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CLUSTER STRUCTURES ON HIGHER TEICHMULLER SPACES FOR CLASSICAL GROUPS

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Abstract

Let S be a surface, G a simply connected classical group, and G' the associated adjoint form of the group. We show that the moduli spaces of framed local systems $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$, which were constructed by Fock and Goncharov ['Moduli spaces of local systems and higher Teichmuller theory', *Publ. Math. Inst. Hautes Études Sci.* **103** (2006), 1–212], have the structure of cluster varieties, and thus together form a cluster ensemble. This simplifies some of the proofs in that paper, and also allows one to quantize higher Teichmuller space, which was previously only possible when G was of type A.

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1. Introduction

Let S be a topological surface with nonempty boundary, let G be a simply connected semisimple group, and let G' be the adjoint form of this group. We are interested in the space $\mathcal{L}_{G,S}$, the moduli space of G-local systems on the surface S, or, equivalently, the space of representations of $\pi_1(S)$ into G. There are two closely related spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$, which were constructed by Fock and Goncharov [FG1]. Both the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ are variations on the space $\mathcal{L}_{G,S}$; they parameterize local systems with certain types of framing at the boundary of S.

One advantage of studying the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ is that they are simpler; for example, they have rational coordinate charts, while $\mathcal{L}_{G,S}$ does not, in general. In our view, the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ are perhaps more fundamental, and allow one to understand properties of the space $\mathcal{L}_{G,S}$ that are otherwise hard to see.

In [FG1], Fock and Goncharov show that each of the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ has an atlas of coordinate charts such that all transition functions involve only addition, multiplication and division. In other words, these spaces each have a *positive atlas* and may be called *positive varieties*. The main ingredient in the construction of this positive atlas of coordinate charts is Lusztig's theory of total positivity. The fact that $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ are positive varieties is quite striking; it is the key result in [FG1].

For the spaces $\mathcal{X}_{PGL_n,S}$ and $\mathcal{A}_{SL_n,S}$, Fock and Goncharov show more: these spaces have cluster-like structures on their rings of functions, and thus the fact that they are positive varieties is a consequence of the fact that they are (up to codimension two, and up to divisors that may be spelled out precisely) *cluster varieties*. Moreover, the pair of spaces $(\mathcal{A}_{SL_n,S}, \mathcal{X}_{PGL_n,S})$ are tightly related, and form what they call a *cluster ensemble*.

Let us review some of the consequences of the theory. The moduli space of representations of $\pi_1(S)$ into a split real group $G(\mathbb{R})$ has several components, one of which is called higher Teichmuller space, because in many ways it behaves like Teichmuller space. Because $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ are positive varieties, it makes sense to take the positive $\mathbb{R}_{>0}$ points of these moduli spaces. The resulting spaces $\mathcal{X}_{G',S}(\mathbb{R}_{>0})$ and $\mathcal{A}_{G,S}(\mathbb{R}_{>0})$ parameterize positive representations. With some work, Fock and Goncharov identify positive representations with higher Teichmuller space, thus giving it an algebrogeometric description. The theory gives an explicit parameterization of positive representations, and it becomes manifest that the space of representations is contractible. Moreover, one can show that all positive representations are discrete and faithful [FG1, L].

In this paper, we will construct cluster ensemble structures on $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ for G a classical group, that is, for G of type B, C and D. In other words, we show that $(\mathcal{A}_{G,S}, \mathcal{X}_{G',S})$ forms a cluster ensemble. In doing so, we give what we think to be a beautiful set of functions on the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ which generalizes the 'canonical functions' when $G = SL_n$. The canonical functions for SL_n have a long history, see [RR, S, HK, Hen]. As in the case of $G = SL_n$, our functions will induce positive structures on $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$, which will agree with the positive structures defined in [FG1].

The cluster structures on double Bruhat cells and in flag varieties are well known [BFZ], and they form a building block for the cluster structures on $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$. The cluster structures on $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ are constructed via ideal triangulations of S. The cluster structure on each triangle is close to the cluster structure on the Borel subgroup B for the group G. The first main problem is to correctly identify the cluster structure on each triangle and to understand how these structures glue together. We were inspired in our solution to this problem by the idea of 'amalgamation' developed in [FG4].

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For each triangulation of the surface S and for any ordering of the vertices in each triangle of the triangulation, one can write down an associated seed for the cluster ensemble. (This seed can be constructed out of a reduced word for the longest element w_0 of the Weyl group of G.) When $G = SL_n$ or PGL_n these functions (for a particular choice of the reduced word for w_0) somewhat miraculously exhibit S_3 symmetry. This is not the case for general groups G. We give the sequence of mutations that relates seeds associated to different orderings of the vertices in a given triangle in the triangulation.

Moreover, we give a sequence of mutations relating seeds that correspond to different triangulations. This problem reduces to the problem of relating seeds coming from triangulations that differ by a 'flip,' which is the change of triangulation that results from replacing one diagonal of a quadrilateral by another. When G has type A, this sequence of mutations comes from the octahedron recurrence, which has been very well-studied [**RR**, **S**, **HK**]. We will give an analogue of the octahedron recurrence for G is a classical group.

Both the sequences of mutations realizing S_3 symmetries and flips of triangulations are relatives of the tetrahedron recurrence and the octahedron recurrence [**S**]. The tetrahedron recurrence has appeared in many guises—it is related to the Jeu de Taquin operation on tableaux, the action of the cactus group on crystals [**HK**, **Hen**]—and plays a role in the work of Goncharov and Shen on DT-invariants [**GS2**]. The computation of these sequences of mutations forms the technical heart of this paper. These computations have the following consequences:

(1) We get an alternative construction of $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ as positive varieties. This allows somewhat simpler proofs of many of the results in [**FG1**].

(2) The system of coordinate charts is naturally mapping class group equivariant. We see therefore get an injection of the mapping class group into the cluster modular group for $(\mathcal{A}_{G,S}, \mathcal{X}_{G',S})$.

(3) Using the formalism of [**FG2**, **FG3**, **FG5**], we can construct the quantization of $\mathcal{X}_{G',S}$ for G any classical group, a result previously only known for $G' = PGL_n$. This quantization of a $*$-algebra, and one also gets quantum representations of this algebra. One can also construct the symplectic double of $\mathcal{X}_{G',S}$.

(4) One observes that for classical groups, the cluster ensembles $(\mathcal{A}_{G,S}, \mathcal{X}_{G',S})$ and $(\mathcal{A}_{(G')^\vee,S}, \mathcal{X}_{G^\vee,S})$ have Langlands dual seeds. Here $G^\vee$ is the Langlands dual group of G. This is some manifestation of mirror symmetry, and gives support for conjectures of [**GS**].

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(5) One sees that the deformed algebras $\mathcal{X}_{q,G',S}$ and $\mathcal{X}_{q^\vee,G^\vee,S}$ are closely related, as predicted in [**FG2**].

(6) One can extend the approach of [**Le**] to define higher laminations for all classical groups. Another definition of higher laminations was given in [**GS**].

In fact, for a general reductive group, our constructions allow us to construct the cluster ensemble structure for $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$. We discuss this further in [**Le2**]. However, in the exceptional cases, we do not know how to realize the S_3 symmetries and the flip. This is the only piece of the puzzle missing that prevents us from getting a mapping class group equivariant quantization of higher Teichmuller space and all the associated results above.

Let us explain the structure of this paper. The structure of the paper is modular, so that the main Sections 3–5 can be read independently of each other with a minimal amount of cross-referencing. Section 2 contains all the relevant background. Following Fock and Goncharov, we define the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ and relate them to spaces of configurations of points in G/B and spaces of twisted configurations of points in G/U: $\operatorname{Conf}_m \mathcal{B}_G$ and $\operatorname{Conf}_m \mathcal{A}_G$. We also recall the necessary facts about cluster algebras.

In Sections 3–5, we give the cluster algebra structure on the spaces $\operatorname{Conf}_m \mathcal{A}_G$ for $G = Sp_{2n}, Spin_{2n+1}, Spin_{2n+2}$. The structure of these three sections is very much parallel. This is in part so that the sections may be read independently, and in part to emphasize the similarities (and differences) between these three different cases. A reader only interested in one case may skip the others. At the same time, it is our hope that the reader who absorbs any one of Sections 3–5 could quite rapidly absorb the others. These sections form the technical heart of the paper. We have included complete formulas in all three cases though in some of the proofs of these formulas, we refer to previous sections if an identity has already been proven. Thus in Sections 4 and 5, we emphasize those aspects of the proofs of the mutation identities which are novel to these particular cases.

In each of these sections, the most important subsections are the first and the last two, consisting of the construction of the seed for the cluster algebra structure on $\operatorname{Conf}_m \mathcal{A}_G$, as well the sequences of mutations that realize the S_3 symmetries on each triangle and the flip of a triangulation. For example, in Section 4, one could read Sections 4.1, 4.5, 4.6 and skip Sections 4.2–4.4 on a first reading.

The formulas and mutation sequences can become quite involved. We try to alleviate this in two ways. First, we include many diagrams. None of the diagrams is essential—all information contained in the diagrams is replicated in

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the text. Nevertheless, they should serve as a useful aid. Moreover, almost all of the important information in our computations is succinctly captured in the diagrams. Once one is comfortable with reading the diagrams, they can serve as a useful reference/summary that can be used independently of the text. All the general phenomena can be seen in small cases, like Sp_8, $Spin_9$ or $Spin_{10}$. These representative examples are treated in full in the diagrams.

Second, throughout Sections 3–5, we try to explain the conceptual underpinnings of the calculations. This is done in several ways. We show how the cluster algebra structure is related to reduced words in the Weyl group; we show how the cactus sequence of mutations plays a role in constructing the seeds and computing some of the S_3 symmetries; we explain how folding the seed for SL_{2n} gives the seed for Sp_{2n}, how Langlands duality relates the seeds for Sp_{2n} and $Spin_{2n+1}$, and how unfolding the seed for $Spin_{2n+1}$ gives the seed for $Spin_{2n+2}$. The latter fact is perhaps most important: all the sequences of mutations that we consider arise from the ones for Sp_{2n} by Langlands duality or unfolding.

The remainder of the paper proceeds as follows. Section 6 treats the $\mathcal{X}$-variety structure on $\mathrm{Conf}_m\,\mathcal{B}_G$. Section 7 gives applications to the quantization of higher Teichmuller space.

2. Background

2.1. Setup. Let S be a compact oriented surface with boundary, and possibly with a finite number of marked points on each boundary component. We will refer to this whole set of data—the surface and the marked points on the boundary—by S. We will always take S to be hyperbolic, meaning it either has negative Euler characteristic, or contains enough marked points on the boundary (in other words, we can give it the structure of a hyperbolic surface such that the boundary components that do not contain marked points are cusps, and all the marked points are also cusps). If we consider the boundary of S and remove the marked points on the boundary, we will call the resulting set the *punctured boundary*.

Let G be a semisimple algebraic group. When G is adjoint, that is, has trivial center (for example, when $G = PGL_m$), we can define a higher Teichmuller space $\mathcal{X}_{G,S}$. On the other hand, for G simply connected (for example, when $G = SL_m$), we can define the higher Teichmuller space $\mathcal{A}_{G,S}$. They will be the space of local systems of S with structure group G with some extra structure of a framing of the local system at the boundary components of S. Alternatively, these spaces describe homomorphisms of $\pi_1(S)$ into G modulo conjugation plus some extra data.

The spaces $\mathcal{X}_{G,S}$ and $\mathcal{A}_{G,S}$ have a distinguished collection of coordinate systems, equivariant under the action of the mapping class group of S. Using an elaboration of Lusztig's work on total positivity, one can show that all the transition functions between these coordinate systems are subtraction-free, and give a *positive atlas* on the corresponding moduli space. This positive atlas gives the spaces $\mathcal{X}_{G,S}$ and $\mathcal{A}_{G,S}$ the structure of a *positive variety*.

The existence of these extraordinary positive coordinate charts depends on G. Lusztig's theory of positivity in semisimple Lie groups [**Lu**, **Lu2**], and is a reflection of the cluster algebra structure of the ring of functions on these spaces.

2.2. Definition of the spaces $\mathcal{X}_{G,S}$ and $\mathcal{A}_{G,S}$. The data of a framing of a local system involves the geometry of the flag variety associated to a group. Let B be a Borel subgroup, a maximal solvable subgroup of G. Then $\mathcal{B} = G/B$ is the flag variety. Let $U := [B, B]$ be a maximal unipotent subgroup in G. Then we will call $\mathcal{A} = G/U$ the 'principal affine space' (sometimes also referred to as the 'base affine space'). We will refer to elements of $\mathcal{A}$ as 'principal flags.'

Let $\mathcal{L}$ be a G-local system on S. For any space X equipped with a G-action, we can form the associated bundle $\mathcal{L}_X$. For $X = G/B$ we get the associated flag bundle $\mathcal{L}_{\mathcal{B}}$, and for $X = G/U$, we get the associated principal flag bundle $\mathcal{L}_{\mathcal{A}}$.

DEFINITION 2.1. A *framed G-local system on S* is a pair $(\mathcal{L}, \beta)$, where $\mathcal{L}$ is a G-local system on S, and β a flat section of the restriction of $\mathcal{L}_{\mathcal{B}}$ to the punctured boundary of S.

The space $\mathcal{X}_{G',S}$ is the moduli space of framed G-local systems on S.

The definition of the space $\mathcal{A}_{G,S}$ is slightly more complicated. The definition involves twisted local systems, which we will now define.

Let G be simply connected. The maximal length element w_0 of the Weyl group of G has a natural lift to G, denoted $\overline{w}_0$. Let $s_G := \overline{w}_0^2$. It turns out that s_G is in the center of G and that $s_G^2 = e$. Depending on G, s_G will have order one or order two. For example, for $G = SL_{2k}$, s_G has order two, while for $G = SL_{2k+1}$, s_G has order one. In type C_n, s_G has order 2, while in types B_n and D_n, the order of s_G depends on $n \mod 4$ (see equations (5.1) and (5.2)).

The fundamental group $\pi_1(S)$ has a natural central extension by $\mathbb{Z}/2\mathbb{Z}$. We see this as follows. For a surface S, let $T'S$ be the tangent bundle with the zero section removed. $\pi_1(T'S)$ is a central extension of $\pi_1(S)$ by $\mathbb{Z}$:

$$\mathbb{Z} \to \pi_1(T'S) \to \pi_1(S).$$

The quotient of $\pi_1(T'S)$ by the central subgroup $2\mathbb{Z} \subset \mathbb{Z}$, gives $\overline{\pi}_1(S)$ which is a central extension of $\pi_1(S)$ by $\mathbb{Z}/2\mathbb{Z}$:

$$\mathbb{Z}/2\mathbb{Z} \to \overline{\pi}_1(S) \to \pi_1(S).$$

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Let $\sigma_S \in \overline{\pi}_1(S)$ denote the nontrivial element of the center.

A *twisted G-local system* is a representation $\overline{\pi}_1(S)$ in G such that σ_S maps to s_G. Such a representation gives a local system on $T'S$.

Now we must describe the framing data for a twisted local system. Let $\mathcal{L}$ be a twisted G-local system on S. Such a twisted local system gives an associated principal affine bundle $\overline{\mathcal{L}}_{\mathcal{A}}$ on the punctured tangent bundle $T'S$. For any boundary component of S, we will construct sections of the punctured tangent bundle above these boundary components. Given any boundary component, consider the outward pointing unit tangent vectors along this component—this gives a section of the punctured tangent bundle above each boundary component of S. We get a bunch of loops and arcs in $T'S$ that lie over the punctured boundary of S. Call this the *lifted boundary*.

DEFINITION 2.2. A *decorated G-local system* on S consists of $(\mathcal{L}, \alpha)$, where $\mathcal{L}$ is a twisted local system on S and α is a flat section of $\overline{\mathcal{L}}_{\mathcal{A}}$ restricted to the lifted boundary.

The space $\mathcal{A}_{G,S}$ is the moduli space of decorated G-local systems on S.

