{k})}(\mathbf{r}_1,\mathbf{r}_2,z_1,z_2)\frac{\varepsilon^k z_2}{q^{\mathbf{r}_1}z_1}\left(1-\frac{\varepsilon^k z_2}{q^{\mathbf{r}_1}z_1}\right)^{-1}\left(1-\frac{q^{\mathbf{r}_2}z_2}{\varepsilon^i z_1}\right)^{-1}. \tag{2.17}$$

One may easily show that

$$R^{(\bar{i})}_{(\bar{k})}(\mathbf{r}_1,\mathbf{r}_2,z_1,z_2) = R^{(\bar{k})}_{(\bar{i})}(\mathbf{r}_2,\mathbf{r}_1,z_2,z_1). \tag{2.18}$$

Moreover, we have

LEMMA 2.19. *For* $\mathbf{r}_1,\mathbf{r}_2\in\Lambda$, $1\leqslant i,k\leqslant n$,

$$\lim_{z_2\to\varepsilon^{-k}q^{\mathbf{r}_1}z_1} :X^{(\bar{i})}(\mathbf{r}_1,z_1)X^{(\bar{k})}(\mathbf{r}_2,z_2): = X^{(\bar{i}+\bar{k})}(\mathbf{r}_1+\mathbf{r}_2,\varepsilon^{-k}z_1) \tag{2.20}$$

and

$$\lim_{z_2\to\varepsilon^{-k}q^{\mathbf{r}_1}z_1} R^{(\bar{i})}_{(\bar{k})}(\mathbf{r}_1,\mathbf{r}_2,z_1,z_2) = (1-\varepsilon^{-i}q^{\mathbf{r}_1})(1-\varepsilon^{-k}q^{\mathbf{r}_2})X^{(\bar{i}+\bar{k})}(\mathbf{r}_1+\mathbf{r}_2,\varepsilon^{-k}z_1). \tag{2.21}$$

The following basic result is similar to (3.34) in [G] whose proof is straightforward.

LEMMA 2.22. *If* $q^{r_1+r_2} \neq \varepsilon^{i+k}$, *then*

$$\begin{aligned}&\left(1-\frac{\varepsilon^k z_2}{q^{r_1} z_1}\right)^{-1}\left(1-\frac{q^{r_2} z_2}{\varepsilon^i z_1}\right)^{-1}-\frac{q^{r_1} z_1}{\varepsilon^k z_2}\frac{\varepsilon^i z_1}{q^{r_2} z_2}\left(1-\frac{\varepsilon^i z_1}{q^{r_2} z_2}\right)^{-1}\left(1-\frac{q^{r_1} z_1}{\varepsilon^k z_2}\right)^{-1}\\&\quad=(1-\varepsilon^{-i-k}q^{r_1+r_2})^{-1}\frac{q^{r_1} z_1}{\varepsilon^k z_2}\left(\delta\left(\frac{\varepsilon^k z_2}{q^{r_1} z_1}\right)-\delta\left(\frac{\varepsilon^i z_1}{q^{r_2} z_2}\right)\right)\end{aligned}$$

Now we are in the position to show our first commutator relation:

PROPOSITION 2.23. *If* $q^{r_1+r_2} \neq \varepsilon^{i+k}$, *then*

$$\begin{aligned}&[X^{(\bar{i})}(\boldsymbol{r}_1, z_1), X^{(\bar{k})}(\boldsymbol{r}_2, z_2)]\\&\quad=(1-\varepsilon^{-i}q^{r_1})(1-\varepsilon^{-k}q^{r_2})(1-\varepsilon^{-i-k}q^{r_1+r_2})^{-1}\times\\&\qquad\times\left(X^{(\bar{i}+\bar{k})}(\boldsymbol{r}_1+\boldsymbol{r}_2, \varepsilon^{-k}z_1)\delta\left(\frac{\varepsilon^k z_2}{q^{r_1} z_1}\right)-X^{(\bar{i}+\bar{k})}(\boldsymbol{r}_1+\boldsymbol{r}_2, \varepsilon^{-i}z_2)\delta\left(\frac{\varepsilon^i z_1}{q^{r_2} z_2}\right)\right).\end{aligned}$$

Proof. By (2.15), (2.18) and Lemma 2.22, we have

$$\begin{aligned}&[X^{(\bar{i})}(\boldsymbol{r}_1, z_1), X^{(\bar{k})}(\boldsymbol{r}_2, z_2)]\\&\quad=X^{(\bar{i})}(\boldsymbol{r}_1, z_1)X^{(\bar{k})}(\boldsymbol{r}_2, z_2)-X^{(\bar{k})}(\boldsymbol{r}_2, z_2)X^{(\bar{i})}(\boldsymbol{r}_1, z_1)\\&\quad=R^{(\bar{i})}_{(\bar{k})}(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, z_2)\frac{\varepsilon^k z_2}{q^{r_1} z_1}\left(1-\frac{\varepsilon^k z_2}{q^{r_1} z_1}\right)^{-1}\left(1-\frac{q^{r_2} z_2}{\varepsilon^i z_1}\right)^{-1}-\\&\qquad-R^{(\bar{k})}_{(\bar{i})}(\boldsymbol{r}_2, \boldsymbol{r}_1, z_2, z_1)\frac{\varepsilon^i z_1}{q^{r_2} z_2}\left(1-\frac{\varepsilon^i z_1}{q^{r_2} z_2}\right)^{-1}\left(1-\frac{q^{r_1} z_1}{\varepsilon^k z_2}\right)^{-1}\\&\quad=R^{(\bar{i})}_{(\bar{k})}(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, z_2)\frac{\varepsilon^k z_2}{q^{r_1} z_1}\\&\qquad\times\left(\left((1-\frac{\varepsilon^k z_2}{q^{r_1} z_1}\right)^{-1}\left(1-\frac{q^{r_2} z_2}{\varepsilon^i z_1}\right)^{-1}-\frac{q^{r_1} z_1}{\varepsilon^k z_2}\frac{\varepsilon^i z_1}{q^{r_2} z_2}\left(1-\frac{\varepsilon^i z_1}{q^{r_2} z_2}\right)^{-1}\left(1-\frac{q^{r_1} z_1}{\varepsilon^k z_2}\right)^{-1}\right)\times\\&\quad=R^{(\bar{i})}_{(\bar{k})}(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, z_2)(1-\varepsilon^{-i-k}q^{r_1+r_2})^{-1}\left(\delta\left(\frac{\varepsilon^k z_2}{q^{r_1} z_1}\right)-\delta\left(\frac{\varepsilon^i z_1}{q^{r_2} z_2}\right)\right).\end{aligned}\tag{2.24}$$

Applying Lemma 2.19 completes the proof. □

Next, if $q^{r_1+r_2} = \varepsilon^{i+k}$ (so $q^{r_2} = q^{-r_1}$ and $\varepsilon^i = \varepsilon^{-k}$) we have

$$\begin{aligned}&[X^{(\bar{i})}(\boldsymbol{r}_1, z_1), X^{(\bar{k})}(\boldsymbol{r}_2, z_2)]\\&\quad=R^{(\bar{i})}_{(\bar{k})}(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, z_2)\frac{\varepsilon^k z_2}{q^{r_1} z_1}\left(1-\frac{\varepsilon^k z_2}{q^{r_1} z_1}\right)^{-2}-\end{aligned}$$

