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THE HYPERCENTRE AND THE n-CENTRE OF THE UNIT GROUP OF AN INTEGRAL GROUP RING

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ABSTRACT. In this paper, we first show that the central height of the unit group of the integral group ring of a periodic group is at most 2. We then give a complete characterization of the n-centre of that unit group. The n-centre of the unit group is either the centre or the second centre (for $n \geq 2$).

1**. Introduction.** Let n be an integer. Two elements x, y in a group G n-commute (see R. Baer [3, 4]) if

$$(xy)^n = x^n y^n \quad \text{and} \quad (yx)^n = y^n x^n.$$

A group is n-abelian if any two elements n-commute. In [3], R. Baer introduced the n-centre $Z(G, n)$ of a group G as the set of those elements which n-commute with every element in the group. Later L. C. Kappe and M. L. Newell [14] proved that $(ax)^n = a^n x^n$ for all $x \in G$ implies $(xa)^n = x^n a^n$ for all $x \in G$, and vice versa. Thus only one of the n-commutativity conditions suffices to define the n-centre $Z(G, n)$.

The n-centre, which can readily be seen to be a characteristic subgroup, shares many properties with the centre, some of which already have been explored in R. Baer [3, 4]. For example, it follows from Corollary 1 in R. Baer [4] that a group is n-abelian if the quotient modulo its n-centre is (locally) cyclic. In [14], L. C. Kappe and M. L. Newell shed further light on these similarities by investigating various characterizations and embedding properties of the n-centre. They characterized the n-centre as the margin of the n-commutator word $(xy)^n y^{-n} x^{-n}$(see also L. C. Kappe [12, 13] and G. T. Hogan and W. P. Kappe [11]), and their result yields some interesting connections with a conjecture of P. Hall on margins (see P. Hall, [9, 10]).

Let $\mathbb{Z}G$ denote the integral group ring of a group G and $\mathcal{U} = \mathcal{U}(\mathbb{Z}G)$ the unit group of such a group ring. In this paper, we investigate the n-centre of the unit group of an integral group ring $\mathbb{Z}G$ for a periodic group G. We first consider, in Section 2, the hypercentre of the unit group $\mathcal{U}$ of the integral group ring of a periodic group. We prove that the central height of $\mathcal{U}$ is always at most 2. When G is finite, this result was proved by S. R. Arora, A. W. Hales and I. B. S. Passi in [1] (Theorem 2.6). We then, in Section 3, apply this to the n-centre of $\mathcal{U}$. It is obvious that the 2-centre of a group is equal to its centre. It turns out that the 3-centre of the unit group of an integral group ring of a periodic group also coincides with the centre of that unit group (Theorem 3). Our main result is to give a complete characterization of the n-centre of the unit group $\mathcal{U}$ for any integer n. We

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prove that this n-centre (for $n \geq 2$) coincides with either the centre $Z_1(\mathcal{U})$ of the unit group or the second centre $Z_2(\mathcal{U})$ of the unit group (Theorem 5). Combined with ([2], Theorem 3.1), this yields that the n-centre (for $n \geq 2$) is either the centre $Z_1(\mathcal{U})$ or the product of the centre with torsion hypercentral units, $Z_1(\mathcal{U})T$, when G is finite.

The work described here is part of the author's doctoral dissertation.

2. The hypercentre of the unit group.

Let G be an arbitrary group and let

$$(1) = Z_0(\mathcal{U}) \leq Z_1(\mathcal{U}) \leq \cdots \leq Z_n(\mathcal{U}) \leq \cdots$$

be the upper central series of the unit group $\mathcal{U}$. Let $\tilde{Z}(\mathcal{U}) = \bigcup_{n=1}^{\infty} Z_n(\mathcal{U})$. Then $\tilde{Z}(\mathcal{U})$ is a normal subgroup of $\mathcal{U}$ and is called the *hypercentre* of $\mathcal{U}$. Let G be a periodic group and let $T = T\big(\tilde{Z}(\mathcal{U})\big)$ denote the set of all torsion units in $\tilde{Z}(\mathcal{U})$ having augmentation 1. Since $T = \bigcup_{n=1}^{\infty} T\big(Z_n(\mathcal{U})\big)$, and $T_n' = \big\{\pm u \mid u \in T\big(Z_n(\mathcal{U})\big)\big\}$ is a characteristic subgroup of $Z_n(\mathcal{U})$ for each n, it follows that T is a periodic normal subgroup of $\mathcal{U}$.