Note that in the case where $s_G = e$, a decorated local system is just a local system on S along with a flat section of $\mathcal{L}_{\mathcal{A}}$ restricted to the boundary. One can generally pretend that this is the case without much danger. It is sometimes convenient to choose one representative point in each component of the punctured boundary, and imagine the framings as flags sitting at those points.

2.3. Relation to configurations of flags. Let us take S to be a disc with m marked points. We will now analyze the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ in this fundamental case. Let us take the components of the punctured boundary of S and number them $1, 2, \ldots, m$, starting at some component and moving counterclockwise.

We first treat the space $\mathcal{X}_{G',S}$. Any local system on a disc is trivial, so we may choose some trivialization. Then the decorations along each component of the punctured boundary can be canonically identified with elements of $\mathcal{B} = G'/B$. Changing the trivialization results in acting on each of these flags by some element $g \in G'$. Thus we have a natural identification

$$\mathcal{X}_{G',S} \simeq \mathrm{Conf}_m \mathcal{B} := G' \backslash (G'/B)^m.$$

If we had chosen a different boundary component to label as 1, we would have a different identification $\mathcal{X}_{G',S} \simeq \mathrm{Conf}_m \mathcal{B}$ which would differ from our original identification by some power of the cyclic shift map

$$T : \mathrm{Conf}_m \mathcal{B} \to \mathrm{Conf}_m \mathcal{B},$$

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$$T(B_1, B_2, \dots, B_n) = (B_n, B_1, \dots, B_{n-1}).$$

The space $\mathcal{A}_{G,S}$ is slightly more complicated. A twisted local system on the disc is a local system on the punctured tangent bundle $T'S$ such that the monodromy around a loop in the fiber is s_G. There is a unique such local system. Choose one point in each of the boundary components $1, 2, \dots, m$. Over that point, consider the outward pointing normal unit vector in $T'S$. Call these points $x_1, x_2, \dots, x_m$, respectively. Now trivialize the local system at x_1. The decoration along the boundary component 1 gives us an element of G/U at x_1. Along each of the other boundary components, the decoration gives us a point of the associated flag bundle. We may parallel transport along any path from x_i to x_1 that rotates the vector in the tangent bundle clockwise the minimal amount. Then we get m elements of G/U. Changing the trivialization changes these flags by the diagonal action of G on $(G/U)^m$. Thus we get a noncanonical identification

$$\mathcal{A}_{G',S} \simeq \operatorname{Conf}_m \mathcal{A} := G\backslash(G/U)^m.$$

If we had chosen a different boundary component to label as 1, we would have a different identification $\mathcal{A}_{G,S} \simeq \operatorname{Conf}_m \mathcal{A}$ which would differ from our original identification by some power of the twisted cyclic shift map

$$T : \operatorname{Conf}_m \mathcal{A} \to \operatorname{Conf}_m \mathcal{A},$$

$$T(U_1, U_2, \dots, U_n) = (s_G \cdot U_n, U_1, \dots, U_{n-1}).$$

In the case of a disc with m marked points, we will think of $\mathcal{A}_{G,S}$ as $\operatorname{Conf}_m \mathcal{A}$ equipped with the cyclic shift map.

2.4. Positive Structures. In this section, we explain how to construct the positive structures on $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$.

Let $T = \operatorname{Spec} \mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ be a split algebraic torus. A positive rational function on T is a nonzero rational function on T which can be written as a ratio of two polynomials in the x_i with positive rational coefficients.

A positive rational morphism $\phi : T_1 \to T_2$ of two split tori is a morphism such that for any character χ of T_2, $\phi^*(\chi)$ is a positive rational function.

A positive atlas on an irreducible space $\mathcal{Y}$ over $\mathbb{Q}$ is a nonempty collection of split tori $T_{\mathbf{c}}$ along with birational isomorphisms over $\mathbb{Q}$,

$$\alpha_{\mathbf{c}} : T_{\mathbf{c}} \dashrightarrow \mathcal{Y}$$

for every pair $\mathbf{c}, \mathbf{c}'$, further satisfying the condition that the map

$$\phi_{\mathbf{c},\mathbf{c}'} := \alpha_{\mathbf{c}}^{-1} \circ \alpha_{\mathbf{c}'} : T_{\mathbf{c}'} \dashrightarrow T_{\mathbf{c}}$$

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is a positive birational morphism. Each map $\alpha_{\mathbf{c}}$ gives a positive coordinate chart on $\mathcal{Y}$.

A positive rational function F on $\mathcal{Y}$ is a rational function given by a subtraction-free rational function in one of the coordinate charts. If a function is subtraction-free on one chart, it is subtraction-free in all the positive coordinate charts of the positive atlas.

A positive rational map $\mathcal{Y} \to \mathcal{Z}$ is a rational map that is a positive rational function in one, and hence in all, positive coordinate systems.

When S is a disc with m marked points, the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ become the spaces of configurations of points in the flag variety $\mathrm{Conf}_m \mathcal{B}$ and twisted configurations of points in the affine variety, $\mathrm{Conf}_m \mathcal{A}$. We will first describe the positive structures on these spaces.

We start with some notation. Let U^+ and U^- denote the upper and lower unipotent subgroups of G'. Also let B^+ and B^- denote the upper and lower Borel subgroups of G'. We can identify points in $\mathcal{B}$ with Borel subgroups. If B is a Borel subgroup, we let $g \cdot B$ denote the Borel subgroup gBg^{-1}. The positive structures on these spaces are defined as follows.

There is a map

$$\underbrace{U^- \times \cdots \times U^-}_{n \text{ copies}} / H \longrightarrow \mathrm{Conf}_{n+2}(\mathcal{B})$$

that sends

$$(u_1, \ldots, u_n) \to (B^-, B^+, u_1 \cdot B^+, u_1 u_2 \cdot B^+, \ldots, u_1 \ldots u_n \cdot B^+). \tag{2.1}$$

Here H acts diagonally on the U^- by conjugation. There is a natural positive structure on U^- [FZ], and this positive structure is preserved by conjugation. Thus $(U^-)^n/H$ is a positive variety, and birational equivalence induces a positive structure on $\mathrm{Conf}_{n+2}(\mathcal{B})$.

Similarly, there is a map

$$H \times \underbrace{B^- \times \cdots \times B^-}_{n \text{ copies}} \longrightarrow \mathrm{Conf}_{n+2}(\mathcal{A})$$

that sends

$$(h, b_1, \ldots, b_n) \to (U^-, h \cdot \overline{w_0} U^-, b_1 \cdot \overline{w_0} U^-, b_1 b_2 \ldots \overline{w_0} U^-, \ldots, b_1 \cdots b_n \cdot \overline{w_0} U^-). \tag{2.2}$$

The natural positive structure on $H \times (B^-)^n$ induces a positive structure on $\mathrm{Conf}_{n+2}(\mathcal{A})$. (Note that $\overline{w}$ denotes a particular lift of the element w of the Weyl

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group to G. See [**BFZ**] or [**FG1**]. Here, w_0 denotes the longest element of the Weyl group.)

We have the following important fact:

THEOREM 2.3 [**FG1**]. *The positive structures on* $\mathrm{Conf}_{n+2}(\mathcal{B})$ *and* $\mathrm{Conf}_{n+2}(\mathcal{A})$ *are invariant under cyclic shift and twisted cyclic shift, respectively.*

We now explain how to construct a positive coordinate chart on $\mathrm{Conf}_m(\mathcal{B})$ for each triangulation of an m-gon. If we place the m flags at the vertices of an m-gon, then to each triangle in the triangulation of the m-gon, we get a configuration of three flags, and to this configuration of three flags, we attach some *face functions*. Each face function will depend on all three flags. The face functions give a positive coordinate chart on $\mathrm{Conf}_3(\mathcal{B})$.

Any edge in the triangulation belongs to two triangles. We attach to each edge a set of *edge functions*, which depend on the four flags at the corners of the two triangles. For any edge, its edge functions along with the face functions for the triangles sharing that edge form a positive coordinate chart on $\mathrm{Conf}_4(\mathcal{B})$. Thus the face functions give invariants of three flags, while the edge functions tell us how to glue two configurations of three flags into a configuration of four flags. If we take the edge and face functions for any triangulation, we get a positive coordinate chart on $\mathrm{Conf}_m \mathcal{B}$. This positive structure agrees with the one described above, and is independent of triangulation.

There are also positive coordinate charts on $\mathrm{Conf}_m(\mathcal{A})$ attached to each triangulation of an m-gon. These are constructed slightly differently. To each edge in the triangulation, we attach a set of *edge functions* which depend on the two flags at the ends of the edge. To each triangle in the triangulation, we get a configuration of three principal flags, and in addition to the edge functions of each pair of edges in the triangle, we attach some *face functions*, which depend on all three of the principal flags (not just two at a time). Thus whereas for $\mathrm{Conf}_m(\mathcal{B})$, the edge functions give us the data for gluing configurations of three flags, for $\mathrm{Conf}_m(\mathcal{A})$, two configurations of three principal flags can be glued along an edge only if the edge functions along that edge are identical. Thus we exchange gluing data for restrictions on when we can glue.

Thus, if we take the edge and face functions for any triangulation, we get a positive coordinate chart on $\mathrm{Conf}_m \mathcal{A}$. This positive structure agrees with the one described above, and is independent of triangulation.

2.5. Reduction to the case of $\mathrm{Conf}_m \mathcal{A}$ or $\mathrm{Conf}_m \mathcal{B}$. We now describe the positive structures on the spaces $\mathcal{X}_{G',S}$ and $\mathcal{A}_{G,S}$ for a general surface.

Let S be a hyperbolic surface. Give it some hyperbolic structure such that the

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boundary components that do not contain marked points are cusps. Moreover, on all other boundary components, choose a representative point, and suppose that these representative points are also cusps. The particular choice of hyperbolic structure will turn out not to matter. An *ideal triangulation* of S consists of a triangulation of S that has vertices at the cusps of S. As we only consider ideal triangulations, we will sometimes refer simply to *triangulation*.

Note that cutting the surface S along any edge of an ideal triangulation gives us another surface S' with an ideal triangulation. We can restrict any framed or decorated local system from S to S'. In particular, we may restrict any framed local system to a triangle in the ideal triangulation or to a pair of triangles sharing an edge. This gives maps from $\mathcal{X}_{G',S}$ to $\mathrm{Conf}_3\,\mathcal{B}$ and $\mathrm{Conf}_4\,\mathcal{B}$ for every face and edge of the triangulation, respectively. Similarly, we may restrict any decorated local system to a triangle or an edge to get maps from $\mathcal{A}_{G,S}$ to $\mathrm{Conf}_3\,\mathcal{A}$ and $\mathrm{Conf}_2\,\mathcal{A}$ for every face and edge of the triangulation, respectively.

This allows us to construct face and edge functions on $\mathcal{X}_{G',S}$ (respectively $\mathcal{A}_{G,S}$) for every face and edge of an ideal triangulation simply by pulling back functions from the spaces $\mathrm{Conf}_3\,\mathcal{B}$ and $\mathrm{Conf}_4\,\mathcal{B}$ (respectively, $\mathrm{Conf}_3\,\mathcal{A}$ and $\mathrm{Conf}_2\,\mathcal{A}$).

This set of functions forms a positive coordinate chart for $\mathcal{X}_{G',S}$ (respectively $\mathcal{A}_{G,S}$). The coordinate charts coming from different triangulations of the surface give a positive atlas on $\mathcal{X}_{G',S}$ (respectively $\mathcal{A}_{G,S}$).

One of the goals of this paper will be to show that these edge and face functions can be realized as part of a cluster ensemble structure on the pair of spaces $(\mathcal{X}_{G',S}, \mathcal{A}_{G,S})$.

2.6. Cluster algebras. We review here the basic definitions of cluster algebras, following [W]. Cluster algebras are commutative rings that come equipped with a collection of distinguished sets of generators, called *cluster variables* or $\mathcal{A}$-coordinates. One can obtain one set of generators from another set of generators by a process called mutation.

Each set of generators belongs to a seed, which roughly consists of the set of generators along with a B-matrix. The B-matrix encodes how one mutates from one seed to any adjacent seed. Starting from any initial seed, the process of mutation gives all the seeds (and all the sets of generators) for the cluster algebra. The cluster variables are coordinates on the $\mathcal{A}$-space.

The same combinatorial data underlying a seed gives rise to a second, related, algebraic structure, an algebra generated by coordinates called $\mathcal{X}$-coordinates. The $\mathcal{X}$-coordinates are functions on the $\mathcal{X}$ space. The $\mathcal{A}$-coordinates and $\mathcal{X}$-coordinates are related by a canonical monomial transformation, which gives a map from the $\mathcal{A}$-space to the $\mathcal{X}$-space. Together, the data of the $\mathcal{A}$-space and the $\mathcal{X}$-space, along with their distinguished sets of coordinates, is called a cluster ensemble.

Cluster algebras and $\mathcal{X}$-coordinates are both defined by seeds. A seed $\Sigma = (I, I_0, B, d)$ consists of the following data:

(1) An index set I with a subset $I_0 \subset I$ of 'frozen' indices.

(2) A rational $I \times I$ *exchange matrix* B. We require that $b_{ij} \in \mathbb{Z}$ unless both i and j are frozen.

(3) A set $d = \{d_i\}_{i \in I}$ of positive integers that skew-symmetrize the matrix B. In other words, we have that $b_{ij} d_j = -b_{ji} d_i$ for all $i, j \in I$.

For most purposes, the values of d_i are only important up to simultaneous scaling. Also note that the values of b_{ij} where i and j are both frozen will play no role in the cluster algebra, though it is sometimes convenient to assign values to b_{ij} for bookkeeping purposes. These values become important in *amalgamation*, where one unfreezes some of the frozen variables.

Two seeds are said to be isomorphic if one can find a bijection between their index sets preserving frozen (and unfrozen) indices, as well as the values of d_i and b_{ij}.

Let $k \in I \setminus I_0$ be an unfrozen index of a seed Σ. We now describe how to mutate the seed Σ to obtain another seed $\Sigma' = \mu_k(\Sigma)$. Σ' has the same index set and frozen indices, and the same values d_i. However, the exchange matrix changes, and the new exchange matrix B' of Σ' satisfies

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k, \\ b_{ij} & b_{ik} b_{kj} \leqslant 0, \\ b_{ij} + |b_{ik}| b_{kj} & b_{ik} b_{kj} > 0. \end{cases} \tag{2.3}$$

Two seeds Σ and Σ' are said to be mutation-equivalent if they are related by a finite sequence of mutations.

To each seed Σ we associate a collection of *cluster variables* $\{A_i\}_{i \in I}$, which are in turn functions on a split algebraic torus $\mathcal{A}_\Sigma := \operatorname{Spec} \mathbb{Q}[A_I^{\pm 1}]$, where $\mathbb{Q}[A_I^{\pm 1}]$ denotes the ring of Laurent polynomials in the cluster variables. If Σ' is obtained from Σ by mutation at the index $k \in I \setminus I_0$, we get a birational map, called a *cluster transformation*, $\mu_k : \mathcal{A}_\Sigma \dashrightarrow \mathcal{A}_{\Sigma'}$. This is defined by the *exchange relation*

$$\mu_k^*(A'_i) = \begin{cases} A_i & i \neq k, \\ A_k^{-1} \left(\prod_{b_{kj} > 0} A_j^{b_{kj}} + \prod_{b_{kj} < 0} A_j^{-b_{kj}} \right) & i = k. \end{cases} \tag{2.4}$$

Composing these transformations allows us to glue any tori $\mathcal{A}_{\Sigma}$ and $\mathcal{A}_{\Sigma'}$ that come from mutation-equivalent seeds Σ and Σ'. The $\mathcal{A}$-space $\mathcal{A}_{|\Sigma|}$ is defined as the scheme obtained from gluing together all the tori of seeds mutation-equivalent to some initial seed Σ.