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$$
\begin{aligned}
&- R^{(\bar{k})}_{(\bar{i})}(\boldsymbol{r}_2, \boldsymbol{r}_1, z_2, z_1)\frac{\varepsilon^i z_1}{q^{\boldsymbol{r}_2} z_2}\left(1 - \frac{q^{\boldsymbol{r}_1} z_1}{\varepsilon^k z_2}\right)^{-2} \\
&= R^{(\bar{i})}_{(\bar{k})}(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, z_2)\left(\frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1} z_1}\left(1 - \frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1} z_1}\right)^{-2} - \frac{q^{\boldsymbol{r}_1} z_1}{\varepsilon^k z_2}\left(1 - \frac{q^{\boldsymbol{r}_1} z_1}{\varepsilon^k z_2}\right)^{-2}\right) \\
&= R^{(\bar{i})}_{(\bar{k})}(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, z_2)(D\delta)\left(\frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1} z_1}\right)
\end{aligned}
$$

here we use the following well-known identity:

$$
z(1-z)^{-2} - z^{-1}(1-z^{-1})^{-2} = (D\delta)(z). \tag{2.26}
$$

By Proposition 2.2.4 in [FLM] and (2.25), we obtain

$$
\begin{aligned}
&[X^{(\bar{i})}(\boldsymbol{r}_1, z_1), X^{(\bar{k})}(\boldsymbol{r}_2, z_2)] \\
&\quad = R^{(\bar{i})}_{(\bar{k})}(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, \varepsilon^{-k} q^{\boldsymbol{r}_1} z_1)(D\delta)\left(\frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1} z_1}\right) - \\
&\quad\quad - (D_{z_2} R^{(\bar{i})}_{(\bar{k})})(r_1, r_2, z_1, \varepsilon^{-k} q^{\boldsymbol{r}_1} z_1)\delta\left(\frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1} z_1}\right) \\
&\quad = G_1 (D\delta)\left(\frac{\varepsilon^k z_2}{q^{r_1} z_1}\right) - G_2 \delta\left(\frac{\varepsilon^k z_2}{q^{r_1} z_1}\right),
\end{aligned}
$$

where

$$
G_1 = R^{(\bar{i})}_{(\bar{k})}(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, \varepsilon^{-k} q^{r_1} z_1) \tag{2.28}
$$

and

$$
G_2 = \left(D_{z_2} R^{(\bar{i})}_{(\bar{k})}\right)(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, \varepsilon^{-k} q^{\boldsymbol{r}_1} z_1). \tag{2.29}
$$

From (2.5) and (2.21), we have

$$
G_1 = (1 - \varepsilon^{-i} q^{\boldsymbol{r}_1})(1 - \varepsilon^{-k} q^{\boldsymbol{r}_2}). \tag{2.30}
$$

To compute G_2, we first have

$$
\begin{aligned}
&D_{z_2} R^{(\bar{i})}_{(\bar{k})}(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, z_2) \\
&\quad = \left(\sum_{j \in -\mathbb{Z}_+} (\varepsilon^{-kj} - q^{-\boldsymbol{r}_2 j}) E(j) z_2^{-j}\right) R^{(\bar{i})}_{(\bar{k})}(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, z_2) + \\
&\quad\quad + R^{(\bar{i})}_{(\bar{k})}(\boldsymbol{r}_1, \boldsymbol{r}_2, z_1, z_2) \sum_{j \in \mathbb{Z}_+} (\varepsilon^{-kj} - q^{-\boldsymbol{r}_2 j}) E(j) z_2^{-j} +
\end{aligned}
$$

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$$+:X^{(\bar{i})}(\boldsymbol{r}_1,z_1)X^{(\bar{k})}(\boldsymbol{r}_2,z_2):\left(-\frac{\varepsilon^k z_2}{\varepsilon^i z_1}\right)\left(1-\frac{q^{\boldsymbol{r}_2}z_2}{q^{\boldsymbol{r}_1}z_1}\right)\frac{q^{\boldsymbol{r}_1}z_1}{\varepsilon^k z_2}+$$
$$+:X^{(\bar{i})}(\boldsymbol{r}_1,z_1)X^{(\bar{k})}(\boldsymbol{r}_2,z_2):\left(1-\frac{\varepsilon^k z_2}{\varepsilon^i z_1}\right)\left(-\frac{q^{\boldsymbol{r}_2}z_2}{q^{\boldsymbol{r}_1}z_1}\right)\frac{q^{\boldsymbol{r}_1}z_1}{\varepsilon^k z_2}+$$
$$+:X^{(\bar{i})}(\boldsymbol{r}_1,z_1)X^{(\bar{k})}(\boldsymbol{r}_2,z_2):\left(1-\frac{\varepsilon^k z_2}{\varepsilon^i z_1}\right)\left(1-\frac{q^{\boldsymbol{r}_2}z_2}{q^{\boldsymbol{r}_1}z_1}\right)\left(-\frac{q^{\boldsymbol{r}_1}z_1}{\varepsilon^k z_2}\right).$$

Thus, it follows from (2.5) and Lemma 2.19 that

$$\begin{aligned}
&(D_{z_2}R^{(\bar{i})}_{(\bar{k})})(\boldsymbol{r}_1,\boldsymbol{r}_2,z_1,\varepsilon^{-k}q^{\boldsymbol{r}_1}z_1)\\
&\quad=(1-\varepsilon^{-i}q^{\boldsymbol{r}_1})(1-\varepsilon^{-k}q^{\boldsymbol{r}_2})\sum_{j\in-\mathbb{Z}_+}(\varepsilon^{-kj}-q^{-\boldsymbol{r}_2 j})E(j)(\varepsilon^{-k}q^{\boldsymbol{r}_1}z_1)^{-j}+\\
&\qquad+(1-\varepsilon^{-i}q^{\boldsymbol{r}_1})(1-\varepsilon^{-k}q^{\boldsymbol{r}_2})\sum_{j\in\mathbb{Z}_+}(\varepsilon^{-kj}-q^{-\boldsymbol{r}_2 j})E(j)(\varepsilon^{-k}q^{\boldsymbol{r}_1}z_1)^{-j}+\\
&\qquad+(-\varepsilon^{-i}q^{\boldsymbol{r}_1})(1-\varepsilon^{-k}q^{\boldsymbol{r}_2})+(1-\varepsilon^{-i}q^{\boldsymbol{r}_1})(-\varepsilon^{-k}q^{\boldsymbol{r}_2})\\
&\qquad+(1-\varepsilon^{-i}q^{\boldsymbol{r}_1})(1-\varepsilon^{-k}q^{\boldsymbol{r}_2})(-1)\\
&\quad=(1-\varepsilon^{-i}q^{\boldsymbol{r}_1})(1-\varepsilon^{-k}q^{\boldsymbol{r}_2})\sum_{j\in\mathbb{Z}}E(j)q^{-\boldsymbol{r}_1 j}z_1^{-j}-\\
&\qquad+-(1-\varepsilon^{-i}q^{\boldsymbol{r}_1})(1-\varepsilon^{-k}q^{\boldsymbol{r}_2})\sum_{j\in\mathbb{Z}}E(j)\varepsilon^{kj}z_1^{-j},
\end{aligned}\tag{2.32}$$

and so

$$\begin{aligned}
&G_2\delta\left(\frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1}z_1}\right)\\
&\quad=(1-\varepsilon^{-i}q^{\boldsymbol{r}_1})(1-\varepsilon^{-k}q^{\boldsymbol{r}_2})\\
&\qquad\times\left(-E(\varepsilon^{-k}z_1)\delta\left(\frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1}z_1}\right)+E(\varepsilon^{-i}z_2)\delta\left(\frac{\varepsilon^i z_1}{q^{\boldsymbol{r}_2}z_2}\right)\right).
\end{aligned}$$

Therefore, we have proved our second commutator relation:

PROPOSITION 2.33. *If* $q^{\boldsymbol{r}_1+\boldsymbol{r}_2}=\varepsilon^{-i-k}=1$, *then*

$$\begin{aligned}
&[X^{(\bar{i})}(\boldsymbol{r}_1,z_1),X^{(\bar{k})}(\boldsymbol{r}_2,z_2)]\\
&\quad=(1-\varepsilon^{-i}q^{\boldsymbol{r}_1})(1-\varepsilon^{-k}q^{\boldsymbol{r}_2})\left(E(\varepsilon^{-k}z_1)\delta\left(\frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1}z_1}\right)-E(\varepsilon^{-i}z_2)\delta\left(\frac{\varepsilon^i z_1}{q^{\boldsymbol{r}_2}z_2}\right)\right)+\\
&\qquad+(1-\varepsilon^{-i}q^{\boldsymbol{r}_1})(1-\varepsilon^{-k}q^{\boldsymbol{r}_2})(D\delta)\left(\frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1}z_1}\right).
\end{aligned}$$

Define

$$Y^{(\bar{i})}(\boldsymbol{r}, z) = \frac{1}{1-\varepsilon^{-i}q^{\boldsymbol{r}}} X^{(\bar{i})}(\boldsymbol{r}, z), \tag{2.34}$$

where $\bar{i} \neq \bar{0}$ and $\boldsymbol{r} \in \Lambda$, or $\bar{i} = \bar{0}$ but $\boldsymbol{r} \in \Lambda \setminus \Lambda_0$. Summarizing the above, we have