Now the results of A. A. Bovdi [5,6] apply to give the following:

THEOREM 1. *Let G be a periodic group. Then exactly one of the following occurs:*
(1) G is a Hamiltonian 2-group and $T = G$;
(2) $T = Z_1(G)$;
(3) G has an Abelian normal subgroup H of index 2 containing an element a of order 4 such that for each $g \in G \setminus H$, $g^2 = a^2$ and $ghg^{-1} = h^{-1}$ for all $h \in H$, and $T = \langle a\rangle \oplus E = Z_2(\mathcal{U}) \cap Z_2(G)$ where E is an elementary Abelian 2-group.

The proof of Theorem 1 is similar to that given by S. R. Arora, A. W. Hales and I. B. S. Passi in [1].

Recalling Theorem 12.5.4 in M. Hall [8], we have the following:

REMARK 1. In case (1) of Theorem 1, $G = Q \oplus E$ where Q is the quaternion group of order 8 and E is an elementary Abelian 2-group. Furthermore, $\mathcal{U} = \pm G$.

COROLLARY 1. *Let G be a periodic group. Then $T \leq Z_2(G)$.*

We first prove the following lemma which is needed for proving the main Theorem 2.

LEMMA 1. *Let G be any periodic group. Then $Z_2(\mathcal{U}) \subseteq N_{\mathcal{U}}(G)$.*

PROOF. Let $v \in Z_2(\mathcal{U})$, and $g \in G$. Then $[v,g] = vgv^{-1}g^{-1} = c \in Z_1(\mathcal{U})$. It follows that $o(cg) = o(vgv^{-1}) < \infty$, and therefore c is of finite order. In view of S. K. Sehgal ([17], p. 46), we conclude that c is a trivial unit. Consequently, $vgv^{-1} = cg \in G$ and this leads to the desired result. ∎

If H and K are subsets of a group G, then we denote by $[H,K]$ the subgroup of G generated by the commutors $[h,k] = hkh^{-1}k^{-1}$, $h \in H$, $k \in K$. Now we prove the main result of this section—the central height of the unit group of an integral group ring of a periodic group is at most 2.

THEOREM 2. *Let G be a periodic group. Then $Z_3(\mathcal{U}) = Z_2(\mathcal{U})$.*

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PROOF. First we prove that

$$[Z_2(\mathcal{U}), \mathcal{U}] \subseteq Z_1(G). \tag{2.1}$$

For any $u_2 \in Z_2(\mathcal{U})$, we have $u_2^2 = ag$ for some $a \in Z_1(\mathcal{U})$ and $g \in G$, by Lemma 1 and S. K. Sehgal ([18], Proposition 9.5). Hence there exists a positive integer $n(u_2)$ such that $(u_2)^{n(u_2)} \in Z_1(\mathcal{U})$. Now for any $u \in \mathcal{U}$, we have that $[u_2, u] = u_2uu_2^{-1}u^{-1} = c$ so $uu_2^{-1}u^{-1} = u_2^{-1}c$, where c is a central unit. By taking the $n(u_2)$-th power of both sides of the above identity, we obtain that $uu_2^{-n(u_2)}u^{-1} = u_2^{-n(u_2)}c^{n(u_2)}$. This forces $c^{n(u_2)} = 1$ since $u_2^{n(u_2)}$ is a central unit and therefore, $c \in Z_1(G)$ by S. K. Sehgal ([17], p. 46). Finally, we conclude that $[Z_2(\mathcal{U}), \mathcal{U}] \subseteq Z_1(G)$.

Next we prove that

$$Z_{n+1}^2(\mathcal{U}) \subseteq Z_n(\mathcal{U}) \quad \text{for all } n \geq 1. \tag{2.2}$$

We first prove that $Z_2^2(\mathcal{U}) \subseteq Z_1(\mathcal{U})$ by contradiction. Assume that $Z_2^2(\mathcal{U}) \not\subseteq Z_1(\mathcal{U})$. Since $Z_2^2(\mathcal{U}) \subseteq N_{\mathcal{U}}^2(G) \subseteq GZ_1(\mathcal{U})$ as seen earlier, there exists a group element $g \in Z_2(\mathcal{U}) \setminus Z_1(\mathcal{U})$. Let $u \in \mathcal{U}$. Then $[u, g] = g_0 \in Z_1(G)$. Hence there exists a positive integer $n = n(u)$ such that $u^n g u^{-n} = g$. It follows from Theorem 1.2 of M. M. Parmenter [15] that the exponent of $Z_1(G)$ is 2. Therefore, for all $u_2 \in Z_2(\mathcal{U})$ and all $g' \in G$, we have $[u_2^2, g'] = [u_2, g']^2 = (g_0')^2 = 1$. This means that u_2^2 is a central unit, forcing $Z_2^2(\mathcal{U}) \subseteq Z_1(\mathcal{U})$. This contradiction finishes the proof.