DEFINITION 2.4. Let Σ be a seed. The *cluster algebra* $\mathcal{A}(\Sigma)$ is the $\mathbb{Z}$-subalgebra of the function field of $\mathcal{A}_{|\Sigma|}$ that is generated by all cluster variables of seeds mutation-equivalent to Σ. The *upper cluster algebra* $\overline{\mathcal{A}}(\Sigma)$ is defined by

$$\overline{\mathcal{A}}(\Sigma) := \mathbb{Z}[\mathcal{A}_{|\Sigma|}] = \bigcap_{\Sigma' \sim \Sigma} \mathbb{Z}[\mathcal{A}_{\Sigma'}] \subset \mathbb{Q}(\mathcal{A}_{|\Sigma|}).$$

Here, $\Sigma' \sim \Sigma$ means that Σ and Σ' are mutation-equivalent. The left-hand side is therefore the intersection of the Laurent polynomial rings for all seeds Σ' mutation-equivalent to Σ.

The Laurent phenomenon tells us that the cluster algebra is contained inside the upper cluster algebra.

The seed Σ also gives rise to a second algebraic torus $\mathcal{X}_{\Sigma} := \operatorname{Spec} \mathbb{Z}[X_{I\setminus I_0}^{\pm 1}]$, where $\mathbb{Z}[X_{I\setminus I_0}^{\pm 1}]$ again denotes the Laurent polynomial ring in the variables $\{X_i\}_{i \in I\setminus I_0}$. For any index $k \in I \setminus I_0$, mutation of the seed Σ at k gives an associated birational map $\mu_k : \mathcal{X}_{\Sigma} \to \mathcal{X}_{\Sigma'}$ defined by

$$\mu_k^*(X_i') = \begin{cases} X_i X_k^{[b_{ik}]_+}(1 + X_k)^{-b_{ik}} & i \neq k, \\ X_k^{-1} & i = k, \end{cases} \tag{2.5}$$

where $[b_{ik}]_+ := \max(0, b_{ik})$. The $\mathcal{X}$-space $\mathcal{X}_{|\Sigma|}$ is defined as the scheme obtained from gluing together all tori associated to seeds mutation-equivalent with an initial seed Σ.

Since B is skew-symmetrizable, there is a canonical Poisson structure on each $\mathcal{X}_{\Sigma}$ given by

$$\{X_i, X_j\} = b_{ij} d_j X_i X_j.$$

One can easily check that this Poisson bracket is preserved by cluster transformations, and hence the definition above is independent of the chart we chose.

Now we will describe the natural map from $\mathcal{A}_{\Sigma}$ to $\mathcal{X}_{\Sigma}$. Let us assume that the entries of the B-matrix are all integers. Then we can define $p : \mathcal{A}_{\Sigma} \to \mathcal{X}_{\Sigma}$ by

$$p^*(X_i) = \prod_{j \in I} A_j^{B_{ij}}.$$

This formula appears to depend on the seed, but it actually intertwines the mutation of both the $\mathcal{A}$-coordinates and the $\mathcal{X}$-coordinates. In other words, if Σ' is obtained from Σ by mutation at k, there is a commutative diagram

$$\begin{tikzcd} \mathcal{A}_{\Sigma} \arrow[r, dashed, "\mu_k"] \arrow[d, "p"] & \mathcal{A}_{\Sigma'} \arrow[d, "p'"] \\ \mathcal{X}_{\Sigma} \arrow[r, dashed, "\mu_k"] & \mathcal{X}_{\Sigma'}. \end{tikzcd}$$

If B is not integral, we can instead consider the space $\mathcal{X}^*_{\Sigma}$ obtained from $\mathcal{X}_{\Sigma}$ by projecting to the unfrozen variables. We will then have a well-defined map from $\mathcal{A}_{\Sigma}$ to $\mathcal{X}^*_{\Sigma}$.

Throughout this paper, we will encode B-matrices by means of quivers. These quivers will have black and white vertices. The black vertices will correspond to indices $i \in I$ such that $d_i = 1$. The white vertices will correspond to indices $i \in I$ such that $d_i = 2$.

The quivers will have both solid arrows and dotted arrows. The dotted arrows are in a sense 'half' a solid arrow. The B-matrix is read off from the quiver by the following rules:

- An arrow from j to i means that $b_{ij} > 0$.
- If there is a solid arrow between i and j, and if $d_i = 2$ and $d_j = 1$, then $|b_{ij}| = 2$.
- In all other cases, if there is a solid arrow between i and j, then $|b_{ij}| = 1$.
- A dotted arrow means that b_{ij} is half the value it would be if the arrow were solid. In other words, $|b_{ij}| = 1$ if $d_i = 2$ and $d_j = 1$ and $|b_{ij}| = \frac{1}{2}$ otherwise.

Note that we will only allow dotted arrows to go between frozen vertices, thus the entries b_{ij} of the B-matrix are integral unless i and j are both frozen, and thus the B-matrix defines a cluster algebra. Our first example will be the quiver depicted in Figure 4.

Finally, let us review some facts about the cluster structure on B. We will later relate the cluster structure to B to the cluster structure on $\mathrm{Conf}_3 \mathcal{A}$. These cluster structures induce the positive structures on B and $\mathrm{Conf}_3 \mathcal{A}$, respectively.

Recall from equation (2.2) that the positive structure on $\mathrm{Conf}_3 \mathcal{A}$ is given by a map $H \times B^- \longrightarrow \mathrm{Conf}_3(\mathcal{A})$ that sends

$$(h, b) \to (U^-, h \cdot \overline{w_0} U^-, b \cdot \overline{w_0} U^-).$$

Then the natural positive structure on $H \times B$ induces a positive structure on $\mathrm{Conf}_3 \mathcal{A}$. This positive structure coincides with the one given in (2.2). Let us

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now restrict our attention to triples of principal flags of the form $(U^-, \overline{w_0}U^-, b \cdot \overline{w_0}U^-)$. We can consider the map

$$i : b \in B^- \to (U^-, \overline{w_0}U^-, b \cdot \overline{w_0}U^-) \in \mathrm{Conf}_3 \mathcal{A}.$$

For u, v elements of the Weyl group W of G, we have the double Bruhat cell

$$G^{u,v} = B^+ \cdot u \cdot B^+ \cap B^- \cdot v \cdot B^-.$$

The cell $G^{w_0,e}$ is the open part of B^-.

We will show in all the cases we consider that the cluster structure we construct on $\mathrm{Conf}_3 \mathcal{A}$, when restricted to the image of i, coincides with the cluster algebra structure given in [**BFZ**] on the cell $G^{w_0,e} \subset B^-$.

We will now recall the cluster variables on $G^{w_0,e}$ given in [**BFZ**].

Let $G_0 = U^-HU^+ \subset G$ be the open subset of elements of G having Gaussian decomposition $x = [x]_-[x]_0[x]_+$. Then for any two elements $u, v \in W$, and any fundamental weight ω_i, we can define the *generalized minor* $\Delta_{u\omega_i, v\omega_i}(x)$. It is a rational function on G which is given generically by the formula

$$\Delta_{u\omega_i, v\omega_i}(x) := ([\overline{u}^{-1} x \overline{v}]_0)^{\omega_i}.$$

The formula gives a well-defined value when $\overline{u}^{-1} x \overline{v} \in G_0$, but may have poles elsewhere.

The cluster structure on B^- has cluster variables which are generalized minors for $v = e$ and u some initial subword of our reduced word for w_0. In particular, the cluster functions on B^- given in [**BFZ**] are $\Delta_{\omega_i, \omega_i}$ for all i (these are the functions associated to $u, v = e$), and

$$\Delta_{u\omega_j, \omega_j},$$

where u ranges over subwords of w_0 ending in s_j.

2.7. Amalgamation.

The cluster structure for $\mathcal{A}_{G,S}$ comes from triangulating S and then attaching the cluster structure on $\mathrm{Conf}_3 \mathcal{A}$ to each triangle. We will work with the case of S a disc with m marked points for simplicity, and describe the cluster structure on $\mathrm{Conf}_m \mathcal{A}$.

Note that the frozen vertices for $\mathrm{Conf}_3 \mathcal{A}$ are attached to functions which only depend on two of the three flags.

To form the quiver for $\mathrm{Conf}_m \mathcal{A}$, we first take a triangulation of an m-gon. On each of the $m-2$ triangles, attach any one of the six quivers formed from performing S_3 symmetries on the quiver for $\mathrm{Conf}_3 \mathcal{A}$ described above. Each edge of each of these triangles has n frozen vertices. Let us describe how to glue two triangles together.

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Let T_1, T_2 be two triangles with edges e_1, e_2, e_3 and e_4, e_5, e_6, respectively. Suppose that we would like to glue the edges e_1 and e_4. e_1 and e_4 each have r frozen vertices, where r is the rank of G. We will glue these $2r$ vertices together in pairs to form r vertices. Each frozen vertex is glued to another vertex that shares the same function. These vertices then become unfrozen. If vertices i and j are glued with i' and j' to get new vertices i'' and j'', then we declare that

$$b_{i''j''} = b_{ij} + b_{i'j'}.$$

In other words, two dotted arrows in the same direction glue to give us a solid arrow, whereas two dotted arrows in the opposite direction cancel to give us no arrow. One can easily check that any gluing will result in no dotted arrows using the unfrozen vertices. The arrows involving vertices that were not previously frozen remain the same. Figure 20 shows one gluing between two triangles for Sp_8.

Repeat this procedure for each interior edge of the triangulation, and one arrives at the quiver for $\mathrm{Conf}_m \mathcal{A}$. The procedure for gluing triangles is very reminiscent of the 'amalgamation' procedure in [**FG4**].

2.8. Webs. In this section we define webs for SL_N. Webs are planar diagrams that encode tensor invariants for some group, which we take here to be SL_N. We work over a field F of characteristic 0, and we let V be the standard N-dimensional representation of SL_N.

It is well known that the functions on $\mathcal{A}_G$ are naturally isomorphic to

$$\bigoplus_{\lambda \in \Lambda_+} V_\lambda.$$

Here the sum is taken over the set of dominant weights, Λ_+. Thus we have that the functions on $\mathrm{Conf}_3 \mathcal{A}_G$ are naturally isomorphic to

$$\bigoplus_{(\lambda,\mu,\nu)\in\Lambda_+^3} [V_\lambda \otimes V_\mu \otimes V_\nu]^G.$$

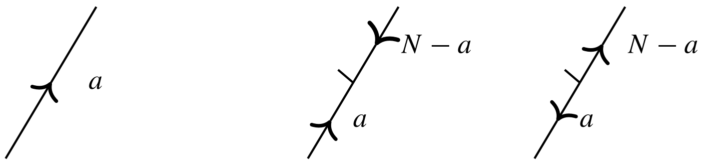
It turns out that all the cluster variables on $\mathrm{Conf}_3 \mathcal{A}_G$ that we will consider in this paper will be invariants of triple tensor products of representations of G, that is, they will lie inside

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^G$$

for some λ, μ, ν. Here, V_λ, V_μ, V_ν correspond to a graded subspace of the vector space of functions on the first, second, and third flags, respectively.

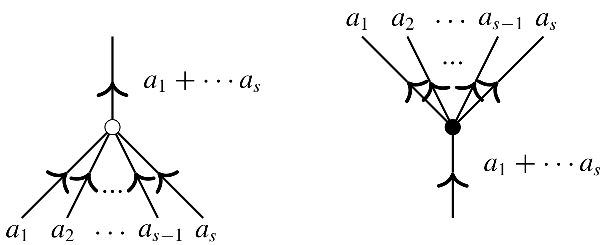
Webs turn out to be a convenient tool both for encoding cluster variables and for computing with them. We are not the first to observe this: there are some fairly extensive conjectures on the relationship between webs and cluster variables in the work of Fomin and Pylyavskyy [FP].

DEFINITION 2.5. A *SL_N-web* W is a planar graph in the disk subject to the following. First, each edge of W is directed and labeled by a multiplicity in one of the three following ways



pair tag source tag . (2.6)

The 'tiny edges' decorating the second and third edge types are called *tags* and they provide us with a preferred choice of side for the given edge. Second, we require that each interior vertex of W is modeled on one of the following two pictures:



wedge shuffle . (2.7)

Thirdly and finally, we require that each boundary edge in W is a source. The *degree* $\lambda(W) = (\lambda_1, \ldots, \lambda_m)$ of W is the sequence of edge multiplicities of the boundary edges.

Now we explain how an SL_r-web W of degree λ determines a tensor invariant

$$W \in [V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_m}]^{SL_N}.$$

Here V_λ is the fundamental representation of SL_N of weight ω_λ, or $\wedge^\lambda(V)$.

An edge with multiplicity $m(e)$ in W encodes a copy of $\bigwedge^{m(e)}(V)$. Each of the interior vertices (2.7) encodes an SL_N-invariant map between the indicated tensor powers of fundamental representations. The first vertex encodes the exterior product map

$$\bigwedge^{a_1}(V) \otimes \cdots \otimes \bigwedge^{a_s}(V) \to \bigwedge^{a_1+\cdots+a_s}(V) \tag{2.8}$$

given by $x_1 \otimes x_2 \otimes \cdots \otimes x_s \mapsto x_1 \wedge x_2 \wedge \cdots \wedge x_s$. The second vertex encodes the map

$$\bigwedge^{a_1+\cdots+a_s}(V) \to \bigwedge^{a_s}(V) \otimes \cdots \otimes \bigwedge^{a_1}(V) \tag{2.9}$$

given by sending the wedge product $x_1 \wedge \cdots \wedge x_b \in \bigwedge^b(V)$ to the signed sum of shuffles

$$\begin{aligned} &\sum \pm(x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_{a_s}}) \otimes \cdots \otimes (x_{i_{b-a_1+1}} \wedge x_{i_{b-a_1+2}} \wedge \cdots \wedge x_{i_b}) \\ &\quad \in \bigwedge^{a_s}(V) \otimes \cdots \otimes \bigwedge^{a_1}(V) \end{aligned} \tag{2.10}$$

where the summation is over permutations $(i_1, i_2, \ldots, i_b)$ of $(1, 2, \ldots, b)$ such that indices are increasing in each block: $i_1 < i_2 < \cdots < i_{a_s}, \ldots, i_{b-a_1+1} < \cdots < i_b$. The sign $\pm$ is the sign of the permutation $(i_1, i_2, \ldots, i_b)$. Note that in both cases, the cyclic order of the edges in (2.7) is crucial for specifying signs.

The tagged edges in (2.6) should be thought of as degenerate cases of the maps (2.8) and (2.9), where the tag encodes a copy of $\bigwedge^N(V)$, which we canonically identify with F using the volume form. Thus the pair tag $\bigwedge^a \otimes \bigwedge^{N-a} \to \bigwedge^N \cong F$ produces a number obtained by pairing the two incoming tensors, and source tag 'creates' two tensors using the shuffle $\bigwedge^N \to \bigwedge^a \otimes \bigwedge^{N-a}$. It turns out that we will not need to use the source tag in this paper. Note that the side of the edge that the tag occurs on matters, because, for example, the maps $\bigwedge^a \otimes \bigwedge^{N-a} \to \bigwedge^N \cong F$ and $\bigwedge^{N-a} \otimes \bigwedge^a \to \bigwedge^N \cong F$ will differ by a sign if both $N-a$ and a are odd.

To evaluate W on a simple tensor $x_1 \otimes \cdots \otimes x_n \in \bigotimes_{i=1}^n \bigwedge^{\lambda_i}(V)$, we imagine placing each tensor x_i at boundary vertex i. We repeatedly compose the four basic morphisms—wedge, shuffle, pair, and source—as indicated by the arrows in W to obtain the value $W(x_1 \otimes \cdots \otimes x_n)$.

All the functions in this paper will be described as webs using the procedure described above. In types B and D, we will sometimes need to take square roots of a function described by an SL_N-web. There is a unique branch of the square root which is dictated by positivity. This is explored in detail in Section 5.2, though the discussion there is rather technical and can be skipped on a first reading.