PROPOSITION 2.35.

$$[E(k), Y^{(\bar{i})}(\boldsymbol{r}, z)] = (\varepsilon^{ik} - q^{\boldsymbol{r}k}) z^k Y^{(\bar{i})}(\boldsymbol{r}, z), \tag{2.36}$$

$$[d_0, Y^{(\bar{i})}(\boldsymbol{r}, z)] = -D Y^{(\bar{i})}(\boldsymbol{r}, z). \tag{2.37}$$

If $q^{\boldsymbol{r}_1+\boldsymbol{r}_2} \neq \varepsilon^{i+k}$*, then*

$$\begin{aligned}&[Y^{(\bar{i})}(\boldsymbol{r}_1, z_1), Y^{(\bar{k})}(\boldsymbol{r}_2, z_2)]\\ &\quad = Y^{(\bar{i}+\bar{k})}(\boldsymbol{r}_1+\boldsymbol{r}_2, \varepsilon^{-k}z_1)\delta\left(\frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1}z_1}\right) - Y^{(\bar{i}+\bar{k})}(\boldsymbol{r}_1+\boldsymbol{r}_2, \varepsilon^{-i}z_2)\delta\left(\frac{\varepsilon^i z_1}{q^{\boldsymbol{r}_2}z_2}\right).\end{aligned} \tag{2.38}$$

If $q^{\boldsymbol{r}_1+\boldsymbol{r}_2} = \varepsilon^{-i-k} (= 1)$*, then*

$$\begin{aligned}&[Y^{(\bar{i})}(\boldsymbol{r}_1, z_1), Y^{(\bar{k})}(\boldsymbol{r}_2, z_2)],\\ &E(\varepsilon^{-k}z_1)\delta\left(\frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1}z_1}\right) - E(\varepsilon^{-i}z_2)\delta\left(\frac{\varepsilon^i z_1}{q^{\boldsymbol{r}_2}z_2}\right) + (D\delta)\left(\frac{\varepsilon^k z_2}{q^{\boldsymbol{r}_1}z_1}\right).\end{aligned} \tag{2.39}$$

To conclude this section, write

$$Y^{(\bar{i})}(\boldsymbol{r}, z) = \sum_{j\in\mathbb{Z}} y^{(\bar{i})}(\boldsymbol{r}, j) z^{-j}, \tag{2.40}$$

and let $\mathcal{V}(\Lambda, q)$ be the $\mathbb{C}$-linear span of operators $E(j), d_0, 1$ and $y^{(\bar{i})}(\boldsymbol{r}, j)$, where $j \in \mathbb{Z}, \bar{i} \neq \bar{0}$ and $\boldsymbol{r} \in \Lambda$, or $\bar{i} = \bar{0}$ but $\boldsymbol{r} \in \Lambda \setminus \Lambda_0$. From Proposition 2.35, we see that

PROPOSITION 2.41. $\mathcal{V}(\Lambda, q)$ *is a Lie subalgebra of* $\mathrm{gl}(S(\widehat{H}^-))$.

The following lemma will be used later. Its proof is easy.

LEMMA 2.42. *For any* $\boldsymbol{r} \in \Lambda \setminus \Lambda_0$,

$$y^{(\bar{0})}(\boldsymbol{r}, 0)1 = \frac{1}{1-q^{\boldsymbol{r}}} 1.$$

Remark 2.43. Notice that our vertex operators $Y^{(\bar{i})}(0, z)$, for $1 \leqslant i \leqslant n-1$, are same as the vertex operators in (4.8) of [KKLW].

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3. Realizations

In this section we will find a realization for the Lie algebra $\mathcal{V}(\Lambda, q)$. If (Λ, q) is generic, we further lift $\mathcal{V}(\Lambda, q)$ to a Lie algebra $\mathcal{W}(\Lambda, q)$ on the enlarged Fock space $W_\Lambda = \mathbb{C}[\Lambda] \otimes_{\mathbb{C}} S(\widehat{H}^-)$.

Let $\mathcal{R} = \mathbb{C}[\Lambda] = \sum_{\boldsymbol{r}\in\Lambda} \oplus \mathbb{C}e^{\boldsymbol{r}}$ be the semigroup algebra of Λ. Let σ be the automorphism of $\mathcal{R}$ given by $\sigma(e^{\boldsymbol{r}}) = q^{\boldsymbol{r}} e^{\boldsymbol{r}}$, for $\boldsymbol{r} \in \Lambda$. Then we can form skew polynomial algebras:

$$\mathcal{R}[t_0, t_0^{-1}; \sigma] = \sum_{i\in\mathbb{Z}} \oplus t_0^i \mathcal{R} \quad \text{and} \quad \mathcal{R}[s_0, s_0^{-1}; \sigma^n] = \sum_{i\in\mathbb{Z}} \oplus s_0^i \mathcal{R} \tag{3.1}$$

with multiplication defined as $at_0^i = t_0^i \sigma^i(a)$ (resp. $as_0^i = s_0^i \sigma^{ni}(a)$), for $a \in \mathcal{R}, i \in \mathbb{Z}$. That is,

$$e^{\boldsymbol{r}} t_0^i = q^{i\boldsymbol{r}} t_0^i e^{\boldsymbol{r}} \,(\text{resp.} e^{\boldsymbol{r}} s_0^i = q^{i\boldsymbol{r}n} s_0^i e^{\boldsymbol{r}}), \quad \text{for } \boldsymbol{r} \in \Lambda, \mathrm{i} \in \mathbb{Z}. \tag{3.2}$$

Define $\kappa, \chi : \mathcal{R}[t_0, t_0^{-1}; \sigma] \,(\text{resp.} \mathcal{R}[s_0, s_0^{-1}; \sigma^n]) \to \mathbb{C}$ to be the $\mathbb{C}$-linear functions given by

$$\begin{aligned} \kappa(t_0^i e^{\boldsymbol{r}})(\text{resp.} \kappa(s_0^i e^{\boldsymbol{r}})) &= \begin{cases} 1, & \text{if } i = 0 \text{ and } \boldsymbol{r} \in \Lambda_0, \\ 0, & \text{otherwise;} \end{cases} \\ \chi(t_0^i e^{\boldsymbol{r}})(\text{resp.} \kappa(s_0^i e^{\boldsymbol{r}})) &= \begin{cases} 1, & \text{if } i = 0 \text{ and } \boldsymbol{r} = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.3}$$

Let d_0, d_i be the degree operators on $\mathcal{R}[t_0, t_0^{-1}; \sigma]$ (resp. $\mathcal{R}[s_0, s_0^{-1}; \sigma^n]$) defined by

$$d_0(t_0^j e^{\boldsymbol{r}}) = j t_0^j e^{\boldsymbol{r}}, \;\; d_i(t_0^j e^{\boldsymbol{r}}) = r_i t_0^j e^{\boldsymbol{r}} \,(\text{resp.} d_0(s_0^j e^{\boldsymbol{r}}) = j s_0^j e^{\boldsymbol{r}}, \;\; d_i(s_0^j e^{\boldsymbol{r}}) = r_i s_0^j e^{\boldsymbol{r}})$$

for $j \in \mathbb{Z}$, $\boldsymbol{r} = (r_1, \cdots, r_N) \in \Lambda$ and $1 \leqslant i \leqslant N$.

For any associative algebra $\mathcal{A}$, we have the matrix algebra $M_n(\mathcal{A})$ with entries from $\mathcal{A}$. Let $gl_n(\mathcal{A})$ be the Lie algebra $M_n(\mathcal{A})^-$ as usual.