The proof continues by induction on n. We just proved that the result is true for $n = 1$. Assume that the result is also true for $n = k - 1 \geq 1$. Now consider the case $n = k$. Let $u \in \mathcal{U}$ and $u_{k+1} \in Z_{k+1}(\mathcal{U})$. Then $[u_{k+1}, u] = u_k \in Z_k(\mathcal{U})$. It in turn yields that $[u_{k+1}^2, u] = [u_{k+1}, u_k]u_k^2 \in Z_{k-1}(\mathcal{U})$ by the inductive assumption, and therefore we conclude $u_{k+1}^2 \in Z_k(\mathcal{U})$. We are done.

Moreover, in view of the fact that for any $u_3 \in Z_3(\mathcal{U})$, $u \in \mathcal{U}$, $[u_3, u]^2 = [u_3, [u_3, u]]^{-1}[u_3^2, u] \in Z_1(G)$ by (2.1) and (2.2), we conclude that

$$[Z_3(\mathcal{U}), \mathcal{U}] \subseteq T. \tag{2.3}$$

Now we are ready to prove our main result: $Z_3(\mathcal{U}) = Z_2(\mathcal{U})$.

According to Theorem 1, we need to deal with the following three cases.

(a) Suppose that G is a Hamiltonian 2-group. Then $\mathcal{U} = \pm G = \pm T = Z_2(\mathcal{U})$ and we are done.

(b) Suppose that T is a central subgroup of $\mathcal{U}$. Then the result follows immediately from (2.3).

(c) Suppose that T is abelian but not a central subgroup. Then $G = \langle H, g \rangle$ is a group of the type (3) in Theorem 1 and therefore, $T = \langle a \rangle \oplus E = Z_2(\mathcal{U}) \cap Z_2(G)$. In this case, we first observe that $Z_1(G) = \{x \in G \mid x^2 = 1\}$ and the exponent of T is 4.

Let $\mathcal{B}_1$ denote the subgroup of $\mathcal{U}(\mathbb{Z}G)$ generated by all bicyclic units. Next we prove the following result:

$$[Z_3(\mathcal{U}), \mathcal{B}_1] = 1. \tag{2.4}$$

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We first show that $[Z_2(\mathcal{U}), \mathcal{B}_1] = 1$. Let $u_2 \in Z_2(\mathcal{U})$ and $u_{b,a} = 1 + (1-b)a\hat{b}$ be a bicyclic unit. Then $[u_2, u_{b,a}] = c_0 \in Z_1(G)$ by (2.1). Therefore, there exists a positive integer n such that $[u_2, u_{b,a}]^n = c_0^n = 1$. It in turns yields that $[u_2, u_{b,a}^n] = 1$ since $[u_2, u_{b,a}^n] = [u_2, u_{b,a}]u_{b,a}\,[u_2, u_{b,a}^{n-1}]u_{b,a}^{-1} = [u_2, u_{b,a}][u_2, u_{b,a}^{n-1}]$ (since $[u_2, u_{b,a}^{n-1}] \in Z_1(G)$ by (2.1)) $= [u_2, u_{b,a}]\,[u_2, u_{b,a}]^{n-1}$ (by inductive assumption) $= [u_2, u_{b,a}]^n$. Observing that $u_{b,a}^n = 1 + n(1-b)a\hat{b}$, we obtain that $[u_2, u_{b,a}] = 1$ and this leads to the desired result. Next let $u_3 \in Z_3(\mathcal{U})$ and b be a bicyclic unit. Then $[u_3, b] \in T$ by (2.3), and hence $[u_3, b]^n = 1$ for some positive integer n. Note that $[u_3, b^n] = [u_3, b^{n-1}]b^{n-1}[u_3, b]b^{-(n-1)} = [u_3, b^{n-1}]\big[b^{n-1}, [u_3, b]\big][u_3, b]$ and, $[b^{n-1}, [u_3, b]] = 1$ since $[u_3, b] \in Z_2(\mathcal{U})$. We conclude, by induction, that $[u_3, b^n] = [u_3, b^{n-1}][u_3, b] = [u_3, b]^{n-1}[u_3, b] = [u_3, b]^n = 1$. Therefore, $[u_3, b] = 1$ as seen earlier and we are done.