Let us now illustrate how a web gives rise to a function. Let us remark that the conventions for evaluating webs vary widely from author to author. The examples below will illustrate our particular choice of conventions. In these examples, we

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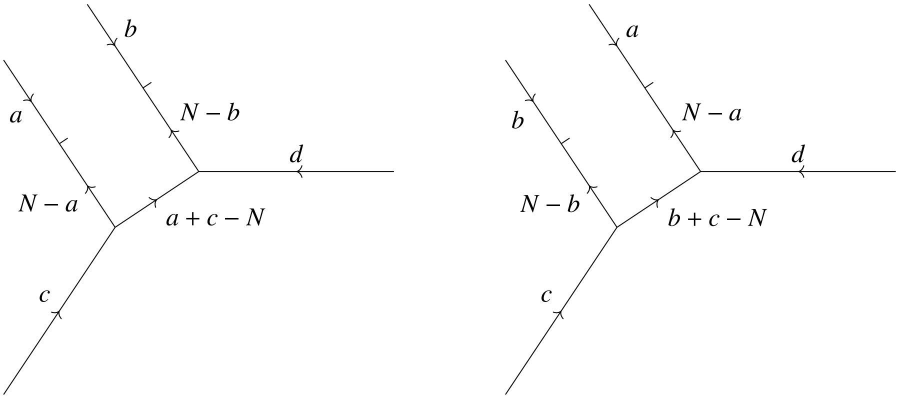

Figure 1. Two equivalent webs for the function $\begin{pmatrix} a,b & \\ & d \\ c & \end{pmatrix}$.

will define some functions on $\mathrm{Conf}_3\,\mathcal{A}_{SL_N}$. We will start with the web diagram, and extract from it a linear algebraic expression, more precisely, an expression involving the exterior algebra of V. This can be done for all the webs in this paper.

Let $1 \leqslant a, b, c, d < N$ be four integers satisfying $a > N-c > b$, $a > N-d > b$ and $a+b+c+d = 2N$. Then there is a one-dimensional space of invariants inside the representation

$$[V_{\omega_a+\omega_b} \otimes V_{\omega_c} \otimes V_{\omega_d}]^{SL_N}.$$

We pick out the function given by the two equivalent webs in Figure 1.

This web encodes a function on the space $\mathrm{Conf}_3\,\mathcal{A}_G$ which we will call $\begin{pmatrix} a,b & \\ & d \\ c & \end{pmatrix}$. Here is how to evaluate the function. Given three flags

$$\begin{gathered} u_1, \ldots, u_N; \\ v_1, \ldots, v_N; \\ w_1, \ldots, w_N; \end{gathered}$$

first consider the forms

$$\begin{aligned} U_a &:= u_1 \wedge \cdots \wedge u_a, \\ U_b &:= u_1 \wedge \cdots \wedge u_b, \\ V_c &:= v_1 \wedge \cdots \wedge v_c, \\ W_d &:= w_1 \wedge \cdots \wedge w_d. \end{aligned}$$

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There is a natural map

$$\phi_{a+c-N,N-a} : \bigwedge^{c} V \to \bigwedge^{a+c-N} V \otimes \bigwedge^{N-a} V.$$

There are also natural maps

$$U_b \wedge - \wedge W_d : \bigwedge^{a+c-N} V \to \bigwedge^{N} V \simeq F$$

and

$$U_a \wedge - : \bigwedge^{N-a} V \to \bigwedge^{N} V \simeq F.$$

Applying these maps to the first and second factors of $\phi_{a+c-N,N-a}(V_c)$, respectively, and then multiplying, we get the value of our function. This is a function on $\mathrm{Conf}_3 \mathcal{A}_{SL_N}$. Let $N = 2n$. Then pulling back gives a function on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$. We will use the notation $\left(\begin{smallmatrix} a,b & \\ & d \\ c & \end{smallmatrix}\right)$ to denote this function on either of those two spaces.

An equivalent way to calculate the function, associated to the second web above, is to use the natural map

$$\phi_{N-a,a+d-N} : \bigwedge^{d} V \to \bigwedge^{N-a} V \otimes \bigwedge^{a+d-N} V.$$

There are natural maps

$$U_b \wedge V_c \wedge - : \bigwedge^{a+d-N} V \to \bigwedge^{N} V \simeq F$$

and

$$U_a \wedge - : \bigwedge^{N-a} V \to \bigwedge^{N} V \simeq F.$$

Applying these maps to the second and first factors of $\phi_{N-a,a+d-N}$, respectively, and then multiplying, we get the value of our function. We will use $\left(\begin{smallmatrix} b,a & \\ & d \\ c & \end{smallmatrix}\right)$ to denote this function. Note that $\left(\begin{smallmatrix} b,a & \\ & d \\ c & \end{smallmatrix}\right) = \left(\begin{smallmatrix} a,b & \\ & d \\ c & \end{smallmatrix}\right)$.

Now let us define another, simpler, set of functions. Let $0 \leqslant a, b, c < N$ be three integers such that $a + b + c = N$. Then there is a one-dimensional space of invariants inside the representation $[V_{\omega_a} \otimes V_{\omega_b} \otimes V_{\omega_c}]^{SL_N}$. We pick out the function given by the web in Figure 2:

The function can be calculated as follows. Given three flags

$$u_1, \ldots, u_N;$$
$$v_1, \ldots, v_N;$$
$$w_1, \ldots, w_N;$$

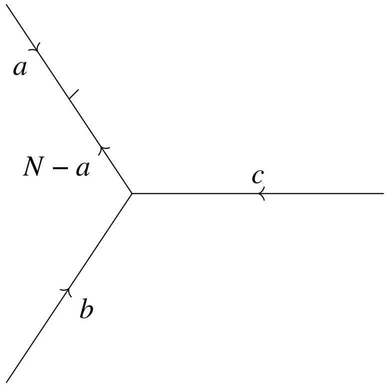

Figure 2. Web for the function $\begin{pmatrix} a & \\ & c \\ b & \end{pmatrix}$, where $a + b + c = N$.

first consider the forms

$$U_a := u_1 \wedge \cdots \wedge u_a,$$
$$V_b := v_1 \wedge \cdots \wedge v_b,$$
$$W_c := w_1 \wedge \cdots \wedge w_c.$$

Then $U_a \wedge V_b \wedge W_c$ is a multiple of $e_1 \wedge \cdots \wedge e_N$, and this multiple is the value of our function. Call this function $\begin{pmatrix} a & \\ & c \\ b & \end{pmatrix}$. In the case that one of a, b, c is 0, we will leave a blank space instead of writing 0. Moreover, when $c = 0$, we will write $\binom{a}{b}$ instead of $\begin{pmatrix} a & \\ & \\ b & \end{pmatrix}$.

Let us collect here for convenience a list of the web identities that we shall use throughout the paper:

- The itemized list in Proposition 3.2 consisting of the octahedron recurrence and degeneracies. These are the identities used in Theorems 3.8, 4.8, 5.8, which concern the first transposition.
- The dualities (3.1), (4.1), (5.3) are used throughout.
- Theorems 3.10 and 3.12, on the second transposition and the flip for Sp_{2n}, use particular specializations of the octahedron recurrence, plus some degeneracies. The specializations are listed in the proofs of the theorems.
- In Theorems 4.9 and 4.11, on the second transposition and the flip for $Spin_{2n+1}$, we need to derive one new identity, which we reduce to ones previously discussed (octahedron, degeneracy, duality).

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- In Theorems 5.8 and 5.9, on the first and second transpositions for $Spin_{2n+2}$, we need a simple product identity (5.5).
- In Theorem 5.13, on the flip for $Spin_{2n+2}$, we again need to derive one new identity, which we reduce to ones previously discussed (octahedron, degeneracy, duality, and the product identity).

2.9. Brief outline. We can now give a brief outline of the goals of the this paper. Our primary goal will be to construct the cluster structure on the spaces $\mathcal{X}_{G'S}$ and $\mathcal{A}_{G,S}$ in types B, C and D. For the bulk of the paper, we focus on the space $\mathcal{A}_{G,S}$. The cluster structure for $\mathcal{X}_{G'S}$ can easily be deduced from the one on $\mathcal{A}_{G,S}$ and we explain how to do this in Section 6.

We give a cluster on $\mathcal{A}_{G,S}$ for any choice of an ideal triangulation of S and any ordering of the vertices of each triangle in the ideal triangulation. We do this by giving a set of edge and face coordinates for the ideal triangulation along with the ordering of vertices in each triangle. Let us point out that there is no reason *a priori* that the cluster structure must exist. Fomin and Pylyavskyy did extensive work on configurations of *partial* flags for SL_3, and they show that there exist configurations whose ring of functions do not admit a cluster structure [**FP**].

Sections 3.1, 4.1, and 5.1 contain the construction of the seed. Figures 4, 22, and 32 depict the quiver for our preferred seed, and Figures 5, 23, and 33 depict the functions. Those sections also explain how to explicitly compute these functions.

We thus obtain a cluster for each triangulation of S with the data of an ordering of the vertices in each triangulation. We then need to relate all these clusters by mutations. We break this into two steps. The first step will be to relate the clusters associated to different orderings of the vertices of a triangle. We will say that these sequences of mutations realize S_3-symmetries. The second step will be to relate the clusters coming from different triangulations. It is sufficient to treat the case of a flip of triangulation. We know of no *a priori* reasons that the sequences of mutations realizing S_3-symmetries or flips should exist.

Let us examine the first step in detail. As discussed previously, in the case of a triangle, the space $\mathcal{A}_{G,S}$ may be viewed as $\mathrm{Conf}_3\,\mathcal{A}$ equipped with the twisted cyclic shift map T where $T(A, B, C) = (s_G \cdot C, A, B)$. Here, A, B, C are the three flags giving the decoration of the local system, labeled going counterclockwise. Our initial cluster consists of a quiver with the frozen vertices corresponding to edge variables and the nonfrozen vertices corresponding to face variables. The edge variables are associated with the edges of the triangle AB, BC, CA.

Let us describe now how to apply an S_3 symmetry to obtain other cluster structures. Let us first describe the quivers. At the cost of planarity, we may draw the quiver inside a triangle such that the edge functions lie on the corresponding

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edge of the triangle. We can arrange the face functions in the interior of the face however we like. If the S_3 symmetry is even, that is, a rotation, the new quiver is obtained by rotating the triangle. If the S_3 symmetry is odd, that is, a transposition, the quiver is obtained by transposing the triangle and reversing the arrows.

Let us now describe the functions. Let us first suppose that the S_3 symmetry is even. For concreteness, let us suppose that it comes from rotation of the quiver clockwise. Then if the function attached to a vertex in the original quiver was f, the function attached to the corresponding vertex in the new quiver would be $T^*(f)$, so that $T^*(f)(A, B, C) = f(s_G \cdot C, A, B)$. Similarly, if the quiver were rotated counterclockwise, the function attached to the corresponding vertex would be $(T^2)^*(f)$.

Now let us suppose the S_3 symmetry is odd, for example, the transposition switching the second and third flags. Then if the function attached to a vertex in the original quiver was $f(A, B, C)$, the function attached to the corresponding vertex in the new quiver would be $\pm f(A, C, B)$. The sign here is subtle, and is related to the fact that switching two flags is not a positive map on $\mathrm{Conf}_3 \mathcal{A}$.

We give an example in Figure 5 of two quivers related by a transposition of the second and third flags.

For example, there is a cluster for $\mathrm{Conf}_3 SL_N$ involving a function of the form $\begin{pmatrix} a,b & \\ & d \\ c & \end{pmatrix}$ defined above. The rotations of this cluster will involve the functions

$$\begin{pmatrix} c & \\ & a,b \\ d & \end{pmatrix} := T^* \begin{pmatrix} a,b & \\ & d \\ c & \end{pmatrix},$$

$$\begin{pmatrix} d & \\ & c \\ a,b & \end{pmatrix} := (T^2)^* \begin{pmatrix} a,b & \\ & d \\ c & \end{pmatrix}.$$

Moreover, there is a cluster for $\mathrm{Conf}_3 SL_N$ involving the functions $\begin{pmatrix} a & \\ & c \\ b & \end{pmatrix}$ where $a + b + c = N$. This cluster is S_3-symmetric in the sense that performing any of the S_3 symmetries brings us back to the same cluster. Note that the functions $\begin{pmatrix} a & \\ & c \\ b & \end{pmatrix}$ and $\begin{pmatrix} a & \\ & b \\ c & \end{pmatrix}$ are related by switching the second and third flags only up to a sign.

REMARK 2.6. Throughout this paper, functions that come from rotating the arguments will be related by the twisted cyclic shift map.

We will exhibit a sequence of mutations that corresponds to transposition of the first and third flags, as well a sequence of mutations that corresponds to transposition of the second and third flags. We call these the first and second transpositions, respectively. They generate all S_3 symmetries.

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Sections 3.4.1, 4.5.1, and 5.5.1 deal with the sequence of mutations for the first transposition. The sequence of mutations for the first transposition is given in equations (3.3), (4.2), and (5.4), respectively. These sections make use of the cactus sequence from [**HK**]. The discussion of the cactus sequence in Section 3.3 is important for these sections. On the other hand, the parallel discussion of the cactus sequence in Section 4.4, though it points to some interesting phenomena, is inessential to the remainder of the paper and can be skipped.

Sections 3.4.2, 4.5.2, and 5.5.2 deal with the sequence of mutations for the second transposition. The sequence of mutations for the second transposition is given in equations (3.4), (4.3), and (5.6), respectively. Section 3.4.3 discusses an alternative approach to the third transposition (valid for all types), which can logically substitute for Sections 3.4.2, 4.5.2, and 5.5.2. The approach in Section 3.4.3 is quite different from the rest of the paper, but it gives a somewhat shorter proof. We have decided to keep Sections 3.4.2, 4.5.2, and 5.5.2 because they have the advantage of identifying the intermediate functions that occur in the mutation sequence. The sequence of mutations there is also more efficient.

Now let us treat the second step. Any two different triangulations are related by a composition of flips of a triangulation. Thus it suffices to relate two clusters coming from the two triangulations of a quadrilateral. Moreover, it is enough to do this for any pair of clusters coming from these two triangulations; we may consider clusters related to any ordering of the vertices within each triangle.

Sections 3.5, 4.6, and 5.6 treat the sequence of mutations for the flip. The sequences of mutations are given in equations (3.5), (4.4), and (5.7). Those sections contain proofs that these sequences of mutations realize the maps of charts corresponding to a flip. Section 3.5.1 contains an alternative proof that this sequence of mutations is correct. The proof is more conceptual and less computational, though it is less explicit, and it relies on the previous work in [**BFZ**]. We believe there are advantages to both approaches, though a reader interested in efficiency may read Section 3.5.1 as a substitute for the rest of Sections 3.5, 4.6 and 5.6. Again, the approaches in the first part of Sections 3.5, 4.6 and 5.6 are more computational, but do have the advantage of identifying the intermediate functions that occur in the mutation sequence.

Once these two steps are completed, we have related by sequences of mutations any clusters coming from a triangulation of the surface and an ordering in each triangle. This gives us a mapping class group equivariant cluster structure on $\mathcal{A}_{G,S}$ and $\mathcal{X}_{G',S}$.

To show that we have given the correct sequences of mutations, we compute the functions at each stage of the mutation sequences explicitly with the aid of webs. In contrast with the situation when $G = SL_2$, the cluster structure on $\mathcal{A}_{G,S}$ is typically wild, so that finding these sequences is nontrivial. We explain

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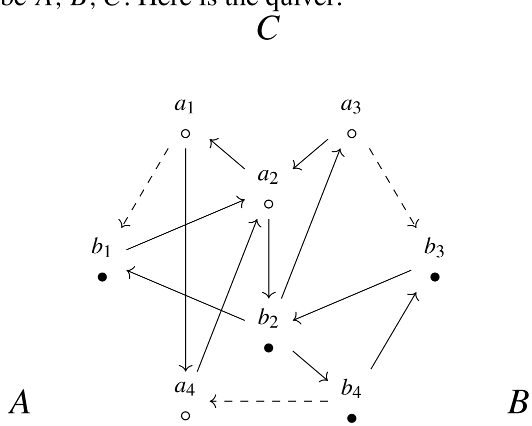
how a careful consideration of the cactus group [HK] and Langlands duality motivates some of these mutation sequences. In particular, we explain how the cluster structure type A folds to give the one in type C; how the cluster structures in types B and C are Langlands dual; and how one folds the cluster structure in type D to obtain the one in type B.