Now we form an $(N+1)$-dimensional central extension of $\mathrm{gl}_\mathrm{n}(\mathcal{R}[s_0, s_0^{-1}; \sigma^n])$,

$$\mathcal{G}^0(\Lambda) = \mathrm{gl}_\mathrm{n}(\mathcal{R}[s_0, s_0^{-1}; \sigma^n]) \oplus \mathbb{C}z_0 \oplus \mathbb{C}z_1 \oplus \cdots \oplus \mathbb{C}z_N \tag{3.4}$$

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with Lie bracket

$$
\begin{aligned}
&[E_{ij}(s_0^{n_1}e^{\boldsymbol r}), E_{kl}(s_0^{n_2}e^{\boldsymbol r'})]\\
&\quad = E_{ij}(s_0^{n_1}e^{\boldsymbol r})E_{kl}(s_0^{n_2}e^{\boldsymbol r'}) - E_{kl}(s_0^{n_2}e^{\boldsymbol r'})E_{ij}(s_0^{n_1}e^{\boldsymbol r})+\\
&\qquad + \delta_{jk}\delta_{il}\kappa((d_0 s_0^{n_1}e^{\boldsymbol r})s_0^{n_2}e^{\boldsymbol r'})z_0 + \delta_{jk}\delta_{il}\sum_{m=1}^{N}\chi((d_m s_0^{n_1}e^{\boldsymbol r})s_0^{n_2}e^{\boldsymbol r'})z_m\\
&\quad = \delta_{jk}q^{n_2\boldsymbol r n}E_{il}(s_0^{n_1+n_2}e^{\boldsymbol r+\boldsymbol r'}) - \delta_{il}q^{n_1\boldsymbol r' n}E_{kj}(s_0^{n_1+n_2}e^{\boldsymbol r+\boldsymbol r'})+\\
&\qquad + n_1\delta_{jk}\delta_{il}\delta_{n_1+n_2,0}q^{n_2\boldsymbol r n}\kappa(e^{\boldsymbol r+\boldsymbol r'})z_0+\\
&\qquad + \delta_{jk}\delta_{il}\delta_{n_1+n_2,0}q^{n_2\boldsymbol r n}\chi(e^{\boldsymbol r+\boldsymbol r'})\sum_{m=1}^{N}r_m z_m,
\end{aligned}
\tag{3.5}
$$

for $\boldsymbol r = (r_1, \cdots, r_N) \in \Lambda, \boldsymbol r' \in \Lambda, n_1, n_2 \in \mathbb{Z}, 1 \leqslant i, j, k, l \leqslant n$.

Let

$$
\mathcal{G}(\Lambda) = \mathcal{G}^0(\Lambda) \oplus \mathbb{C}d_0 \oplus \mathbb{C}d_1 \oplus \cdots \oplus \mathbb{C}d_N \tag{3.6}
$$

be the semi-direct product of $\mathcal{G}^0(\Lambda)$ and the degree derivations $d_0, d_1, \cdots, d_N$, where $z_0, z_1, \cdots, z_N$ are central elements of $\mathcal{G}_\Lambda$.

Note that $\mathbb{C}[s_0, s_0^{-1}]$ is a subalgebra of $\mathcal{R}[s_0, s_0^{-1}; \sigma^n]$. Correspondingly, the affinization $\widetilde{\mathrm{gl}}_{\mathrm{n}}$ of gl_{n} is a subalgebra of $\mathcal{G}(\Lambda)$.

Define

$$
\mathcal{G}_p^0(\Lambda) = \sum_{1 \leqslant i,j \leqslant n}\sum_{k\in\mathbb{Z}} \oplus E_{ij}(t_0^{j-i+kn}\mathcal{R}) \oplus \sum_{i=0}^{N}\oplus\mathbb{C}c_i \tag{3.7}
$$

with Lie bracket

$$
\begin{aligned}
&[E_{ij}(t_0^{n_1}e^{\boldsymbol r}), E_{kl}(t_0^{n_2}e^{\boldsymbol r'})]\\
&\quad = E_{ij}(t_0^{n_1}e^{\boldsymbol r})E_{kl}(t_0^{n_2}e^{\boldsymbol r'}) - E_{kl}(t_0^{n_2}e^{\boldsymbol r'})E_{ij}(t_0^{n_1}e^{\boldsymbol r})+\\
&\qquad + \frac{1}{n}\kappa(tr((d_0E_{ij}(t_0^{n_1}e^{\boldsymbol r}))E_{kl}(t_0^{n_2}e^{\boldsymbol r'})))c_0+\\
&\qquad + \sum_{m=1}^{N}\chi(tr((d_mE_{ij}(t_0^{n_1}e^{\boldsymbol r}))E_{kl}(t_0^{n_2}e^{\boldsymbol r'})))c_m\\
&\quad = \delta_{jk}q^{n_2\boldsymbol r}E_{il}(t_0^{n_1+n_2}e^{\boldsymbol r+\boldsymbol r'}) - \delta_{il}q^{n_1\boldsymbol r'}E_{kj}(t_0^{n_1+n_2}e^{\boldsymbol r+\boldsymbol r'})+\\
&\qquad + n_1 n\delta_{jk}\delta_{il}\delta_{n_1+n_2,0}q^{n_2\boldsymbol r}\kappa(e^{\boldsymbol r+\boldsymbol r'})c_0+\\
&\qquad + \delta_{jk}\delta_{il}\delta_{n_1+n_2,0}q^{n_2\boldsymbol r}\chi(e^{\boldsymbol r+\boldsymbol r'})\sum_{m=1}^{N}r_m c_m,
\end{aligned}
\tag{3.8}
$$

where $\boldsymbol r = (r_1, \cdots, r_N), \boldsymbol r' \in \Lambda, n_1, n_2 \in \mathbb{Z}$ and $1 \leqslant i, j, k, l \leqslant n$, $c_0, c_1, \cdots, c_N$ are central elements of $\mathcal{G}_p^0(\Lambda)$. We form the semi-direct product of $\mathcal{G}_p^0(\Lambda)$ and the degree derivations $d_0, d_1, \cdots, d_N$.

$$\mathcal{G}_p(\Lambda) = \mathcal{G}_p^0(\Lambda) \oplus \mathbb{C}d_0 \oplus \mathbb{C}d_1 \oplus \cdots \oplus \mathbb{C}d_N. \tag{3.9}$$

PROPOSITION 3.10. *The Lie algebra $\mathcal{G}^0(\Lambda) \oplus \mathbb{C}d_0$ is isomorphic to $\mathcal{G}_p^0(\Lambda) \oplus \mathbb{C}d_0$ with the isomorphism given by the $\mathbb{C}$-linear map φ:*

$$E_{ij}(s_0^k e^{\mathbf{r}}) \mapsto q^{j\mathbf{r}} E_{ij}(t_0^{j-i+kn} e^{\mathbf{r}}) - \frac{i}{n}\delta_{ij}\delta_{k,0} q^{j\mathbf{r}} \kappa(e^{\mathbf{r}}) c_0,$$
$$z_0 \mapsto c_0, \quad d_0 \mapsto \left(\frac{1}{n}(d_0 + \sum_{i=1}^{n} iE_{ii}\right), \quad z_m \mapsto c_m,$$

for $1 \leqslant i, j \leqslant n$, $k \in \mathbb{Z}$, $\mathbf{r} \in \Lambda$ and $1 \leqslant m \leqslant N$. If (Λ, q) is generic, then φ can be extended to an isomorphism from $\mathcal{G}(\Lambda)$ onto $\mathcal{G}_p(\Lambda)$ by defining $\varphi(d_m) = d_m$ for $1 \leqslant m \leqslant N$.

Proof. It is sufficient to show that φ preserves Lie bracket in the following two cases:

$$\varphi[E_{ij}(s_0^{n_1} e^{\mathbf{r}}), E_{kl}(s_0^{n_2} e^{\mathbf{r}'})] = [\varphi E_{ij}(s_0^{n_1} e^{\mathbf{r}}), \varphi E_{kl}(s_0^{n_2} e^{\mathbf{r}'})], \tag{3.11a}$$

$$\varphi[d_0, E_{ij}(s_0^{n_1} e^{\mathbf{r}})] = [\varphi d_0, \varphi E_{ij}(s_0^{n_1} e^{\mathbf{r}})], \tag{3.11b}$$

for $1 \leqslant i, j, k, l \leqslant n$, $n_1, n_2 \in \mathbb{Z}$, $\mathbf{r}, \mathbf{r}' \in \Lambda$.