Now we claim that

$$[Z_3(\mathcal{U}), G] \subseteq \langle a^2 \rangle. \tag{2.5}$$

The proof of (2.5) is omitted since it is similar to that of Theorem 2.6 in [1] except that (2.4) is used instead of Proposition 2.3(iii) in [1]. Therefore, we have $[Z_3^2(\mathcal{U}), G] = 1$ and

$$Z_3^2(\mathcal{U}) \subseteq Z_1(\mathcal{U}). \tag{2.6}$$

Suppose that there exists $x \in Z_3(\mathcal{U}) \setminus Z_2(\mathcal{U})$. Then, for some $u \in \mathcal{U}$, $[x, u] = t \in T$ is an element of order 4. Clearly, $t \in G'$, the derived group of G (mapping t into $\mathbf{Z}(G/G')$, we obtain that $\bar{t} = [\bar{x}, \bar{u}] = \bar{1}$ since $\mathbf{Z}(G/G')$ is a commutative group ring. Thus $t - 1 \in \triangle(G')\mathbb{Z}G$. This implies $t \in G'$). It is not hard to check that, in this case, $G' = \{h^2 \mid h \in H\}$. Note that $[x, h] \in \langle a^2 \rangle \subseteq Z_1(\mathcal{U})$ for all $h \in H$ by (2.5), so $[x, h]^2 = 1$. It follows that $[x, t] = [x, h^2]$(for some $h \in H$) $= [x, h]h[x, h]h^{-1} = [x, h]^2 = 1$. Hence $[x^2, u] = x[x, u]x^{-1}[x, u] = [x, t]t^2 = t^2 \neq 1$. However, in view of (2.6), we have $[x^2, u] = 1$, a contradiction. Thus we must have $Z_2(\mathcal{U}) = Z_3(\mathcal{U})$ always. ∎

COROLLARY 2. *Let G be a periodic group. If all central units are trivial, then all hypercentral units are trivial too.*

PROOF. Let $u \in \tilde{Z}(\mathcal{U})$. Then $u \in N_{\mathcal{U}}(G)$ by Theorem 2 and Lemma 1. It follows from S. K. Sehgal ([18], Proposition 9.4) that $uu^* = g \in Z_1(G)$ and hence $uu^* = 1$. Consequently, u is trivial and we are done. ∎

By recalling the result in J. Ritter and S. K. Sehgal [16] giving necessary and sufficient conditions for all central units to be trivial when G is finite, we obtain the following necessary and sufficient conditions for all hypercentral units to be trivial.

COROLLARY 3. *Let G be a finite group. All hypercentral units of ZG are trivial if and only if for every $x \in G$ and every natural number j relatively prime to $|G|$, x^j is conjugate to x or x^{-1}.*

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3. **The *n*-centre of the unit group of an integral group ring.** In this section, we apply the result on the hypercentre of $\mathcal{U}$ (Theorem 2) to the n-centre of $\mathcal{U}$. Subsection 3.1 introduces some basic results and notation on the n-centre. It turns out that the 3-centre of the unit group of the integral group ring of a periodic group also coincides with the centre of that unit group. In subsection 3.2, we give a complete characterization of the n-centre of the unit group of the integral group ring of a periodic group.

3.1. *Basic results and notation.* We first introduce some basic definitions and notation. Then we recall some fundamental results which will be needed later in this paper. Other notation follow L. C. Kappe and M. L. Newell [14].

Let

$$S_1(G,n) = \{a \in G \mid (ax)^n = a^n x^n \ \forall x \in G\}$$

and

$$S_2(G,n) = \{a \in G \mid (xa)^n = x^n a^n \ \forall x \in G\}.$$

R. Baer first defined the n-centre in [3] as

$$Z(G,n) = S_1(G,n) \cap S_2(G,n).$$

However, L. C. Kappe and M. L. Newell proved that $S_1(G,n) = S_2(G,n)$ ([14], Theorem 2.1). Thus only one of the n-commutativity conditions suffices to define the n-centre.

The following proposition collects various facts about the elements in the n-centre. Note that $Z(G,1) = Z(G,0) = G$.

PROPOSITION 1 ([14], LEMMA 2.2). *Let $a \in Z(G,n)$. Then*

(1) $[a^{n-1}, x^n] = 1$ *for all* $x \in G$;

(2) $a \in Z(G, 1-n)$ *(Therefore always* $Z(G,n) = Z(G,1-n)$*)*;

(3) $[a^n, x] = [a,x]^n = [a, x^n]$ *for all* $x \in G$;

(4) $1 = [a, x^{n(1-n)}] = [a^{n(1-n)}, x] = [a,x]^{n(1-n)} = [a^n, x^{1-n}]$ *for all* $x \in G$;

(5) $a^n \in Z(G, n-1)$.

It can be easily seen by the definition that the 2-centre of a group coincides with its centre. Even a better result can be obtained when we investigate the 3-centre of the unit group $\mathcal{U}$ of an integral group ring of a periodic group G. We will show that the 3-centre $Z(\mathcal{U},3)$ of the unit group also coincides with its centre $Z_1(\mathcal{U})$. In the next subsection, a complete characterization of $Z(\mathcal{U},n)$ will be obtained for all n.