A secondary goal is to relate the cluster structures we construct on $\mathrm{Conf}_3 \mathcal{A}$ to Berenstein, Fomin and Zelevinsky's cluster structure on B, the Borel in the group G [BFZ]. This will allow us to see that our cluster structure induces a positive structure on $\mathcal{A}_{G,S}$, and moreover that this positive structure on $\mathcal{A}_{G,S}$ is identical to the one given in [FG1]. In fact, it was the comparison between the cluster structures in [BFZ] and [FG1] that initially motivated our constructions of the clusters in this paper.

The cluster structure on B is given by generalized minors of double Bruhat cells, of which B is one example. We will show that the tensor invariants we construct via webs coincide with the set of generalized minors for some reduced word for w_0. We also explain how coordinates on $\mathcal{X}_{G'S}$ are convenient for computing monodromies of the local system, that is, relate these coordinates to 'factorization coordinates.' Thus, we explicitly relate three different viewpoints on cluster variables: tensor invariants, generalized minors, and factorization coordinates. The relationship between these viewpoints is explained in Section 3.2. We note that similar results were obtained in [Z] for rank 2 groups.

2.10. The simplest example. In this section, we explain our results in the simplest case of $G = Sp_4 = Spin_5$.

We first describe the cluster structure on $\mathrm{Conf}_3 \mathcal{A}_{Sp_4}$. Let the three flags in the configuration be A, B, C. Here is the quiver:



Quiver for the cluster structure on $\mathrm{Conf}_3 \mathcal{A}_{Sp_4}$.

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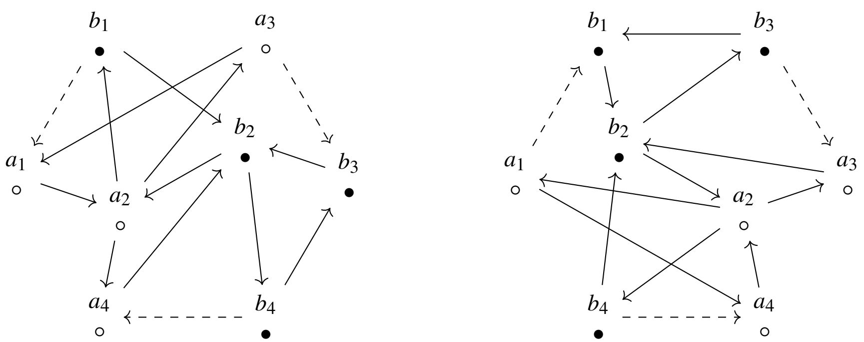
Each function can be described as a tensor invariant. For now, let us just state which tensor invariant spaces the functions lie in. We will say that a function has weight (λ, μ, ν) if it lies in the invariant space $[V_\lambda \otimes V_\mu \otimes V_\nu]^G$. Let α_1, α_2 be the short and long roots, respectively. We can take them to be $(0, 2)$ and $(1, -1)$, for example. Now let ω_1, ω_2 be the fundamental weights, which will then be $(1, 0)$ and $(1, 1)$. V_{ω_1} is the four-dimensional standard representation and V_{ω_2} is the five-dimensional standard representation of $Spin_5$, which is a subrepresentation of $\bigwedge^2 V_{\omega_1}$.

Then, for example, the function attached to a_4 has weight $(\omega_2, \omega_2, 0)$, and the function attached to b_4 has weight $(\omega_1, \omega_1, 0)$. The other edge variables are similar. The most interesting cases are the functions attached to a_2, b_2 which respectively have weights

$$(2\omega_1, \omega_2, \omega_2),$$

$$(\omega_1, \omega_2, \omega_1).$$

2.10.1. S_3 symmetries. Mutating at a_2 and rearranging gives the quiver on the left, which comes from the transposition of the first and third flags. Mutating at b_2 and rearranging gives the quiver on the right, which comes from the transposition of the second and third flags.



The transpositions obtained by mutating at a_2 and b_2, respectively.

In the left diagram, the function attached to b_2 is unchanged, but the function attached to a_2 changes. The functions attached to a_2, b_2 now respectively have weights

$$(\omega_2, \omega_2, 2\omega_1),$$

$$(\omega_1, \omega_2, \omega_1).$$

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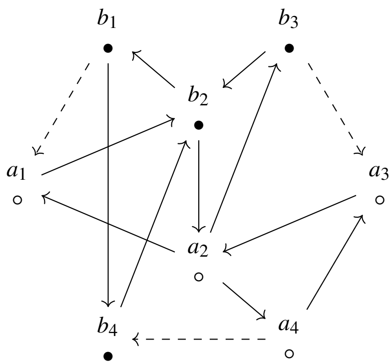
We see that the first and third weights have been exchanged. In the right diagram, a_2 is unchanged, but the function attached to b_2 changes. The functions attached to a_2, b_2 now respectively have weights

$$(2\omega_1, \omega_2, \omega_2),$$
$$(\omega_1, \omega_1, \omega_2).$$

We see that the second and third weights have been exchanged.

2.10.2. Langlands duality. Let us now explain the phenomenon of Langlands duality in this case. Mutating the sequence a_2, b_2, a_2 gives the following quiver, which is Langlands dual to the original quiver:



The Langlands dual quiver.

Note that the quiver is isomorphic to the original, except that the vertices a_i and b_i have been exchanged. The weights attached to the functions at a_2 and b_2 were originally, respectively,

$$w_a := (2\omega_1, \omega_2, \omega_2),$$

$$w_b := (\omega_1, \omega_2, \omega_1).$$

In the Langlands dual quiver, the weights are now

$$w'_a := (\omega_2, 2\omega_1, \omega_2),$$

$$w'_b := (\omega_2, \omega_1, \omega_1).$$

There is a map, canonically defined up to a constant, between the weight space of a group and the weight space of its Langlands dual. In this case, the map takes ω_1 to $\omega_1^\vee$ and ω_2 to $2\omega_2^\vee$. However, as Sp_4 is Langlands self-dual (we are not

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concerned with centers here), we have that $\omega_1^\vee = \omega_2$ and $\omega_2^\vee = \omega_1$. Thus there is a self-map L of the weight space which takes ω_1 to ω_2 and ω_2 to $2\omega_1$. Then we have that

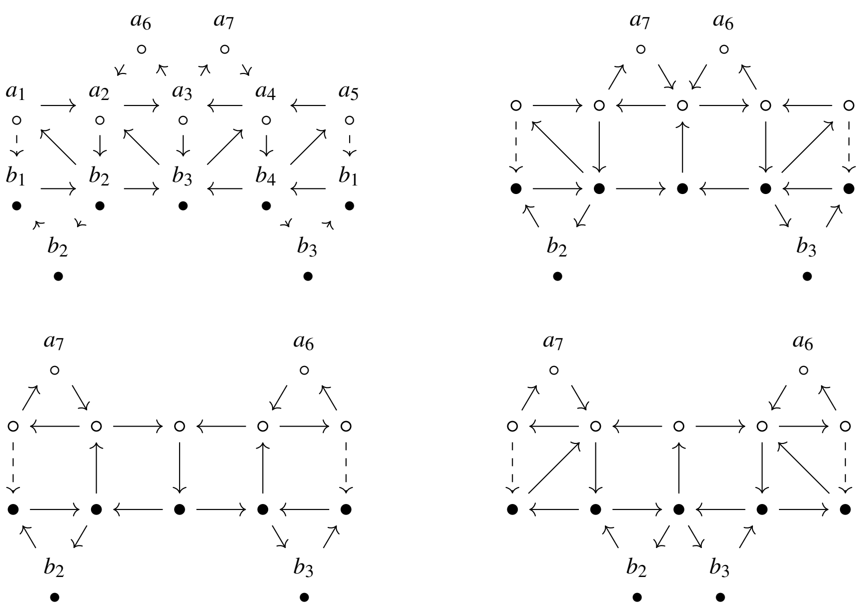
$$w'_a = Lw_b,$$

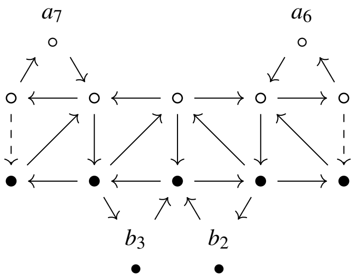
$$w'_b = \tfrac{1}{2}Lw_a.$$

This is an instance of Langlands duality, as explored in-depth in Section 4.2.

2.10.3. The flip. Below we give the sequence of mutations realizing a flip of triangulation. We depict the original quiver, and then the state of the quiver after the following four stages of mutation: a_3; a_2, a_4, b_3; a_3, b_2, b_4; b_3.

If the four flags are A, B, C, D, then a_6, b_6 are edge variables for the edge AB; a_1, b_1 are edge variables for the edge BC; a_5, b_5 are edge variables for the edge CD; and a_7, b_7 are edge variables for the edge DA. At the start of the mutation sequence, a_3, b_3 are edge variables for the edge AC, and by the end of the sequence they become edge variables for the edge BD. Note that throughout there are dotted arrows between b_6 and a_6 as well as between b_7 and a_7. We suppress these arrows for simplicity of the diagrams. The vertices a_6, a_7, b_6, b_7 are labeled throughout, because it is convenient for drawing the diagrams that they are allowed to move. The remaining vertices stay fixed.





3. The cluster algebra structure on $\mathrm{Conf}_m \mathcal{A}$ for $G = Sp_{2n}$

3.1. Construction of the seed. In this section we give the construction of seeds for the cluster structure on $\mathrm{Conf}_m \mathcal{A}$ when $G = Sp_{2n}$. Throughout this section, $G = Sp_{2n}$ unless otherwise noted.

Recall that Sp_{2n} is associated to the type C Dynkin diagram:

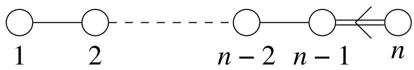


Figure 3. C_n Dynkin diagram.

The nodes of the diagram correspond to $n-1$ short roots, numbered $1, 2, \ldots, n-1$, and one long root, which is numbered n. To describe the cluster structure on $\mathrm{Conf}_3 \mathcal{A}$, we need to give the following data: the set I parameterizing vertices, the functions on $\mathrm{Conf}_3 \mathcal{A}$ corresponding to each vertex, the B-matrix for this seed, and the multipliers d_i for each vertex i.

The B-matrix is encoded via a quiver which consists of $n^2 + 2n$ vertices, of which $n + 2$ are white, while the remaining vertices are black. There are n edge functions for each edge of the triangle, and $n^2 - n$ face functions. There is one white vertex along each edge.

In Figure 4, we see the quiver for Sp_6. The generalization for other values of n should be clear.

We will no longer use single letters like i, j to denote vertices of the quiver, because it will be convenient for us to use the pairs (i, j) to parameterize the vertices of the quiver. In the formulas in the remainder of this section, we will not refer to the particular entries of the B-matrix, b_{ij}. Instead, the values of the entries of the B-matrix will be encoded in quivers. This will hopefully avoid any notational confusion.

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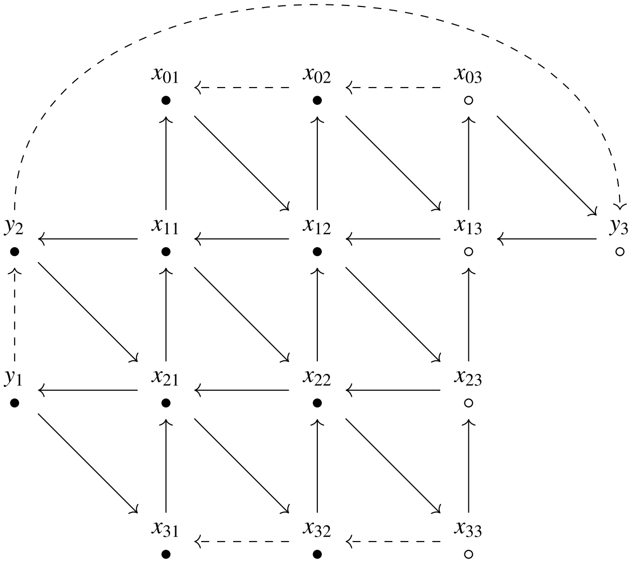

Figure 4. Quiver encoding the cluster structure for $\mathrm{Conf}_3\,\mathcal{A}_{Sp_6}$.

Label the vertices of the quiver x_{ij} and y_k, where $0 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant n$. The white vertices correspond to x_{in} and y_n. The vertices y_k and x_{ij} for $i = 0$ or n are frozen. We will sometimes write $x_{i,j}$ for x_{ij} for orthographic reasons.

Our next goal will be to define the functions attached to the vertices in the quiver. We first need to recall some facts about the representation theory of Sp_{2n}. We will work over a field F of characteristic 0. The fundamental representations of Sp_{2n} are labeled by the fundamental weights $\omega_1, \ldots, \omega_n$. Sp_{2n} has a standard $2n$-dimensional representation V. Let $\langle -, - \rangle$ be the symplectic pairing. Then the representation V_{ω_i} corresponding to ω_i is a direct summand of $\bigwedge^i V$. In fact, it is the kernel of the homomorphism $\bigwedge^i V \to \bigwedge^{i-2} V$ that comes from contracting with the symplectic form. Note that the symplectic form also induces an isomorphism between $\bigwedge^i V$ and $\bigwedge^{2n-i} V$.

Let $\mathcal{A}_G$ denote the principal affine space for G. We will sometimes drop the subscript 'G' if it is clear which group we are referring to. There is a natural embedding of $Sp_{2n} \hookrightarrow SL_{2n}$ given by taking the subgroup of SL_{2n} that fixes the symplectic form $\langle e_i, e_{2n+1-i} \rangle = (-1)^{i-1}$, where the vectors e_i form the standard basis of V (the signs are chosen here to be compatible with the positive structures on Sp_{2n} and SL_{2n}). This induces a natural embedding of $\mathcal{A}_{Sp_{2n}}$ inside $\mathcal{A}_{SL_{2n}}$, which we now describe.

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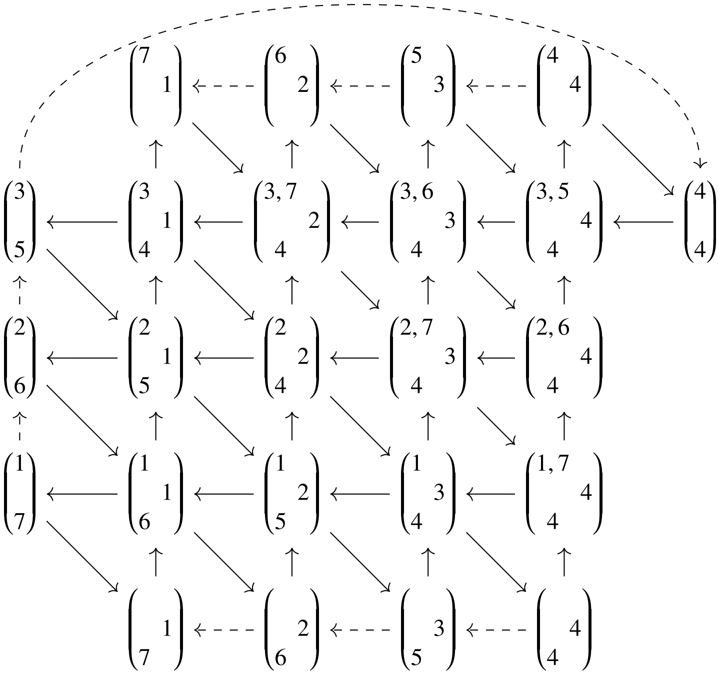

Figure 5a. A cluster structure for $\mathrm{Conf}_3\,\mathcal{A}_{Sp_8}$.

The variety $\mathcal{A}_{Sp_{2n}}$ parameterizes chains of isotropic vector spaces

$$V_1 \subset V_2 \subset \cdots \subset V_n \subset V$$

inside the $2n$-dimensional standard representation V, where $\dim V_i = i$, and where each V_i is equipped with a volume form.

Equivalently, a point of $\mathcal{A}_{Sp_{2n}}$ is given by a sequence of vectors

$$v_1, v_2, \ldots, v_n,$$

where

$$V_i := \langle v_1, \ldots, v_i \rangle$$

is isotropic, and where v_i is only determined up to adding linear combinations of v_j for $j < i$.

The volume form on V_i is then $v_1 \wedge \cdots \wedge v_i$.