We first have

$$\begin{aligned}
&[E_{ij}(t_0^{j-i+n_1 n} e^{\mathbf{r}}), E_{kl}(t_0^{l-k+n_2 n} e^{\mathbf{r}'})] \\
&\quad = \delta_{jk} q^{(l-j+n_2 n)\mathbf{r}} E_{il}(t_0^{l-i+(n_1+n_2)n} e^{\mathbf{r}+\mathbf{r}'}) - \\
&\qquad - \delta_{il} q^{(j-l+n_1 n)\mathbf{r}'} E_{kj}(t_0^{j-k+(n_1+n_2)n} e^{\mathbf{r}+\mathbf{r}'}) + \\
&\qquad + \frac{j-i+n_1 n}{n} \delta_{jk}\delta_{il} q^{(l-j+n_2 n)\mathbf{r}} \kappa(t_0^{(n_1+n_2)n} e^{\mathbf{r}+\mathbf{r}'}) c_0 + \\
&\qquad + \delta_{jk}\delta_{il} q^{(l-j+n_2 n)\mathbf{r}} \chi(t_0^{(n_1+n_2)n} e^{\mathbf{r}+\mathbf{r}'}) \sum_{m=1}^{N} r_m c_m \\
&\quad = \delta_{jk} q^{(l-j+n_2 n)\mathbf{r}} (E_{il}(t_0^{l-i+(n_1+n_2)n} e^{\mathbf{r}+\mathbf{r}'}) - in\delta_{il}\delta_{n_1+n_2,0} \kappa(e^{\mathbf{r}+\mathbf{r}'}) c_0) - \\
&\qquad - \delta_{il} q^{(j-l+n_1 n)\mathbf{r}'} (E_{kj}(t_0^{j-k+(n_1+n_2)n} e^{\mathbf{r}+\mathbf{r}'}) - jn\delta_{jk}\delta_{n_1+n_2,0} \kappa(e^{\mathbf{r}+\mathbf{r}'}) c_0) + \\
&\qquad + n_1 \delta_{jk}\delta_{il}\delta_{n_1+n_2,0} q^{(l-j+n_2 n)\mathbf{r}} \kappa(e^{\mathbf{r}+\mathbf{r}'}) c_0 + \\
&\qquad + \delta_{jk}\delta_{il}\delta_{n_1+n_2,0} q^{(l-j+n_2 n)\mathbf{r}} \chi(e^{\mathbf{r}+\mathbf{r}'}) \sum_{m=1}^{N} r_m c_m.
\end{aligned}$$

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Thus,

$$
\begin{aligned}
&[q^{jr}E_{ij}(t_0^{j-i+n_1n}e^{\mathbf{r}}), q^{l\mathbf{r}'}E_{kl}(t_0^{l-k+n_2n}e^{\mathbf{r}'})] \\
&\quad = \delta_{jk}q^{n_2n\mathbf{r}}q^{l(\mathbf{r}+\mathbf{r}')}(E_{il}(t_0^{l-i+(n_1+n_2)n}e^{\mathbf{r}+\mathbf{r}'}) - in\delta_{il}\delta_{n_1+n_2,0}\kappa(e^{\mathbf{r}+\mathbf{r}'})c_0) - \\
&\quad - \delta_{il}q^{n_1n\mathbf{r}'}q^{j(\mathbf{r}+\mathbf{r}')}(E_{kj}(t_0^{j-k+(n_1+n_2)n}e^{\mathbf{r}+\mathbf{r}'}) - jn\delta_{jk}\delta_{n_1+n_2,0}\kappa(e^{\mathbf{r}+\mathbf{r}'})c_0) + \\
&\quad + n_1\delta_{jk}\delta_{il}\delta_{n_1+n_2,0}q^{n_2n\mathbf{r}}q^{l(\mathbf{r}+\mathbf{r}')}\kappa(e^{\mathbf{r}+\mathbf{r}'})c_0 + \\
&\quad + \delta_{jk}\delta_{il}\delta_{n_1+n_2,0}q^{n_2n\mathbf{r}}q^{l(\mathbf{r}+\mathbf{r}')}\chi(e^{\mathbf{r}+\mathbf{r}'})\sum_{m=1}^{N} r_m c_m,
\end{aligned}
$$

and (3.11a) follows from the fact that $q^{l(\mathbf{r}+\mathbf{r}')}\kappa(e^{\mathbf{r}+\mathbf{r}'}) = \kappa(e^{\mathbf{r}+\mathbf{r}'})$ and $q^{l(\mathbf{r}+\mathbf{r}')}\chi(e^{\mathbf{r}+\mathbf{r}'}) = \chi(e^{\mathbf{r}+\mathbf{r}'})$.

Next we have

$$
\begin{aligned}
&\left[\frac{1}{n}d_0 + \frac{1}{n}\sum_{k=1}^{n} kE_{kk}, E_{ij}(t_0^{j-i+n_1n}e^{\mathbf{r}})\right] \\
&\quad = \frac{j-i+n_1n}{n}E_{ij}(t_0^{j-i+n_1n}e^{\mathbf{r}}) + \frac{1}{n}\sum_{k=1}^{n}[kE_{kk}, E_{ij}(t_0^{j-i+n_1n}e^{\mathbf{r}})] \\
&\quad = \frac{j-i+n_1n}{n}E_{ij}(t_0^{j-i+n_1n}e^{\mathbf{r}}) + \frac{i-j}{n}E_{ij}(t_0^{j-i+n_1n}e^{\mathbf{r}}) \\
&\quad = n_1E_{ij}(t_0^{j-i+n_1n}e^{\mathbf{r}}),
\end{aligned}
$$

which shows (3.11b). □

Remark 3.12. The homomorphism φ is not uniquely determined. Actually, given $a \in \mathbb{C}$ and $c \in \sum_{i=0}^{N} \oplus\mathbb{C}c_i$, one may define a homomorphism φ as follows:

$$
\begin{aligned}
&E_{ij}(s_0^k e^{\mathbf{r}}) \longmapsto q^{j\mathbf{r}}E_{ij}(t_0^{j-i+kn}e^{\mathbf{r}}) - \frac{i+a}{n}\delta_{ij}\delta_{k,0}q^{j\mathbf{r}}\kappa(e^{\mathbf{r}})c_0, \\
&z_0 \longmapsto c_0, \ \ d_0 \longmapsto \frac{1}{n}(d_0 + \sum_{i=1}^{n} iE_{ii} + c), \ \ z_m \longmapsto c_m,
\end{aligned}
$$

for $1 \leqslant i, j \leqslant n$, $k \in \mathbb{Z}$, $\mathbf{r} \in \Lambda$ and $1 \leqslant m \leqslant N$.

Note that $\sum_{\bar{j}-\bar{i}=\bar{k}} \oplus\mathbb{C}E_{ij} = \sum_{i=1}^{n} \oplus\mathbb{C}F^iE^k$, for $1 \leqslant k \leqslant n$. This will enable us to choose a new basis for $\mathcal{G}_p(\Lambda)$ as in the following lemma. The verification of the commutator relation is a routine matter.

LEMMA 3.13.

$$
\mathcal{G}_p^0(\Lambda) = \sum_{i=1}^{n}\sum_{j\in\mathbb{Z}} \oplus F^iE^j(t_0^j\mathcal{R}) \oplus \sum_{i=0}^{N} \oplus\mathbb{C}c_i, \tag{3.14}
$$

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with the Lie bracket

$$
\begin{aligned}
&[F^iE^{j_1}(t_0^{j_1}e^{\mathbf{r}}), F^kE^{j_2}(t_0^{j_2}e^{\mathbf{r}'})] \\
&\quad = \varepsilon^{kj_1}q^{\mathbf{r}j_2}F^{i+k}E^{j_1+j_2}(t_0^{j_1+j_2}e^{\mathbf{r}+\mathbf{r}'}) \\
&\qquad + \frac{1}{n}\kappa(tr((d_0F^iE^{j_1}(t_0^{j_1}e^{\mathbf{r}}))E^kF^{j_2}(t_0^{j_2}e^{\mathbf{r}'})))c_0 + \\
&\qquad + \sum_{m=1}^{N}\chi(tr((d_mF^iE^{j_1}(t_0^{j_1}e^{\mathbf{r}}))E^kF^{j_2}(t_0^{j_2}e^{\mathbf{r}'})))c_m \\
&\quad = \varepsilon^{kj_1}q^{\mathbf{r}j_2}F^{i+k}E^{j_1+j_2}(t_0^{j_1+j_2}e^{\mathbf{r}+\mathbf{r}'}) \\
&\qquad + j_1\delta_{\bar{i}+\bar{k},\bar{0}}\delta_{j_1+j_2,0}\kappa(e^{\mathbf{r}+\mathbf{r}'})\varepsilon^{kj_1}q^{\mathbf{r}j_2}c_0 \\
&\qquad + n\delta_{\bar{i}+\bar{k},\bar{0}}\delta_{j_1+j_2,0}\chi(e^{\mathbf{r}+\mathbf{r}'})\varepsilon^{kj_1}q^{\mathbf{r}j_2}\sum_{m=1}^{N}r_mc_m.
\end{aligned}
\tag{3.15}
$$