THEOREM 3. *Let G be a periodic group. Then*

$$Z(\mathcal{U},3) = Z(\mathcal{U},2) = Z_1(\mathcal{U}).$$

The following proposition due to L. C. Kappe and M. L. Newell is needed in the proof of Theorem 3.

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PROPOSITION 2 ([14] THEOREM 4.3). *Let G be a group. Then*

$$Z(G,3) = \{a \in R_2(G) \mid a^3 \in Z_1(G)\} \quad \text{and} \quad Z(G,3) \subseteq Z_3(G).$$

Here $Z_m(G)$ is the m-th centre of G and $R_m(G) = \{a \in G \mid [a,_m x] = 1\ \forall x \in G\}$ denotes the set of right m- Engel elements, where

$$[x,_m y] = \big[[x,_{m-1} y], y\big] \quad \text{and} \quad [x,_1 y] = [x,y].$$

Now we are ready to prove Theorem 3

PROOF. Recall that $Z(U,3) \subseteq Z_3(U)$ by Proposition 2 and also that $Z_3(U) = Z_2(U)$ and $Z_2^2(U) \subseteq Z_1(U)$ by Theorem 2 and its proof. It follows that for all $u \in Z(U,3)$, $u^2 \in Z_1(U)$. Also note that $u^3 \in Z_1(U)$ by Proposition 2 (or by Proposition 1(5)). Thus $u \in Z_1(U)$ and $Z(U,3) \subseteq Z_1(U)$. We are done. ∎

3.2. *The main result.* In this section, we investigate the n-centre of the unit group of an integral group ring for $n \geq 4$. We first characterize periodic Q^*-groups as precisely those periodic groups which contain a noncentral element lying in the 4-centre of U. Then we turn our attention to studying the set of all torsion units in $Z(U,n)$. Our main result is Theorem 5, which gives a complete characterization of the n-centre of the unit group of an integral group ring for any periodic group.

A group G is said to be a *Q^*-group* if G has an Abelian normal subgroup A of index 2 which has an element a of order 4 such that for all $h \in A$ and all $g \in G \setminus A$, $g^2 = a^2$ and $g^{-1}hg = h^{-1}$. We note that finite Q^*-groups have played a significant role in work by S. R. Arora and I. B. S. Passi [2] (see also[1]), where they are characterized as precisely those groups G with the property that U is of central height 2. Such groups also appear in a paper by A. Williamson [19], who showed that Q^* groups are exactly those groups containing a noncentral element a which has finitely many conjugates in U. Recently, M. M. Parmenter [15] showed that a weaker conjugation condition also characterizes these groups. For our purpose, we characterize Q^* groups by the 4-centre of the unit group.

THEOREM 4. *Let G be a periodic group. Then the following are equivalent:*

(1) G is a Q^*-group;

(2) G contains a noncentral element a such that $a \in Z(U,4)$;

(3) G contains a noncentral element a such that $a \in Z(U,n)$ for some $n \geq 4$.

To prove Theorem 4, we need the following results. The first one is proved by M. M. Parmenter in [15] (Theorem 1.2).

PROPOSITION 3. *Let G be a periodic group. Then the following are equivalent:*

(1) G contains a noncentral element a with the property that given any unit u in U, there exists a positive integer $n = n(u)$ such that $u^n a u^{-n}$ belongs to G.

(2) G is a Q^*-group.

The following proposition establishes a relationship between the 4-centre and the second centre of the unit group of an integral group ring.

PROPOSITION 4. *Let G be a periodic group. Then $Z_2(U) \subseteq Z(U,4)$.*

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PROOF. Let $u \in Z_2(\mathcal{U})$ and $v \in \mathcal{U}$. Then we have $[u,v] \in Z_1(\mathcal{U})$ (*) and $u^2 \in Z_2^2(\mathcal{U}) \subseteq Z_1(\mathcal{U})$ (**) by the proof of Theorem 2. It follows that

$$[u,v]^2 = [u,v]u(vu^{-1}v^{-1}) = u[u,v](vu^{-1}v^{-1}) = u^2vu^{-2}v^{-1} = 1.$$

Therefore,

$$uvuv = uvu^{-1}v^{-1}vu^2v = [u,v]v^2u^2 \quad \text{since } u^2 \in Z_1(\mathcal{U}).$$

Consequently,

$$(uv)^4 = (uvuv)(uvuv) = ([u,v]v^2u^2)([u,v]v^2u^2) = [u,v]^2u^4v^4 = u^4v^4.$$

This leads to $u \in Z(\mathcal{U},4)$ and we are done. ∎

Now we are ready to prove Theorem 4.