From the sequence of vectors $v_1, \ldots, v_n$, we can complete to a symplectic basis $v_1, v_2, \ldots, v_{2n}$, where $\langle v_i, v_{2n+1-i} \rangle = (-1)^{i-1}$, and $\langle v_i, v_j \rangle = 0$ otherwise.

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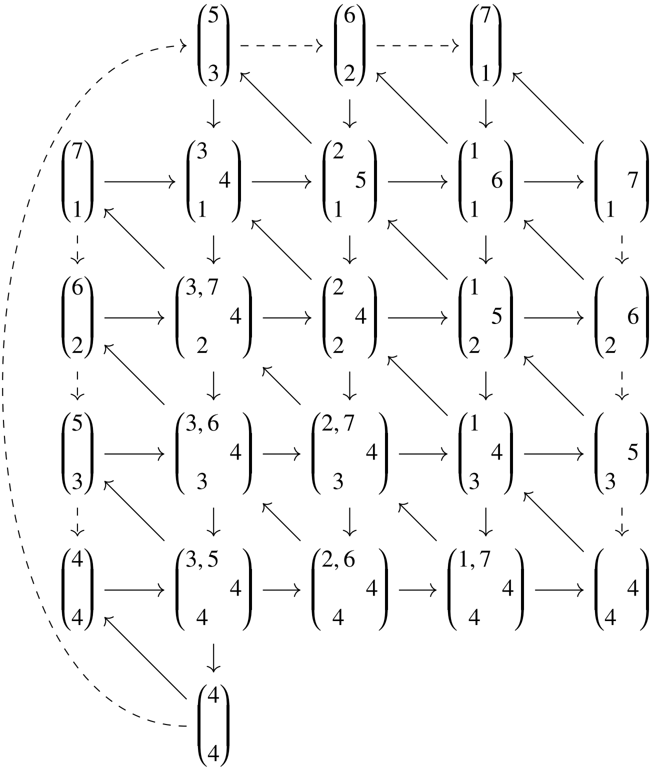

Figure 5b. Another cluster structure for $\mathrm{Conf}_3 \mathcal{A}_{Sp_8}$ related by an S_3 symmetry.

Equivalently, the symplectic form induces an isomorphism $\langle -, - \rangle : V \to V^*$. At the same time, there are perfect pairings

$$\bigwedge^k V \times \bigwedge^k V^* \to F$$

$$\bigwedge^{2n-k} V \times \bigwedge^k V \to F$$

that induce an isomorphism

$$\bigwedge^{2n-k} V \simeq \bigwedge^k V^*.$$

Composing this with the inverse of the isomorphism

$$\langle -, - \rangle : \bigwedge^k V \to \bigwedge^k V^*$$

gives an isomorphism

$$\bigwedge^{2n-k} V \simeq \bigwedge^{k} V^* \simeq \bigwedge^{k} V.$$

Then $v_{n+1}, \ldots, v_{2n}$ are chosen so that this isomorphism takes $v_1 \wedge \cdots \wedge v_k$ to $v_1 \wedge \cdots \wedge v_{2n-k}$.

Then $v_1, v_2, \ldots, v_{2n}$ determines a point of $\mathcal{A}_{SL_{2n}}$, as $\mathcal{A}_{SL_{2n}}$ parameterizes chains of vector subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_{2n} \subset V$$

along with volume forms $v_1 \wedge \cdots \wedge v_i$, $1 \leqslant i \leqslant 2n$.

From the embedding

$$\mathcal{A}_{Sp_{2n}} \hookrightarrow \mathcal{A}_{SL_{2n}},$$

one naturally gets an embedding $\mathrm{Conf}_m \mathcal{A}_{Sp_{2n}} \hookrightarrow \mathrm{Conf}_m \mathcal{A}_{SL_{2n}}$. We will define the cluster functions on $\mathrm{Conf}_m \mathcal{A}_{Sp_{2n}}$ via cluster functions on $\mathrm{Conf}_m \mathcal{A}_{SL_{2n}}$. In fact, we will see later in this paper that the entire cluster algebra structure on $\mathrm{Conf}_m \mathcal{A}_{Sp_{2n}}$ comes from *folding* the cluster algebra structure on $\mathrm{Conf}_m \mathcal{A}_{SL_{2n}}$.

We previously constructed functions $\begin{pmatrix} a & \\ b & c \end{pmatrix}$ and $\begin{pmatrix} a,b & \\ c & d \end{pmatrix}$ on $\mathrm{Conf}_m \mathcal{A}_{SL_N}$. Now set $N = 2n$. We can pull back these functions to $\mathrm{Conf}_m \mathcal{A}_{Sp_{2n}}$. We are now ready to specify the functions attached to the vertices of the quiver. Here are the rules:

(1) We assign the function $\begin{pmatrix} k & \\ 2n-k & \end{pmatrix}$ to y_k.

(2) When $i \geqslant j$, we assign the function $\begin{pmatrix} n-i & \\ n+i-j & j \end{pmatrix}$ to x_{ij}.

(3) When $i < j$ and $i \neq 0$, we assign the function $\begin{pmatrix} n-i,2n+i-j & \\ n & j \end{pmatrix}$ to x_{ij}.

(4) When $i = 0$, we assign the function $\begin{pmatrix} 2n-j & \\ & j \end{pmatrix}$ to x_{ij}.

This completely describes the cluster structure on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$. Note that the cluster structure is not symmetric with respect to the three flags. Performing various S_3 symmetries, we obtain six different possible cluster structures on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$. We will later see that these six structures are related by sequences of mutations. In Figure 5, we depict two of the cluster structures for $\mathrm{Conf}_3 \mathcal{A}_{Sp_8}$: the standard cluster described above and the one obtained by applying the first transposition described below, which switches the second and third flags.

The functions attached to the vertices come from permuting the arguments in our notation for the function. For example, rotating the function $\begin{pmatrix} n-i,2n+i-j & \\ n & j \end{pmatrix}$

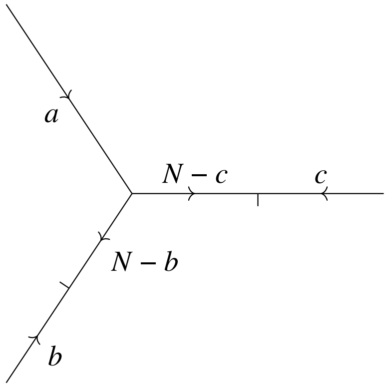

Figure 6. Web for the function $\begin{pmatrix} a & \\ & c \\ b & \end{pmatrix}$, where $a+b+c=N$.

gives the function $\begin{pmatrix} j & \\ & n \\ n-i,2n+i-j & \end{pmatrix}$, while transposing the first two arguments gives $\begin{pmatrix} n & \\ & j \\ n-i,2n+i-j & \end{pmatrix}$.

Another set of functions will be useful to us. Let $0 \leqslant a, b, c < N$ be three integers such that $a+b+c=2N$. Then there is a one-dimensional space of invariants inside the representation $[V_{\omega_a} \otimes V_{\omega_b} \otimes V_{\omega_c}]^{SL_N}$. We pick out the function given by the web in Figure 6.

Call this function $\begin{pmatrix} a & \\ & c \\ b & \end{pmatrix}$, whether it is viewed as a function on $\mathrm{Conf}_3 \mathcal{A}_{SL_N}$ or $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ when $N=2n$. Note that we have the following equalities of functions:

$$\begin{pmatrix} k & \\ & \\ 2n-k & \end{pmatrix} = \begin{pmatrix} 2n-k & \\ & \\ k & \end{pmatrix}$$
$$\begin{pmatrix} n-i & \\ & j \\ n+i-j & \end{pmatrix} = \begin{pmatrix} n+i & \\ & 2n-j \\ n-i+j & \end{pmatrix} \tag{3.1}$$
$$\begin{pmatrix} n-i,2n+i-j & \\ & j \\ n & \end{pmatrix} = \begin{pmatrix} n+i,j-i & \\ & 2n-j \\ n & \end{pmatrix}.$$

These equalities arise because the symplectic form induces a duality isomorphism between $\bigwedge^i V$ and $\bigwedge^{2n-i} V$.

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3.2. Reduced words. The goal of this section is to relate the cluster structure on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ given in the previous section to Berenstein, Fomin and Zelevinsky's cluster structure on B, the Borel in the group G [BFZ].

We consider the map

$$i : b \in B^- \to (U^-, \overline{w_0}U^-, b \cdot \overline{w_0}U^-) \in \mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}.$$

PROPOSITION 3.1. *The cluster algebra constructed above on* $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$*, when restricted to the image of* i*, coincides with the cluster algebra structure given in* [BFZ] *on the cell* $G^{w_0,e} \subset B^-$.

Proof. We choose a reduced word for w_0. In the numbering of the nodes of the Dynkin diagram given above for Sp_{2n}, we choose the reduced word expression

$$w_0 = (s_n s_{n-1} \cdots s_2 s_1)^n.$$

Here our convention is that the above word corresponds to the string $i_1, i_2, \ldots, i_{n-1}, i_n$ repeated n times.

Then the cluster functions on B^- given in [BFZ] are $\Delta_{\omega_i,\omega_i}$ for $1 \leqslant i \leqslant n$ (these are the functions associated to $u, v = e$), and

$$\Delta_{u_{ij}\omega_j,\omega_j},$$

which are the functions associated to $v = e$ and $u = u_{ij} = (s_n s_{n-1} \cdots s_2 s_1)^i s_n s_{n-1} \cdots s_j$. Note that u_{ij} is the subword of u that stops on the ith iteration of s_j.

We have the following claims:

(1) When $i = 0$, the function we assigned to x_{ij}, $\binom{2n-j}{\quad j}$, is precisely $\Delta_{\omega_j,\omega_j}$.

(2) When $i \geqslant j$, the function we assigned to x_{ij}, $\begin{pmatrix} n-i & \\ & j \\ n+i-j & \end{pmatrix}$, is precisely $\Delta_{u_{ij}\omega_j,\omega_j}$.

(3) When $i < j$, the function we assigned to x_{ij}, $\begin{pmatrix} n-i,2n+i-j & \\ & j \\ n & \end{pmatrix}$, is precisely $\Delta_{u_{ij}\omega_j,\omega_j}$.

The proof of these claims is a straightforward calculation. Let us carry out this calculation. The calculation is not central to this paper, so can be safely skipped.

It is convenient to choose an embedding of Sp_{2n} into SL_{2n}. Moreover, choose the symplectic form so that $\langle e_i, e_{n+i}\rangle = (-1)^i$ and all other pairings of basis elements are zero. The sign is chosen so that the representation $Sp_{2n} \hookrightarrow SL_{2n}$ preserves positive structure. Now choose a pinning such that under the embedding $Sp_{2n} \hookrightarrow SL_{2n}$,

$$E_{\alpha_n} = E_{n,2n}, \quad F_{\alpha_n} = E_{n,2n},$$

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and for $1 \leqslant i < n$,

$$E_{\alpha_i} = E_{i,i+1} + E_{n+i+1,n+i}, \quad F_{\alpha_i} = E_{i+1,i} + E_{n+i,n+i+1}.$$

Here $E_{i,j}$ is the (i, j)-elementary matrix, that is, the matrix with a 1 in the (i, j) position and 0 in all other positions.

With respect to this embedding, we can directly calculate $\Delta_{\omega_j,\omega_j}(x)$ where $x \in B^-$. When x is embedded in SL_{2n}, $\Delta_{\omega_j,\omega_j}(x)$ is simply the determinant of the minor consisting of the first j rows and the first j columns. Similarly, one can calculate that $\Delta_{u_{ij}\omega_j,\omega_j}(x)$ is precisely the determinant of the minor of x consisting of rows $2n+1-i, 2n+2-i, \ldots, 2n+j-i$ (here the indices are taken modulo $2n$) and the first j columns.

We then must calculate the functions $\binom{2n-j \quad}{\quad j}$, $\binom{\quad n-i \quad}{n+i-j \quad j}$ and $\binom{n-i, 2n+i-j \quad}{n \quad j}$ on the triple of flags $(U^-, \overline{w_0}U^-, b \cdot \overline{w_0}U^-)$. Under the embedding $Sp_{2n} \hookrightarrow SL_{2n}$, we should choose the flag U^- to be $-e_{n+1}, e_{n+2}, -e_{n+3}, \ldots, (-1)^n e_{2n}$, so that $\overline{w_0}U^-$ is given by the flag $e_1, e_2, \ldots, e_n$. Direct calculation then shows that $\binom{2n-j \quad}{\quad j}$, $\binom{\quad n-i \quad}{n+i-j \quad j}$ and $\binom{n-i, 2n+i-j \quad}{n \quad j}$ are given by determinants of the minors consisting of the first j rows and the first j columns and determinants of the minors consisting of rows $2n+1-i, 2n+2-i, \ldots, 2n+j-i$ (the indices taken modulo $2n$) and the first j columns. $\square$

The map i a specialization of a map from $B \times H$ to $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ given by

$$(b, h) \to (U^-, h \cdot \overline{w_0}U^-, b \cdot \overline{w_0}U^-)$$

to $B \times e$. The effect of changing the value of h is simple: if a function lies in the invariant space

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^G,$$

then the function becomes multiplied by $\mu(h)$. Thus the cluster coordinates on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ are given by generalized minors up to a monomial action of H.

We give a useful formula here that is related to the formula (6.1) given later on.

Consider the quiver as depicted in Figure 4 attached to the triple of flags (A, B, C). Now let us temporarily relabel y_n as $x_{1,n+1}$, and for $k \neq n$, let us temporarily relabel y_k as $x_{n-k,0}$. Now for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$ let us define

$$a_{ij} = \frac{x_{i,j+1}x_{i-1,j-1}}{x_{ij}x_{i-1,j}}.$$

We take $x_{ij} = 1$ in the above formula in any instance where x_{ij} has not yet been defined (this is just to take care of boundary cases). Suppose that (A, B) is a pair of principal flags. Then we may let $\overline{B} \in G/B$ be the projection of B to the flag variety. Then $(A, \overline{B})$ determines a frame for the group G. There is a unique $g \in G$ that translates $(C, \overline{B})$ to $(C, \overline{A})$. If we choose C to be the standard principal flag and $\overline{B}$ to be the opposite flag, then $g \in U$. We have the following formula from [**BFZ**]:

$$g = E_1(a_{n1})E_2(a_{n2}) \cdots E_n(a_{nn})E_1(a_{n-1,1}) \cdots E_n(a_{n-1,n}) \cdots E_1(a_{11}) \cdots E_n(a_{1n}). \tag{3.2}$$

In fact, this formula was shown to hold in the situation that the frozen variables formerly called y_k were all 1. In this situation, the cluster variables as we have defined them can be identified with generalized minors as in Section 3.2. The formulas in [**BFZ**] are in terms of generalized minors. To see that it holds in general, consider that action of H on B which preserves $\overline{B}$, and make the following three observations:

- The formulas for a_{ij} are invariant under the action of H on B.
- The element g only depends on $\overline{B}$.
- Using the action of H on B, we may arrange that $y_k = 1$.

Similar formulas hold for all reduced words.

3.3. Cactus transformation. In this section, first we collect the necessary facts that we need about the 'tetrahedron recurrence', which can be interpreted as a sequence of mutations on the cluster algebra for $\mathrm{Conf}_3 \mathcal{A}_{SL_N}$. Most of what is in this section can be found in [**Hen**] or [**HK**], though our notation is somewhat different. We will relate this to our construction of the cluster structure on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$. This will be useful for computations that we do later in the paper.

The tetrahedron recurrence is really just a variation on the octahedron recurrence. It is a sequence of mutations on the cluster algebra for $\mathrm{Conf}_3 \mathcal{A}_{SL_N}$ that realizes the operation of replacing three principal flags in the space $V = \mathbb{C}^n$ with the dual principal flags in V^*. Put in another way, there is an outer automorphism of SL_N given by $g \to {}^t g^{-1}$. This automorphism induces an automorphism ϕ on $\mathrm{Conf}_3 \mathcal{A}_{SL_N}$. It turns out (though this is not obvious), that this is an automorphism of the cluster structure on $\mathrm{Conf}_3 \mathcal{A}_{SL_N}$, that is, if x_i are cluster variables in one seed, then $\phi^* x_i$ will be cluster variables in another seed. The sequence of mutations we are interested in is a sequence of mutations transforming from the initial seed x_i to the seed consisting of the functions $\phi^* x_i$.