Set

$$
A^{(\bar{i})}(\mathbf{r}, z) = \sum_{j\in\mathbb{Z}} F^iE^j(t_0^je^{\mathbf{r}})z^{-j}, \tag{3.16}
$$

for $\mathbf{r} \in \Lambda$, $1 \leqslant i \leqslant n$. Then we have

PROPOSITION 3.17. *In $\mathcal{G}_p(\Lambda)$, we have*

$$
[E(j), A^{(\bar{i})}(\mathbf{r}, z)] = (\varepsilon^{ij} - q^{\mathbf{r}j})z^jA^{(\bar{i})}(\mathbf{r}, z) + j\varepsilon^{ij}\delta_{\bar{i},\bar{0}}\kappa(e^{\mathbf{r}})c_0z^j, \tag{3.18}
$$

$$
[d_0, A^{(\bar{i})}(\mathbf{r}, z)] = -DA^{(\bar{i})}(\mathbf{r}, z), \quad [d_m, A^{(\bar{i})}(\mathbf{r}, z)] = r_mA^{(\bar{i})}(\mathbf{r}, z). \tag{3.19}
$$

Moreover,

$$
\begin{aligned}
&[A^{(\bar{i})}(\mathbf{r}, z_1), A^{(\bar{k})}(\mathbf{r}', z_2)] \\
&\quad = A^{(\bar{i}+\bar{k})}(\mathbf{r}+\mathbf{r}', \varepsilon^{-k}z_1)\delta\left(\frac{\varepsilon^kz_2}{q^{\mathbf{r}}z_1}\right) - \\
&\qquad - A^{(\bar{i}+\bar{k})}(\mathbf{r}+\mathbf{r}', \varepsilon^{-i}z_2)\delta\left(\frac{\varepsilon^iz_1}{q^{\mathbf{r}'}z_2}\right) + \\
&\qquad + \delta_{\bar{i}+\bar{k},\bar{0}}\kappa(e^{\mathbf{r}+\mathbf{r}'})(D\delta)\left(\frac{\varepsilon^kz_2}{q^{\mathbf{r}}z_1}\right)c_0 + \\
&\qquad + n\delta_{\bar{i}+\bar{k},\bar{0}}\chi(e^{\mathbf{r}+\mathbf{r}'})\delta\left(\frac{\varepsilon^kz_2}{q^{\mathbf{r}}z_1}\right)\sum_{m=1}^{N}r_mc_m,
\end{aligned}
\tag{3.20}
$$

where $\mathbf{r} = (r_1, \cdots, r_{N}) \in \Lambda$, $\mathbf{r}' \in \Lambda$, $1 \leqslant i, k \leqslant n, j \in \mathbb{Z}$, $1 \leqslant m \leqslant N$.

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Proof. We only check (3.20). It follows from (3.15) that

$$
\begin{aligned}
&[A^{(\bar{i})}(\mathbf{r}, z_1), A^{(\bar{k})}(\mathbf{r}', z_2)] \\
&\quad = \sum_{j_1, j_2 \in \mathbb{Z}} [F^i E^{j_1}(t_0^{j_1} e^{\mathbf{r}}), F^k E^{j_2}(t_0^{j_2} e^{\mathbf{r}'})] z_1^{-j_1} z_2^{-j_2} \\
&\quad = \sum_{j_1, j_2 \in \mathbb{Z}} \varepsilon^{kj_1} q^{\mathbf{r}j_2} F^{i+k} E^{j_1+j_2}(t_0^{j_1+j_2} e^{\mathbf{r}+\mathbf{r}'}) z_1^{-j_1} z_2^{-j_2} \\
&\qquad - \sum_{j_1, j_2 \in \mathbb{Z}} \varepsilon^{ij_2} q^{\mathbf{r}'j_1} F^{i+k} E^{j_1+j_2}(t_0^{j_1+j_2} e^{\mathbf{r}+\mathbf{r}'}) z_1^{-j_1} z_2^{-j_2} \\
&\qquad + \delta_{\bar{i}+\bar{k},\bar{0}} \sum_{j_1, j_2 \in \mathbb{Z}} j_1 \kappa(t_0^{j_1+j_2} e^{\mathbf{r}+\mathbf{r}'}) \varepsilon^{kj_1} q^{\mathbf{r}j_2} z_1^{-j_1} z_2^{-j_2} c_0 \\
&\qquad + n\delta_{\bar{i}+\bar{k},\bar{0}} \sum_{j_1, j_2 \in \mathbb{Z}} \chi(t_0^{j_1+j_2} e^{\mathbf{r}+\mathbf{r}'}) \varepsilon^{kj_1} q^{\mathbf{r}j_2} z_1^{-j_1} z_2^{-j_2} \sum_{m=1}^{N} r_m c_m \\
&\quad = \sum_{j_1, j_2 \in \mathbb{Z}} F^{i+k} E^{j_1+j_2}(t_0^{j_1+j_2} e^{\mathbf{r}+\mathbf{r}'}) (\varepsilon^{-k} z_1)^{-j_1-j_2} \left(\frac{q^{\mathbf{r}} z_1}{\varepsilon^k z_2}\right)^{j_2} \\
&\qquad - \sum_{j_1, j_2 \in \mathbb{Z}} F^{i+k} E^{j_1+j_2}(t_0^{j_1+j_2} e^{\mathbf{r}+\mathbf{r}'}) (\varepsilon^{-i} z_2)^{-j_1-j_2} \left(\frac{q^{\mathbf{r}'} z_2}{\varepsilon^i z_1}\right)^{j_1} \\
&\qquad + \delta_{\bar{i}+\bar{k},\bar{0}} \kappa(e^{\mathbf{r}+\mathbf{r}'}) \sum_{j_1 \in \mathbb{Z}} j_1 \left(\frac{\varepsilon^k z_2}{q^{\mathbf{r}} z_1}\right)^{j_1} c_0 \\
&\qquad + n\delta_{\bar{i}+\bar{k},\bar{0}} \chi(e^{\mathbf{r}+\mathbf{r}'}) \sum_{j_1 \in \mathbb{Z}} \left(\frac{\varepsilon^k z_2}{q^{\mathbf{r}} z_1}\right)^{j_1} \sum_{m=1}^{N} r_m c_m
\end{aligned}
$$

as wanted. □

Comparing Proposition 3.17 with Proposition 2.35 and using Remark 2.5, one can easily show that the following result holds true.

THEOREM 3.21. *The linear map from the subalgebra $\mathcal{G}_p^0(\Lambda) \oplus \mathbb{C}d_0$ of $\mathcal{G}_p(\Lambda)$ to $\mathcal{V}(\Lambda, q)$ given by*

$$
\begin{aligned}
&F^i E^j(t_0^j e^{\mathbf{r}}) \longmapsto y^{(\bar{i})}(\mathbf{r}, j), \ \text{for } 1 \leqslant i \leqslant n-1, \ \mathbf{r} \in \Lambda, \ j \in \mathbb{Z}; \\
&E^j(t_0^j e^{\mathbf{r}}) \longmapsto \begin{cases} E(j), \ \text{for } \mathbf{r} \in \Lambda_0, \ j \in \mathbb{Z}, \\ y^{(\bar{0})}(\mathbf{r}, j), \ \text{for } \mathbf{r} \in \Lambda \setminus \Lambda_0, \ j \in \mathbb{Z}; \end{cases} \\
&c_0 \longmapsto 1, \quad d_0 \longmapsto d_0 \\
&c_m \longmapsto 0, \ \text{for } 1 \leqslant m \leqslant N,
\end{aligned}
$$

is a Lie algebra homomorphism.

Remark 3.22. In the above theorem, if $\Lambda = \{0\}$, we obtain an irreducible vertex operator representation for the affine Lie algebra $\widetilde{\mathrm{gl}}_{\mathrm{n}}$.

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Recall that $\Lambda_0 = \{\mathbf{r} \in \Lambda : q^{\mathbf{r}} = 1\}$. The pair (Λ, q) is said to be *generic* if $\Lambda_0 = \{0\}$. To get a module for $\mathcal{G}_p(\Lambda)$, we need to assume that (Λ, q) is generic. So from now on we suppose that (Λ, q) is generic. That is $\Lambda_0 = \{0\}$.