PROOF. (1) ⇒ (2) If G is a Q^*-group, then G has an Abelian subgroup A of index 2 which has an element a of order 4 such that for all $h \in A$ and all $g \in G \setminus A$, $g^2 = a^2$ and $g^{-1}hg = h^{-1}$. We claim that a is a noncentral element and belongs to $Z_2(\mathcal{U})$. Therefore, Proposition 4 implies that (2) is true.

It is obvious that a is noncentral. To see that $a \in Z_2(\mathcal{U})$, let $f\colon G \longrightarrow \pm 1$ be the orientation homomorphism such that $\operatorname{Ker}(f) = A$ and $f(b) = -1$, where $G = \langle A, b\rangle$ and $b^2 = a^2$. It follows from A. A. Bovdi and S. K. Sehgal [7] that for any unit $u = a_1 + a_2b$, $u^f = a_1^* - a_2a^2b$ is also a unit. We claim that $u^{-1} = u^fc'$ where c' is a central unit. Let $v = uu^f$. Then

$$v^* = (uu^f)^* = a_1a_1^* - a_2a_2^* - a_1a_2b(1-b^2)$$

$$\begin{aligned} vv^* &= (a_1a_1^*)^2 + (a_2a_2^*)^2 - 2(a_1a_1^*a_2a_2^*)b^2 \\ &= (a_1a_1^* - a_2a_2^*b^2)^2 = c^2 \end{aligned}$$

where $c = (a_1a_1^* - a_2a_2^*b^2) = c^* = c^f \in Z_1(\mathcal{U})$.

Let $v_1 = vc^{-1}$, thus

$$v_1v_1^* = vc^{-1}(c^{-1})^*v^* = vv^*c^{-1}(c^*)^{-1} = 1.$$

We conclude that $v_1 = \pm g_0$ for some $g_0 \in G$ and $v = \pm cg_0$.

Let $g_0 = a_0b^i$, $a_0 \in A$, $i = 0$ or 1. If $i = 1$, then $g_0 = a_0b$ and $v = \pm ca_0b$; therefore,

$$\pm c = a_0^{-1}vb^3 = a_0^{-1}(a_1a_1^* - a_2a_2^*)b^3 + a_0^{-1}\big(a_1a_2(1-b^2)\big) \in Z_1(\mathcal{U}).$$

Since $c \in \mathbb{Z}A$ (by the proposition in [7]), we have $a_0^{-1}(a_1a_1^* - a_2a_2^*)b^3 = 0$. However, this is a contradiction since the augmentation of $(a_1a_1^* - a_2a_2^*)$ is ± 1. As a consequence, $i = 0$ and $g_0 = a_0$. Now since

$$a_0^{-1}(a_1a_1^* - a_2a_2^*) + a_0^{-1}\big(a_1a_2(1-b^2)b\big) = a_0^{-1}v = \pm c \in \mathbb{Z}A,$$

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we conclude that $a_0^{-1}\big(a_1a_2(1-b^2)b\big) = 0$, so $a_1a_2(1-b^2)b = 0$. Therefore, $v = uu^f = a_1a_1^* - a_2a_2^* \in Z_1(\mathcal{U})$. This leads to the desired result.

Now we have

$$[a,u] = (aua^{-1})u^{-1} = (a_1 + a_2a^2b)(a_1^* - a_2a^2b)c' = (a_1a_1^* - a_2a_2^*a^2)c' \in Z_1(\mathcal{U}).$$

Hence $a \in Z_2(\mathcal{U})$ and we are done.

(2) $\Rightarrow$ (3). Immediate.

(3) $\Rightarrow$ (1) Suppose $g \in Z(\mathcal{U}, n) \setminus Z_1(\mathcal{U})$. For $u \in \mathcal{U}$, Proposition 1(4) says that

$$[g, u^{n(1-n)}] = [g,u]^{n(1-n)} = [g^{n(1-n)}, u] = 1.$$

Hence $u^{n(n-1)}gu^{-n(n-1)} = g \in G$ for all $u \in \mathcal{U}$ and Proposition 3 gives the desired result. ∎

We can now obtain a different version of Proposition 3.

COROLLARY 4. *Let G be a periodic group. Then the following are equivalent:*

(1) G is a Q^*-group;

(2) G contains a noncentral element a such that for any unit $u \in \mathcal{U}$, $u^4au^{-4} = a$.

PROOF. We need to verify only (1) $\Rightarrow$ (2). By Theorem 4, G contains a noncentral element a such that $a \in Z(\mathcal{U}, 4)$. It follows that for $u \in \mathcal{U}$, Proposition 1(3) implies that

$$[a, u^4] = [a,u]^4 = [a^4, u] = 1$$

for $a^4 \in Z(\mathcal{U}, 3) = Z_1(\mathcal{U})$ by Proposition 1(5) and Theorem 3. Hence $u^4au^{-4} = a \in G$ for all $u \in \mathcal{U}$. ∎

Now we turn to characterizing the n-centre of the unit group. We first study the set of all torsion elements of the n-centre.