The name 'tetrahedron recurrence' comes from the fact that all the cluster variables involved can be put at the integral lattice points of a tetrahedron [**HK**].

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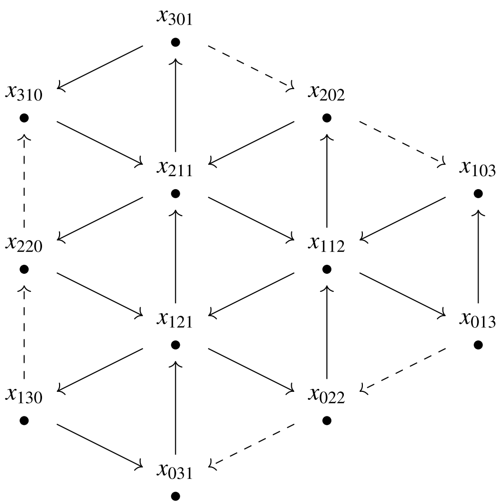

Figure 7. The functions and quiver for the cluster algebra on $\mathrm{Conf}_3\,\mathcal{A}_{SL_4}$.

By performing this sequence of mutations on various triangles in the triangulation of an m-gon, we get the action of the *cactus group* on $\mathrm{Conf}_3\,\mathcal{A}_{SL_N}$. For this reason we will call the sequence of mutations realizing the outer automorphism of SL_N the 'cactus sequence.'

First we need to review the cluster algebra structure on $\mathrm{Conf}_3\,\mathcal{A}_{SL_N}$. The quiver for this cluster algebra has a vertex x_{ijk} for all triples $1 \leqslant i, j, k \leqslant n$ such that $i + j + k = n$ and no more than one of i, j, k is equal to 0. The vertex x_{ijk} can be placed at the point (i, j, k) in the plane $i + j + k = n$. Then the quiver looks as in Figure 7 for SL_4.

We have used dotted arrows in line with the conventions above for amalgamation. The function attached to x_{ijk} will then be $\left(\begin{smallmatrix} i & \\ & k \\ j & \end{smallmatrix}\right)$ in the notation of Section 3.1. The frozen vertices are $x_{i,N-i,0}, x_{0,j,N-j}, x_{i,0,N-i}$.

Now we describe the sequence of mutations. First mutate $x_{N-2,1,1}$. Then mutate $x_{N-3,2,1}$ and $x_{N-3,1,2}$. Then mutate $x_{N-4,3,1}, x_{N-4,2,2}, x_{N-4,1,3}$. Continue in this manner, until we mutate $x_{1,1,N-2}$. Then we start the sequence over again by mutating $x_{N-2,1,1}$, but the second time through, we stop at $x_{2,1,N-3}$. The third time through, we stop at $x_{3,1,N-4}$. We continue in this manner so that the whole sequence of mutations is

$$x_{N-2,1,1}, x_{N-3,2,1}, x_{N-3,1,2}, x_{N-4,3,1}, x_{N-4,2,2}, x_{N-4,1,3}, \ldots x_{1,N-2,1}, \ldots x_{1,1,N-2},$$

$$x_{N-2,1,1}, x_{N-3,2,1}, x_{N-3,1,2}, x_{N-4,3,1}, x_{N-4,2,2}, x_{N-4,1,3}, \ldots x_{2,N-3,1}, \ldots x_{2,1,N-3},$$

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$$x_{N-2,1,1}, x_{N-3,2,1}, x_{N-3,1,2}, x_{N-4,3,1}, x_{N-4,2,2}, x_{N-4,1,3}, \dots x_{3,1,N-4},$$

$$\dots$$

$$x_{N-2,1,1}, x_{N-3,2,1}, x_{N-3,1,2},$$

$$x_{N-2,1,1}.$$

Perhaps it is more useful to describe the sequence in another way. Think of the quiver for $\mathrm{Conf}_3 \mathcal{A}_{SL_N}$ as consisting of rows, where row r consists of all x_{ijk} where $i = r$. Then for $r = N-2, N-3, \dots 2, 1$, there are $N-1-r$ vertices in that row. We start by mutating the one vertex in row $N-2$, then the two vertices in row $N-3$, then the three vertices in row $N-4$, and so on. The sequence of rows that we mutate is

$$N-2, N-3, N-4, \dots, 2, 1, N-2, N-3, \dots, 3, \\ \times 2, N-2, \dots, 3, \dots, \dots, N-2, N-3, N-2.$$

There are $(N-2)(N-1)/2$ terms in the above list. This gives a total of $N(N-1)(N_2)/6$ mutations. Let us think of the sequence of mutations as happening in $N-2$ stages, where at stage r we mutate all the vertices in rows $N-2, N-3, \dots, r$, in that order. It turns out that because of how the quiver transforms, the mutations in any given row can be performed in any order.

There is actually even more freedom in the order in which we perform mutations. Suppose at some point in the sequence of mutations, we want to mutate row i for the jth time. It turns out that any vertex in row i can be mutated only after

- the two vertices in row $i-1$ directly below it have been mutated $j-1$ times and
- the two vertices in row $i+1$ directly above it have been mutated j times.

The combinatorics of this will become clear after we analyze how the quiver transforms under mutations. Thus we have many equivalent sequences of mutations.

For example, we could mutate the rows

$$N-2, N-3, N-2, N-4, N-3, N-2, N-5, \dots, \\ \times N-2, N-6, \dots, \dots, 1, 2, 3, \dots, N-2.$$

Let us now analyze how the quiver transforms. In Figure 8 we picture the quiver for SL_5, the result after mutating rows 3, 2, 1.

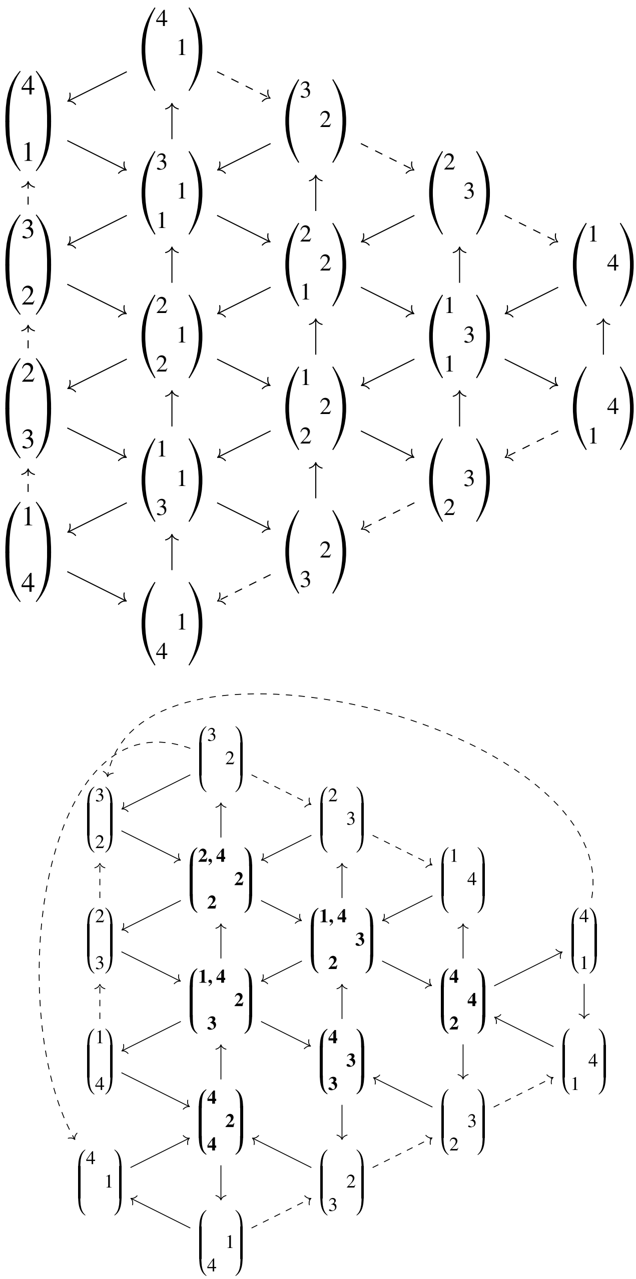

Figure 8. The SL_5 quiver before mutation and after mutating rows 3, 2, 1.

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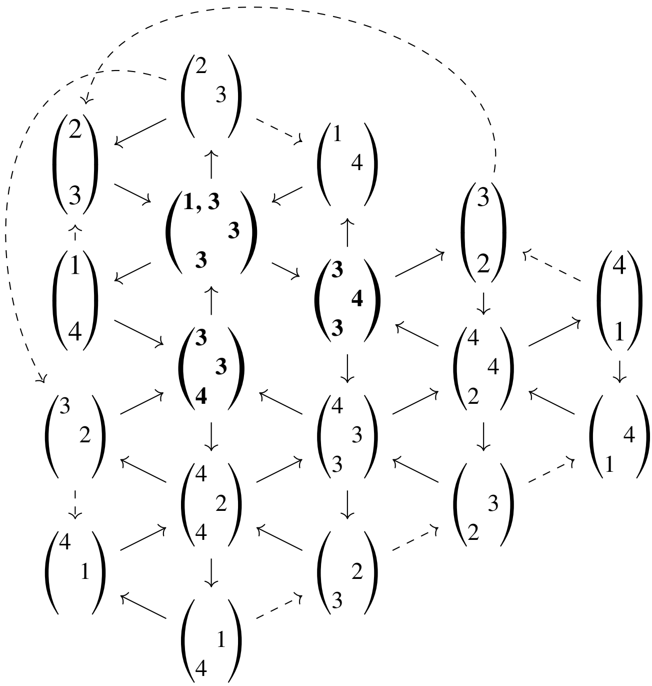

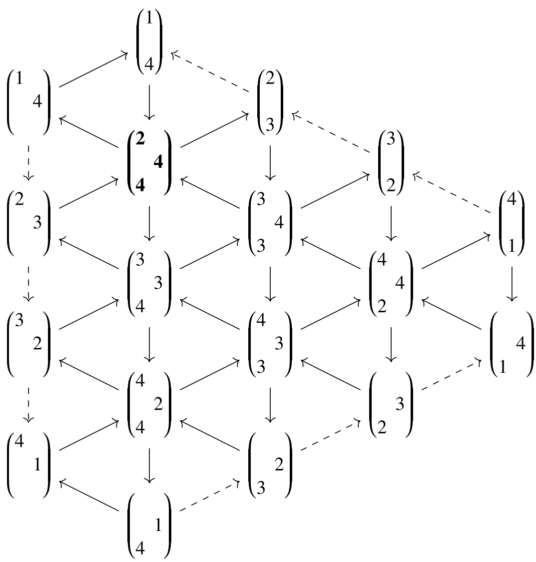

Figure 9. The SL_5 quiver after stages 2 and 3 of the cactus sequence.

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Note that at each stage, we rearrange some of the frozen vertices to make the structure of the quiver more transparent. The nonfrozen vertices, however, do not move.

We see that after the mutations in rows $N-2, N-3, \ldots, 1$ have been performed, we end up with the quiver for SL_{N-1} in the top $N-1$ rows of the diagram. We may then inductively see that after each descending sequence of rows has been mutated, we get the quivers pictured in Figure 9.

The final quiver we arrive at is essentially the original quiver with arrows reversed (additionally, frozen vertices have moved and some arrows have changed between frozen vertices).

Now we must understand the functions attached to the vertices at the quiver at various stages. Note that x_{ijk} is mutated i times.

PROPOSITION 3.2. *The function associated to x_{ijk} transforms as follows:*

$$\begin{pmatrix} i & \\ & k \\ j & \end{pmatrix} \to \begin{pmatrix} n-1, i-1 & \\ & k+1 \\ j+1 & \end{pmatrix} \to \begin{pmatrix} n-2, i-2 & \\ & k+2 \\ j+2 & \end{pmatrix} \to \cdots$$

$$\to \begin{pmatrix} n-i+1, 1 & \\ & k+i-1 \\ j+i-1 & \end{pmatrix} \to \begin{pmatrix} n-i & \\ & k+i \\ j+i & \end{pmatrix} = \begin{pmatrix} n-i & \\ & n-j \\ n-k & \end{pmatrix}.$$

The pattern above is clearer if we note that $\begin{pmatrix} 2n, a & \\ & c \\ b & \end{pmatrix} = \begin{pmatrix} a & \\ & c \\ b & \end{pmatrix} = \begin{pmatrix} a, 0 & \\ & c \\ b & \end{pmatrix}$.

Proof. We have already described the quivers at the various stages of mutation. We must then check that the functions above satisfy the identities of the associated cluster transformations.

We need the following facts:

- Let $1 \leqslant a, b, c, d \leqslant N$, and $a+b+c+d = 2N$.

$$\begin{pmatrix} a, b & \\ & d \\ c & \end{pmatrix} \begin{pmatrix} a-1, b-1 & \\ & d+1 \\ c+1 & \end{pmatrix}$$

$$= \begin{pmatrix} a-1, b & \\ & d \\ c+1 & \end{pmatrix} \begin{pmatrix} a, b-1 & \\ & d+1 \\ c & \end{pmatrix} + \begin{pmatrix} a-1, b & \\ & d+1 \\ c & \end{pmatrix} \begin{pmatrix} a, b-1 & \\ & d \\ c+1 & \end{pmatrix}.$$

- If $a+c = b+d = N$, $\begin{pmatrix} a, b & \\ & d \\ c & \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b & \\ & d \end{pmatrix}$.

- If $a+d = b+c = N$, $\begin{pmatrix} a, b & \\ & d \\ c & \end{pmatrix} = \begin{pmatrix} a & \\ & d \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}$.

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- If $b+c+d=N$, then $\binom{N,b \quad\quad}{c \quad d} = \binom{b \quad}{c \quad d}$.

The first identity is a close relative of the octahedron recurrence. The other are direct consequences of the definition of these functions via webs. We will call such identities 'degeneracies.'

Most of the cluster mutations are precisely the first identity:

$$\begin{pmatrix} a,b & & \\ & & d \\ c & & \end{pmatrix}\begin{pmatrix} a-1,b-1 & & \\ & & d+1 \\ & c+1 & \end{pmatrix} = \begin{pmatrix} a-1,b & & \\ & & d \\ c+1 & & \end{pmatrix}\begin{pmatrix} a,b-1 & & \\ & & d+1 \\ & c & \end{pmatrix} + \begin{pmatrix} a-1,b & & \\ & & d+1 \\ & c & \end{pmatrix}\begin{pmatrix} a,b-1 & & \\ & & d \\ c+1 & & \end{pmatrix}.$$

All other cluster mutations are degenerations of this identity, and are obtained from this one by applying the other three identities. $\square$

REMARK 3.3. This will be a common theme throughout all the main proofs in this paper—the mutation identities will usually come down to one or two applications of the octahedron recurrence used in combination with degeneracies. If we regard degeneracies as identities which are in some sense 'easy,' the octahedron recurrence and its relatives are really the central tool in the proofs of various identities. It is perhaps not always easy to see that an identity reduces to the octahedron recurrence; this is the reason that we try to be careful in our bookkeeping throughout the paper.