Define

$$W_\Lambda = \mathbb{C}[\Lambda] \otimes_{\mathbb{C}} S(\widehat{H}^-), \tag{3.23}$$

and $f \otimes X \in gl(W_\Lambda)$ as

$$(f \otimes X)(g \otimes w) = fg \otimes Xw$$

for $f, g \in \mathbb{C}[\Lambda]$, $X \in \mathcal{V}(\Lambda, q)$, $w \in S(\widehat{H}^-)$. Let $\mathcal{W}(\Lambda, q)$ be the linear span of operators

$$\begin{aligned}
&e^{\mathbf{r}} \otimes y^{(\bar{i})}(\mathbf{r}, j), \quad 1 \leqslant i \leqslant n-1, j \in \mathbb{Z}, \mathbf{r} \in \Lambda;\\
&e^{\mathbf{r}} \otimes y^{(\bar{0})}(\mathbf{r}, j), \quad j \in \mathbb{Z}, \mathbf{r} \in \Lambda \setminus \{0\};\\
&1 \otimes E(j), \quad j \in \mathbb{Z};\\
&1 \otimes 1, \ 1 \otimes d_0,\\
&d_m \otimes 1, \ \mathrm{for} 1 \leqslant \mathrm{m} \leqslant \mathrm{N}.
\end{aligned} \tag{3.24}$$

Then it follows from Proposition 2.35 that those operators satisfy the same derived relations from (2.36) through (2.39). Hence, $\mathcal{W}(\Lambda, q)$ is a Lie subalgebra of $gl(W_\Lambda)$. This Lie algebra $\mathcal{W}(\Lambda, q)$ is the lifting of $\mathcal{V}(\Lambda, q)$.

Now we can state our main theorem.

THEOREM 3.25. *The linear map $\pi : \mathcal{G}_p(\Lambda) \to \mathcal{W}(\Lambda, q)$ given by*

$$\begin{aligned}
&\pi(F^i E^j(t_0^j e^{\mathbf{r}})) = e^{\mathbf{r}} \otimes y^{(\bar{i})}(\mathbf{r}, j), \ \mathit{for}\ 1 \leqslant i \leqslant n-1, j \in \mathbb{Z}, \mathbf{r} \in \Lambda;\\
&\pi(E^j(t_0^j e^{\mathbf{r}})) = \begin{cases} 1 \otimes E(j), \ \mathit{for}\ j \in \mathbb{Z}, \mathbf{r} = 0,\\ e^{\mathbf{r}} \otimes y^{(\bar{0})}(\mathbf{r}, j), \ \mathit{for}\ j \in \mathbb{Z}, \mathbf{r} \in \Lambda \setminus \{0\}; \end{cases}\\
&\pi(c_0) = 1 \otimes 1, \quad \pi(d_0) = 1 \otimes d_0;\\
&\pi(c_m) = 0, \quad \pi(d_m) = d_m \otimes 1, \mathit{for} 1 \leqslant m \leqslant N,
\end{aligned}$$

is a Lie algebra homomorphism. If Λ is a group, then W_Λ is irreducible as $\mathcal{G}_p(\Lambda)$ module.

Proof. It follows from (2.35) and (3.17) that π is a Lie algebra homomorphism. Let us check the irreducibility when Λ is a group.

Let U be a nonzero submodule of $W_\Lambda = \mathbb{C}[\Lambda] \otimes_{\mathbb{C}} S(\widehat{H}^-)$. Since the Heisenberg algebra $\mathfrak{s} = \widehat{H}^+ + \mathbb{C}c_0 + \widehat{H}^-$ is a subalgebra of $\mathcal{G}_p(\Lambda)$, Lemma 9.13 in [K] (or Theorem 1.7.3 in [FLM]) implies that U is completely reducible as $\mathfrak{s}$-module and so $U = V \otimes S(\widehat{H}^-)$ for some subspace V of $\mathbb{C}[\Lambda]$. Thanks to the degree operators d_m for $1 \leqslant m \leqslant N$, we see that $U = \sum_{\mathbf{r} \in \Lambda'} \oplus (e^{\mathbf{r}} \otimes S(\widehat{H}^-))$ for some subset Λ' of Λ.

Assume that $e^{r_0} \otimes 1 \in U$, then for $\boldsymbol{r} \in \Lambda$ and $\boldsymbol{r} \neq \boldsymbol{r}_0$, we have

$$\begin{aligned}&(e^{\boldsymbol{r}-\boldsymbol{r}_0} \otimes y^{(\bar{0})}(\boldsymbol{r}-\boldsymbol{r}_0, 0))(e^{\boldsymbol{r}_0} \otimes 1)\\ &\quad = e^{\boldsymbol{r}} \otimes (y^{(\bar{0})}(\boldsymbol{r}-\boldsymbol{r}_0, 0)1)\\ &\quad = \frac{1}{1-q^{\boldsymbol{r}-\boldsymbol{r}_0}} e^{\boldsymbol{r}} \otimes 1 \in U\end{aligned}$$

by Lemma 2.42. We thus obtain that $\boldsymbol{r} \in \Lambda'$ for all $\boldsymbol{r} \in \Lambda$ and so $U = W_\Lambda$. □

4. Extended Affine Lie Algebras

The notion of extended affine Lie algebras was first introduced in [H-KT] (under the name of irreducible quasi-simple Lie algebras) and systematically studied in [AABGP] and [BGK]. They can be roughly characterized as complex Lie algebras which have a nondegenerate invariant form, a finite-dimensional Cartan subalgebra, a discrete irreducible root system, and ad-nilpotency of nonisotropic root spaces. This new class of Lie algebras is closely related to the extended affine root systems introduced in [Sa] for the study of elliptic singularities, the intersection matrix algebras in [Sl], and the Lie algebras graded by finite root systems studied by [BM], [BZ], [Se] and [N].

In this section, we will apply the results in Section 3 to obtain irreducible representations of extended affine Lie algebras of type A_{n-1} coordinatized by certain quantum tori with ν variables.

Let $(\Lambda, q) = (\mathbb{Z}^{\nu-1}, q)$, where $q = (q_1, \cdots, q_{\nu-1})$. Note that we still assume (Λ, q) is generic.

Let ε_i be the vector in $\mathbb{Z}^{\nu-1}$ which is 1 in the ith entry and 0 everywhere else, for $1 \leqslant i \leqslant \nu-1$. Write $e^{\varepsilon_i} = t_i$. Then

$$\mathcal{R}[s_0, s_0^{-1}; \sigma^n] = \mathbb{C}_{Q_n}[s_0^{\pm 1}, t_1^{\pm 1}, \cdots, t_{\nu-1}^{\pm 1}] \tag{4.1}$$

and

$$\mathcal{R}[t_0, t_0^{-1}; \sigma] = \mathbb{C}_Q[t_0^{\pm 1}, t_1^{\pm 1}, \cdots, t_{\nu-1}^{\pm 1}], \tag{4.2}$$

where $Q = (q_{ij})$ with

$$q_{i0} = q_i, \text{ for } 1 \leqslant i \leqslant \nu - 1 \tag{4.3}$$

and

$$q_{ij} = 1, \text{ for all other } i \text{ and } j,\ 0 \leqslant i, j \leqslant \nu - 1,$$

and $Q_n = (q_{ij}^n)$.

Let $\mathcal{G}^0(\Lambda)$ and $\mathcal{G}(\Lambda)$ be defined as in (3.4) and (3.6) respectively. The nondegenerate invariant form on $\mathcal{G}(\Lambda)$ can be defined as

$$\begin{aligned}&(E_{ij}(u), E_{kl}(v)) = \delta_{jk}\delta_{il}\kappa(uv),\\&(E_{ij}(u), c_m) = (E_{ij}(u), d_m) = 0,\\&(c_m, d_r) = \delta_{mr},\end{aligned} \tag{4.4}$$

for $u, v \in \mathbb{C}_{Q_n}$, $1 \leqslant i, j, k, l \leqslant n$, $0 \leqslant m, r \leqslant \nu - 1$.