PROPOSITION 5. *Let G be a periodic group and $T_n = T\big(Z(\mathcal{U}, n)\big) = \{x \in Z(\mathcal{U}, n) \mid x$ is of finite order and* $\operatorname{aug}(x) = 1\}$. *Then for all $n \geq 2$,*

(1) T_n is a characteristic subgroup of $Z(\mathcal{U}, n)$. Moreover,

$$T_n = Z(\mathcal{U}, n) \cap G,$$

(2) If $u \in Z(\mathcal{U}, n)$, then $[u, v] \in T_n$ for all $v \in \mathcal{U}$,

(3) $Z(\mathcal{U}, n) \subseteq N_{\mathcal{U}}(G)$ and $Z^2(\mathcal{U}, n) \subseteq T_nZ_1(\mathcal{U})$,

(4) $T_n \subseteq T\big(Z_2(\mathcal{U})\big)$. Moreover, $T_4 = T\big(Z_2(\mathcal{U})\big)$,

(5) $Z(\mathcal{U}, n) \subseteq Z_2(\mathcal{U})$. Moreover, $Z(\mathcal{U}, 4) = Z_2(\mathcal{U})$.

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PROOF. (1) Referring to Theorem 3, we need to consider only the situation for $n \geq 4$ because central units of finite order are trivial (S. K. Sehgal [17], p. 46). Note that if $a \in T_n$, then always $a^{-1} \in T_n$ since $o(a^{-1}) = o(a) < \infty$ and $a^{-1} \in Z(\mathcal{U}, n)$. For $a, b \in T_n$, we only need to show that $ab \in T_n$, *i.e.*, $o(ab) < \infty$. We will do it by using induction.

Let $n = 4$ and $a, b \in T_4$. Suppose that $o(a) = l$, $o(b) = m$. Thus

$$(ab)^{4lm} = (a^4b^4)^{lm} = a^{4lm}b^{4lm} = 1 \quad (\text{since } a^4, b^4 \in Z(\mathcal{U}, 3) = Z_1(\mathcal{U})).$$

Therefore, $ab \in T_4$. Consequently, T_4 is a subgroup.

Suppose that for $n = k > 3$, T_k is a subgroup of $Z(\mathcal{U}, k)$.

Now consider that $n = k + 1$. For $a, b \in T_{k+1} \subseteq Z(\mathcal{U}, k + 1)$, observe that $(ab)^{k+1} = a^{k+1}b^{k+1}$. Since a^{k+1}, $b^{k+1} \in Z(\mathcal{U}, k)$ by Proposition 1(5) and both have finite order, we conclude $a^{k+1}, b^{k+1} \in T_k$. It follows from the inductive assumption on T_k that $a^{k+1}b^{k+1} \in T_k$. As a consequence, $o((ab)^{k+1}) = o(a^{k+1}b^{k+1}) < \infty$, so $o(ab) < \infty$. This means that T_{k+1} forms a subgroup. We have proved that T_n is a subgroup of $Z(\mathcal{U}, n)$ for every integer $n \geq 2$.

It can be easily seen that the subgroup $\pm T_n$ is a characteristic subgroup. Hence, since $Z(\mathcal{U}, n)$ is a normal subgroup of the unit group $\mathcal{U}$ so is T_n. It follows from A. A. Bovdi ([3], Theorem 1 and [4], Theorem 3), that $T_n \lhd G$. Therefore, $T_n = Z(\mathcal{U}, n) \cap G$.

(2) Let $u \in Z(\mathcal{U}, n)$ and $v \in \mathcal{U}$. Since $Z(\mathcal{U}, n)$ is a normal subgroup of $\mathcal{U}$, we observe that $vu^{-1}v^{-1} \in Z(\mathcal{U}, n)$; therefore, $[u, v] = uvu^{-1}v^{-1} \in Z(\mathcal{U}, n)$. Moreover

$$[u, v]^{n(n-1)} = ([u, v]^{n(1-n)})^{-1} = 1 \quad \text{by Proposition 1(4).}$$

Hence, $[u, v] \in T_n$ as desired.

(3) The first statement follows directly from (1) and (2). Observing that

$$Z^2(\mathcal{U}, n) \subseteq N^2_{\mathcal{U}}(G) \subseteq GZ_1(\mathcal{U}) \quad \text{(S. K. Sehgal [18], Proposition 9.5),}$$

we easily obtain $Z^2(\mathcal{U}, n) \subseteq T_nZ_1(\mathcal{U})$.

(4) Suppose that for some $n \geq 2$ there exists $a \in T_n$ such that $a \notin T\big(Z_2(\mathcal{U})\big)$, thus a is a noncentral group element. According to Theorem 4, G is a Q^*-group. Next we show that this a is a special element of order 4 in G, as given in the definition of Q^*-groups. Observing the proof of Proposition 3, we find that if $g \in G$, then either

(i) $\langle a, g\rangle$ is Abelian

or

(ii) $\langle a, g\rangle \cong Q$, the group of quaternions.

Setting $A = C_G(a) \subseteq G$ and $g \notin A$, we obtain that $\langle a, g\rangle \cong Q$, thus $a^2 = g^2$. It follows that a has order 4. For any $h \in A$, $g \notin A$, we have $hg \notin A$. Therefore, $\langle a, hg\rangle \cong Q$. It follows that $g^2 = a^2 = hghg$, and so $ghg^{-1} = h^{-1}$ (*). We also note that if $k \notin A$, then $gag^{-1} = a^{-1} = kak^{-1}$. It follows that $ag^{-1}k = g^{-1}ka$ and $g^{-1}k \in C_G(a) = A$, and so A is of index 2 in G. Condition (*) tells that A is Abelian; therefore the element a is a special element as we claimed. However we showed in the proof of Theorem 4

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that $a \in T\big(Z_2(U)\big)$. This contradiction leads to the first result. Moreover, recalling Proposition 4 which gives $T\big(Z_2(U)\big) \subseteq T_4$, we obtain that $T\big(Z_2(U)\big) = T_4$.

(5) Let $u \in Z(U, n)$ and $v \in U$. Then $[u, v] \in T_n$ by (2); therefore, $[u, v] \in T\big(Z_2(U)\big)$ by (4). It follows that $u \in Z_3(U)$ and therefore, $Z(U, n) \subseteq Z_3(U)$. Since $Z_3 = Z_2$ (Theorem 2), we conclude that $Z(U, n) \subseteq Z_2(U)$. In particular, $Z(U, 4) \subseteq Z_2(U)$. Now Proposition 4 finishes the proof. ■

Now we give a complete characterization of the n-centre of the unit group.

THEOREM 5. *Let G be a periodic group. Then*

$$Z(U, n) = \begin{cases} U & \text{for } n = 0 \text{ or } 1 \\ Z_2(U) & \text{for } n = 4k \text{ or } 4k+1,\ k \geq 1 \\ Z_1(U) & \text{for } n = 4k+2 \text{ or } 4k+3,\ k \geq 0 \end{cases}.$$

PROOF. The first equality is obvious.

Now we prove that $Z_2(U) \subseteq Z(U, 4k)$ and $Z_2(U) \subseteq Z(U, 4k+1)$ for all $k \geq 1$. Combined with Proposition 5(5), this leads to the second part.

Let $u \in Z_2(U)$ and $v \in U$. Then $u \in Z(U, 4)$ by Proposition 4, and therefore $u^4 \in Z_1(U)$ by Proposition 1(5) and Theorem 3. It follows that

$$(uv)^{4k} = ((uv)^4)^k = (u^4v^4)^k = u^{4k}v^{4k}.$$

This forces $u \in Z(U, 4k)$, thus $Z_2(U) \subseteq Z(U, 4k)$.

Similarly,

$$(uv)^{4k+1} = (uv)(uv)^{4k} = uvu^{4k}v^{4k} = u^{4k+1}v^{4k+1}.$$

This means that $Z_2(U) \subseteq Z(U, 4k+1)$.

Next suppose that $n = 4k+2$ or $4k+3$, $k \geq 1$. First let us consider $n = 4k+2$. Note that $Z(U, 4k+2) \subseteq Z_2$ by Proposition 5(5) and therefore, $Z(U, 4k+2) \subseteq Z(U, 4k) \cap Z(U, 4k+1)$ by the above. Recall that if an element is contained in 3 consecutive n-centres, then it must be a central element (see L. C. Kappe and M. L. Newell [14]). We are done.

Similar arguments work for the case of $n = 4k+3$. ■

In view of S. R. Arora and I. B. S. Passi ([2], Theorem 3.1), we obtain the following corollary:

COROLLARY 5. *Let G be a finite group. Then*

$$Z(U, n) = \begin{cases} U & \text{for } n = 0 \text{ or } 1 \\ T\big(Z_2(U)\big)Z_1(U) & \text{for } n = 4k \text{ or } 4k+1,\ k \geq 1 \\ Z_1(U) & \text{for } n = 4k+2 \text{ or } 4k+3,\ k \geq 0 \end{cases}.$$

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