$\mathcal{G}(\Lambda)$ has the Cartan subalgebra

$$\mathcal{H} = \mathfrak{h} \oplus \sum_{i=0}^{\nu-1} \oplus \mathbb{C}c_i \oplus \sum_{i=0}^{\nu-1} \oplus \mathbb{C}d_i, \tag{4.5}$$

where $\mathfrak{h} = \sum_{i=1}^{n} \oplus \mathbb{C}E_{ii}$.

Define $\tau_i \in \mathcal{H}^*$ as follows:

$$\tau_i|_{\mathfrak{h}\oplus\sum_{k=0}^{\nu-1}\oplus\mathbb{C}c_k} = 0,\ \tau_i(d_j) = \delta_{ij}, \tag{4.6}$$

for $0 \leqslant i, j \leqslant \nu - 1$. Then the root system of $\mathcal{G}(\Lambda)$ with respect to $\mathcal{H}$ is

$$R = \left(\Delta + \sum_{i=0}^{\nu-1} \mathbb{Z}\tau_i\right) \cup \left(\sum_{i=0}^{\nu-1} \oplus \mathbb{Z}\tau_i\right), \tag{4.7}$$

where $\Delta = \{\theta_i - \theta_j : 1 \leqslant i \neq j \leqslant n\}$ is the root system of type A_{n-1}, and the root space decomposition is as follows:

$$\mathcal{G}(\Lambda) = \sum_{\alpha\in R} \oplus \mathcal{G}_\alpha, \tag{4.8}$$

where

$$\begin{aligned}&\mathcal{G}_0 = \mathcal{H};\\&\mathcal{G}_{\theta_i-\theta_j+m_0\tau_0+\cdots+m_{\nu-1}\tau_{\nu-1}} = \mathbb{C}E_{ij}(s_0^{m_0}t^{}),\\&\text{for } 1 \leqslant i \neq j \leqslant n,\ m_0 \in \mathbb{Z}, = (m_1, \cdots, m_{\nu-1}) \in \Lambda = \mathbb{Z}^{\nu-1},\\&\mathcal{G}_{m_0\tau_0+\cdots+m_{\nu-1}\tau_{\nu-1}} = \sum_{i=1}^{n} \oplus \mathbb{C}E_{ii}(s_0^{m_0}t^{}),\\&\text{for } m_0 \in \mathbb{Z}, \in \Lambda \text{ but } (m_0,) \neq (0, 0).\end{aligned}$$

This Lie algebra $\mathcal{G}(\Lambda)$ is an extended affine Lie algebra of nullity ν (see [AABGP] and [BGK]). $\sum_{i=0}^{\nu-1} \oplus \mathbb{Z}\tau_i$ are called isotropic roots while $\Delta + \sum_{i=0}^{\nu-1} \mathbb{Z}\tau_i$ are nonisotropic roots.

Now from Proposition 3.10 we see that $\mathcal{G}_p(\Lambda) \cong \mathcal{G}(\Lambda)$. Theorem 3.25 immediately gives us the following result.

PROPOSITION 4.10. *For* $(\Lambda, q) = (\mathbb{Z}^{\nu-1}, q)$, W_Λ *is an irreducible* $\mathcal{G}(\Lambda)$*-module.*

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Remark 4.11. Note that the coordinate algebra in $\mathcal{G}_p(\Lambda)$ is the quantum torus $\mathbb{C}_Q$ while the coordinate algebra in $\mathcal{G}(\Lambda)$ is $\mathbb{C}_{Q_n}$, where Q is given in (4.3).

Remark 4.12. It is not difficult to see that W_Λ has a weight space decomposition with respect to the Cartan subalgebra $\mathcal{H}$. Moreover, each weight space is finite-dimensional.

Next we further consider a subalgebra of $\mathcal{G}(\Lambda)$ which is the so-called tame extended affine Lie algebra. The tameness was introduced in [BGK] in order to classify all extended affine Lie algebras (see also [AABGP]).

Set $sl_n(\mathbb{C}_{Q_n}) = \{X \in gl_n(\mathbb{C}_{Q_n}) : \mathrm{tr(X)} \in [\mathbb{C}_{\mathrm{Q_n}}, \mathbb{C}_{\mathrm{Q_n}}]\}$ to be the subalgebra of $\mathrm{gl_n}(\mathbb{C}_{Q_n})$ which is generated by $E_{ij}(u)$, $u \in \mathbb{C}_{Q_n}$, $1 \leqslant i \neq j \leqslant n$. Define

$$\mathcal{L}_c(\Lambda) = sl_n(\mathbb{C}_{Q_n}) \oplus \sum_{i=0}^{\nu-1} \oplus \mathbb{C}c_i \tag{4.13}$$

to be the subalgebra of $\mathcal{G}^0(\Lambda)$, and let

$$\mathcal{L}(\Lambda) = \mathcal{L}_c(\Lambda) \oplus \sum_{i=0}^{\nu-1} \oplus \mathbb{C}d_i \tag{4.14}$$

be the subalgebra of $\mathcal{G}(\Lambda)$. The restriction of the invariant form on $\mathcal{L}(\Lambda)$ is also nondegenerate. This Lie algebra $\mathcal{L}(\Lambda)$ is a tame extended affine Lie algebra. It has the same root system R as $\mathcal{G}(\Lambda)$ and the following root space decomposition:

$$\mathcal{L}(\Lambda) = \oplus_{\alpha \in R} \mathcal{L}_\alpha, \tag{4.15}$$

where

$$\mathcal{L}_0 = \sum_{i=1}^{n-1} \oplus \mathbb{C}(E_{ii} - E_{i+1,i+1}) \oplus \sum_{i=0}^{\nu-1} \oplus \mathbb{C}c_i \oplus \sum_{i=0}^{\nu-1} \oplus \mathbb{C}d_i$$

is the Cartan subalgebra of $\mathcal{L}(\Lambda)$,

$$\mathcal{L}_\alpha = \mathcal{G}_\alpha$$

for $\alpha \in \Delta + \sum_{i=0}^{\nu-1} \mathbb{Z}\tau_i$, and

$$\begin{aligned}\mathcal{L}&_{m_0\tau_0+\cdots+m_{\nu-1}\tau_{\nu-1}} \\ &= \sum_{i=1}^{n-1} \oplus \mathbb{C}(E_{ii} - E_{i+1,i+1})(s_0^{m_0} t) \oplus I_n((\mathbb{C}s_0^{m_0} t) \cap [\mathbb{C}_{Q_n}, \mathbb{C}_{Q_n}])\end{aligned} \tag{4.16}$$

for $(m_0,) = (m_0, m_1, \cdots, m_{\nu-1}) \in \mathbb{Z}^\nu \setminus \{0\}$, where I_n is the $n \times n$ identity matrix.

By taking the restriction, we know that W_Λ is an $\mathcal{L}(\Lambda)$-module.

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THEOREM 4.17. *W_Λ is an irreducible $\mathcal{L}(\Lambda)$-module.*

Proof. To check the irreducibility, we need to show that $t_0^i, t^{\mathbf{r}} \in [\mathbb{C}_Q, \mathbb{C}_Q]$, for $i \in \mathbb{Z} \setminus \{0\}$ and $\mathbf{r} \in \Lambda \setminus \{0\}$. Indeed, if $t_0^i \in [\mathbb{C}_Q, \mathbb{C}_Q]$ for $\in \mathbb{Z} \setminus \{0\}$, then the Heisenberg subalgebra $\mathfrak{s}$ is contained in $\varphi(\mathcal{L}(\Lambda))$. If $t^{\mathbf{r}} \in [\mathbb{C}_Q, \mathbb{C}_Q]$, then we will be able to use the operator $y^{(\bar{0})}(\mathbf{r} - \mathbf{r}_0, 0)$ to prove the irreducibility as was done in Theorem 3.25. Since

$$(1 - q_1^i)t_0^i = (t_0^i t_1^{-1})t_1 - t_1(t_0^i t_1^{-1}) \text{ and } (1 - q^{\mathbf{r}})t^{\mathbf{r}} = t_0(t_0^{-1} t^{\mathbf{r}}) - (t_0^{-1} t^{\mathbf{r}})t_0, \tag{4.18}$$

the proof is thus completed. □

Remark 4.19. Note that if $(\Lambda, q) = (\mathbb{Z}^{\nu-1}, q)$ is generic, then $\mathcal{G}(\Lambda) = \mathcal{L}(\Lambda) \oplus \mathbb{C}I_n$ if and only if $\nu = 2$.

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