We now specialize to $N=2n$. The cactus sequence will allow us to give an alternative way to construct the cluster algebra structure for $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$. We perform the following sequence of mutations on $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$:

$$\begin{gathered}
x_{N-2,1,1}, x_{N-3,1,2}, \ldots, x_{1,1,N-2} \\
x_{N-3,2,1}, x_{N-4,2,2}, \ldots, x_{1,2,N-3} \\
x_{N-2,1,1}, x_{N-3,1,2}, \ldots, x_{2,1,N-3} \\
x_{N-4,3,1}, x_{N-5,3,2}, \ldots, x_{1,3,N-4} \\
x_{N-3,2,1}, x_{N-4,2,2}, \ldots, x_{2,2,N-4} \\
x_{N-2,1,1}, x_{N-3,1,2}, \ldots, x_{3,1,N-4} \\
\ldots \\
x_{n,n-1,1}, x_{n-1,n-1,2}, \ldots, x_{1,n-1,n} \\
\ldots \\
x_{N-3,2,1}, x_{N-4,2,2}, \ldots, x_{n-2,2,n} \\
x_{N-2,1,1}, x_{N-3,1,2}, \ldots, x_{n-1,1,n}.
\end{gathered}$$

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$$\begin{tikzcd}
\begin{pmatrix} 3 & \\ & 1 \\ 4 & \end{pmatrix} \arrow[dr] & \begin{pmatrix} 3,7 & \\ & 2 \\ 4 & \end{pmatrix} \arrow[l] \arrow[dr] & \begin{pmatrix} 3,6 & \\ & 3 \\ 4 & \end{pmatrix} \arrow[l] \arrow[dr] & \begin{pmatrix} 3,5 & \\ & 4 \\ 4 & \end{pmatrix} \arrow[l] \arrow[r] & \begin{pmatrix} 2,5 & \\ & 5 \\ 4 & \end{pmatrix} \arrow[r] \arrow[dl] & \begin{pmatrix} 1,5 & \\ & 6 \\ 4 & \end{pmatrix} \arrow[r] \arrow[dl] & \begin{pmatrix} 5 & \\ & 7 \\ 4 & \end{pmatrix} \arrow[dl] \\
\begin{pmatrix} 2 & \\ & 1 \\ 5 & \end{pmatrix} \arrow[u, dashed] \arrow[dr] & \begin{pmatrix} 2 & \\ & 2 \\ 4 & \end{pmatrix} \arrow[l] \arrow[u] \arrow[dr] & \begin{pmatrix} 2,7 & \\ & 3 \\ 4 & \end{pmatrix} \arrow[l] \arrow[u] \arrow[dr] & \begin{pmatrix} 2,6 & \\ & 4 \\ 4 & \end{pmatrix} \arrow[l] \arrow[u] \arrow[r] & \begin{pmatrix} 1,6 & \\ & 5 \\ 4 & \end{pmatrix} \arrow[u] \arrow[r] \arrow[dl] & \begin{pmatrix} 6 & \\ & 6 \\ 4 & \end{pmatrix} \arrow[u] \arrow[r] \arrow[dl] & \begin{pmatrix} 6 & \\ & 7 \\ 3 & \end{pmatrix} \arrow[u, dashed] \arrow[dl] \\
\begin{pmatrix} 1 & \\ & 1 \\ 6 & \end{pmatrix} \arrow[u, dashed] & \begin{pmatrix} 1 & \\ & 2 \\ 5 & \end{pmatrix} \arrow[l] \arrow[u] & \begin{pmatrix} 1 & \\ & 3 \\ 4 & \end{pmatrix} \arrow[l] \arrow[u] & \begin{pmatrix} 1,7 & \\ & 4 \\ 4 & \end{pmatrix} \arrow[l] \arrow[u] \arrow[r] & \begin{pmatrix} 7 & \\ & 5 \\ 4 & \end{pmatrix} \arrow[u] \arrow[r] & \begin{pmatrix} 7 & \\ & 6 \\ 3 & \end{pmatrix} \arrow[u] \arrow[r] & \begin{pmatrix} 7 & \\ & 7 \\ 2 & \end{pmatrix} \arrow[u, dashed]
\end{tikzcd}$$

Figure 10. The functions and quiver for the cluster $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$ which folds to give a cluster for $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$.

One can picture the above sequence as consisting of $n-1$ stages, where each stage consists of mutating all vertices lying in a parallelogram. The result will be that we get the quiver pictured in Figure 10 (all frozen vertices have been deleted for simplicity)

The function attached to x_{ijk} (here we take $i,\ j,\ k > 0$ to exclude frozen vertices) will be

- $\begin{pmatrix} i & \\ & k \\ j & \end{pmatrix}$ if $j \geqslant n$,
- $\begin{pmatrix} n+j, i+j-n & \\ & k+n-j \\ n & \end{pmatrix}$ if $j < n$ and $k < n$ (in fact for $j \leqslant n$ and $k \leqslant n$; equality cases agree with the cases above and below),
- $(2n-i, i+j; k+i)$ if $k \geqslant n$.

Thus we see that the functions attached to x_{ijk} and x_{ikj} pull back to the same functions on $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$ via the embedding $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}} \hookrightarrow \mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$. Here we use the identities (3.1). In fact, for $j \geqslant k$ and $j \geqslant n$, we have that the vertex x_{ijk} corresponds to the vertex $x_{n-i,k}$ in the quiver for $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$, while for $j \geqslant k$ and $j < n$, we have that the vertex x_{ijk} corresponds to the vertex $x_{k,n+k-j}$. The symmetry along the vertical axis in Figure 10 should be apparent.

The result of this is that we can obtain the cluster structure on $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$ as follows: start with the standard cluster algebra on $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$, which has functions attached to vertices x_{ijk}. Perform series of mutations given above, which is a subsequence of the cactus sequence. Then fold the resulting cluster structure by identifying the following pairs of vertices: $x_{i,2n-i,0}$ and $x_{2n-i,i,0}$; $x_{i,0,2n-i}$ and $x_{2n-i,0,i}$; $x_{0,j,2n-j}$ and $x_{0,2n-j,j}$; x_{ijk} and x_{ikj} for $i,\ j,\ k > 0$.

Let us say a few words about folding. Suppose we have a cluster algebra C such that its quiver is preserved under an involution σ. Note that this forces vertices in the same orbit to have no arrows between them. Then we may form a new cluster algebra C'. The B-matrix for C' comes from quotienting the quiver for C by the involution σ and then coloring the vertices fixed under the involution white and all other vertices black. On the level of algebras, one sets the variables for vertices in the same orbit equal. It is not hard to check that mutating a vertex in C' corresponds to unfolding, mutating all lifts of that vertex in C, and then folding again. Thus we constructed above a cluster for $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$ which folds to give a cluster for $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$. More generally, any cluster on $\mathcal{A}_{Sp_{2n},S}$ unfolds to give a cluster on $\mathcal{A}_{SL_{2n},S}$, and any mutation on $\mathcal{A}_{Sp_{2n},S}$ corresponds to mutation at the lifts of the vertices under folding. This is expressed by the following diagram:

$$\begin{tikzcd}
\mathcal{A}_{SL_{2n},S} \arrow[r, dashed, "\text{lift of mutation}"] \arrow[d, "\text{fold}"] & \mathcal{A}_{SL_{2n},S} \arrow[d, "\text{fold}"] \\
\mathcal{A}_{Sp_{2n},S} \arrow[r, dashed, "\text{mutation}"] & \mathcal{A}_{Sp_{2n},S}.
\end{tikzcd}$$

For more details about the procedure of folding for cluster algebras, and for folding under more general automorphisms, see [FZ2].

More invariantly, there is an outer automorphism of SL_{2n} that has Sp_{2n} as its fixed locus. This gives an involution of $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$ (and more generally $\mathrm{Conf}_m\,\mathcal{A}_{SL_{2n}}$) that has $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$ (respectively, $\mathrm{Conf}_m\,\mathcal{A}_{Sp_{2n}}$) as its fixed locus. It turns out that the cluster algebra structure on $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$ is preserved by this involution, and that, moreover, there is a particular seed, constructed above, that is preserved by this involution. Folding this seed gives the cluster algebra structure on $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$.

Let us make some observations. Let σ be the Dynkin diagram automorphism of A_{2n-1} having quotient C_n. (We call this automorphism σ by abuse of notation: we will see soon that it induces the automorphism of the quiver that we called σ above.) Then σ induces a map on the root system for SL_{2n}, and hence on the fundamental weights and the dominant weights. It also induces an outer automorphism of SL_{2n} having fixed locus Sp_{2n}, and an involution on the spaces $\mathrm{Conf}_m\,\mathcal{A}_{SL_{2n}}$. Let π be the map from the vertices of A_{2n-1} to the vertices of C_n. This induces a map π sending fundamental weights to corresponding fundamental weights, and therefore projects the weight space for SL_{2n} to the weight space for Sp_{2n}.

OBSERVATION 3.4. Let f be a function on $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$ that lies in the invariant space

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^{SL_{2n}}.$$

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Then $\sigma^*(f)$ lies in the invariant space

$$[V_{\sigma(\lambda)} \otimes V_{\sigma(\mu)} \otimes V_{\sigma(\nu)}]^{SL_{2n}}.$$

OBSERVATION 3.5. Let f be a function on $\operatorname{Conf}_3 \mathcal{A}_{SL_{2n}}$ that lies in the invariant space

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^{SL_{2n}}.$$

Then as a function on $\operatorname{Conf}_3 \mathcal{A}_{Sp_{2n}}$, f lies in the invariant space

$$[V_{\pi(\lambda)} \otimes V_{\pi(\mu)} \otimes V_{\pi(\nu)}]^{Sp_{2n}}.$$

OBSERVATION 3.6. Consider a cluster for $\operatorname{Conf}_3 \mathcal{A}_{SL_{2n}}$ which folds to give a cluster for $\operatorname{Conf}_3 \mathcal{A}_{Sp_{2n}}$. Suppose v is a vertex in the quiver for this cluster. Let f_v be the function attached to v. Then

$$f_v \in [V_\lambda \otimes V_\mu \otimes V_\nu]^{SL_{2n}},$$

$$f_{\sigma(v)} \in [V_{\sigma(\lambda)} \otimes V_{\sigma(\mu)} \otimes V_{\sigma(\nu)}]^{SL_{2n}}.$$

However, on $\operatorname{Conf}_3 \mathcal{A}_{Sp_{2n}}$, $f_v = f_{\sigma(v)}$. This means that we must have $\pi(\lambda) = \pi(\sigma(\lambda))$, $\pi(\mu) = \pi(\sigma(\mu))$, and $\pi(\nu) = \pi(\sigma(\nu))$.

It is also clarifying to step back and motivate the sequence of mutations realizing the cactus sequence above. As explained in Section 3.2, the cluster structure on $\operatorname{Conf}_3 \mathcal{A}$ comes from a reduced word for the longest element w_0 in the Weyl group of G. The initial seed for $\operatorname{Conf}_3 \mathcal{A}_{SL_{2n}}$ is built using the reduced word

$$s_1 s_2 \ldots s_{2n-1} s_1 s_2 \ldots s_{2n-2} \ldots s_1 s_2 s_3 s_1 s_2 s_1.$$

Here $s_1, \ldots, s_{2n-1}$ are the generators of the Weyl group for SL_{2n}. It is known how to use cluster transformations to pass between the clusters that are associated to different reduced words [**BFZ**]. The cactus sequence transforms between the cluster above and the cluster associated to the reduced word

$$s_{2n-1} s_{2n-2} \ldots s_1 s_{2n-1} s_{2n-2} \ldots s_2 \ldots s_{2n-1} s_{2n-2} s_{2n-3} s_{2n-1} s_{2n-2} s_{2n-1}.$$

Thus we have the following observation, which we do not believe has been made elsewhere:

OBSERVATION 3.7. The cactus sequence may be viewed as the sequence of mutations relating the cluster structures on $\operatorname{Conf}_3 \mathcal{A}_{SL_N}$ related to the reduced words

$$s_1 s_2 \ldots s_{N-1} s_1 s_2 \ldots s_{N-2} \ldots s_1 s_2 s_3 s_1 s_2 s_1$$

and

$$s_{N-1}s_{N-2}\ldots s_1 s_{N-1}s_{N-2}\ldots s_2\ldots s_{N-1}s_{N-2}s_{N-3}s_{N-1}s_{N-2}s_{N-1}.$$

The subsequence of the cactus sequence given above transforms the initial cluster into the cluster associated with the reduced word

$$(s_n s_{n-1}s_{n+1}s_{n-2}s_{n+2}\ldots s_1 s_{2n-1})^n.$$

This reduced word is in some sense 'intermediate' between the two other reduced words discussed above.

Now let $s'_1, s'_2, \ldots, s'_n$ be the generators of the Weyl group of Sp_{2n}. There is an injection from the Weyl group of Sp_{2n} to the Weyl group of SL_{2n} that takes

$$s'_n \to s_n,$$

$$s'_i \to s_i s_{2n-i}.$$

Under this map,

$$(s'_n s'_{n-1}\ldots s'_1)^n \to (s_n s_{n-1}s_{n+1}s_{n-2}s_{n+2}\ldots s_1 s_{2n-1})^n.$$

Therefore the reduced word for the longest element of the Weyl group of SL_{2n} folds to give the reduced word for the longest element of the Weyl group of Sp_{2n}. So the folding that gives the cluster structure on $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$ from the cluster structure on $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$ really takes place on the level of Weyl groups.

3.4. The sequences of mutations realizing S_3 symmetries. We consider the cluster structure on $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$ described above. In fact, because the cluster structure we gave was not symmetric, we have described six different cluster structures on $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$. We would now like to give sequences of mutations relating these six clusters to show that they are actually all clusters in the same cluster algebra.

From the folding construction of the previous section, it is clear that we could unfold the cluster structure of $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$ to get the cluster structure on $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$, perform a sequence of mutations on $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$, and then refold to obtain one of the other cluster structures on $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$. The question then becomes whether we can realize this sequence of mutations 'upstairs' on $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$ 'downstairs' on $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$. It is by no means obvious that this is possible. Let us give two reasons for this. First, mutating one vertex downstairs generally corresponds to mutating two vertices upstairs (the two that are folded together to give the vertex downstairs), so that this severely restricts the possible sequences of mutations upstairs. Second, note that if we perform the naive

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procedure of passing to $\mathrm{Conf}_3 \mathcal{A}_{SL_{2n}}$ and performing a sequence of mutations there, when we refold, we are generally no longer folding together the same pairs of vertices.

We will realize the S_3 symmetries on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ by exhibiting sequences of mutations that realize two different transpositions in the group S_3. As we shall see, one of these transpositions can actually be realized by going 'upstairs' and using the calculations of the previous section with some care. The other transposition requires a completely different analysis.

3.4.1. The first transposition. Let $(A, B, C) \in \mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ be a triple of flags. Let us first give the sequence of mutations that realizes that S_3 symmetry $(A, B, C) \to (A, C, B)$.

The sequence of mutations is as follows:

$$\begin{gathered} x_{11}, x_{21}, x_{22}, x_{12}, x_{13}, x_{23}, x_{33}, x_{32}, x_{31}, \ldots, x_{1,n-1}, \ldots, x_{n-1,n-1}, \ldots, x_{n-1,1}, \\ x_{11}, x_{21}, x_{22}, x_{12}, \ldots x_{1,n-2}, \ldots, x_{n-2,n-2}, \ldots, x_{n-2,1}, \\ \ldots, \\ x_{11}, x_{21}, x_{22}, x_{12}, \\ x_{11} \end{gathered} \tag{3.3}$$

The sequence can be thought of as follows: At any step of the process, we mutate all nonfrozen x_{ij} such that $\max(i, j)$ is constant. It will not matter in which order we mutate the x_{ij} at each step because the vertices we mutate have no arrows between them. So we first mutate the x_{ij} such that $\max(i, j) = 1$, then the x_{ij} such that $\max(i, j) = 2$ (in any order), then the x_{ij} such that $\max(i, j) = 3$ (in any order), and so on. The sequence of maximums that we use is

$$1, 2, 3, \ldots, n-1, 1, 2, \ldots, n-2, \ldots, 1, 2, 3, 1, 2, 1.$$

In Figure 11, we depict how the quiver and functions for $\mathrm{Conf}_3 \mathcal{A}_{Sp_8}$ change after performing the sequences of mutations of x_{ij} having maximums $1, 2, 3$. The initial quiver is in Figure 5.

In Figure 12, we depict the state of the quiver after performing the sequence of mutations of x_{ij} having maximums $1, 2, 3$; $1, 2, 3, 1, 2$; and $1, 2, 3, 1, 2, 1$.

From these diagrams the various quivers in the general case of $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ should be clear.

THEOREM 3.8. *The sequence of mutations* (3.3) *realizes the S_3 symmetry* $(A, B, C) \to (A, C, B)$.

Proof. We will proceed by directly calculating the functions at each stage of mutation. If $\max(i, j) = k$, then x_{ij} is mutated a total of $n - k$ times. Recall that

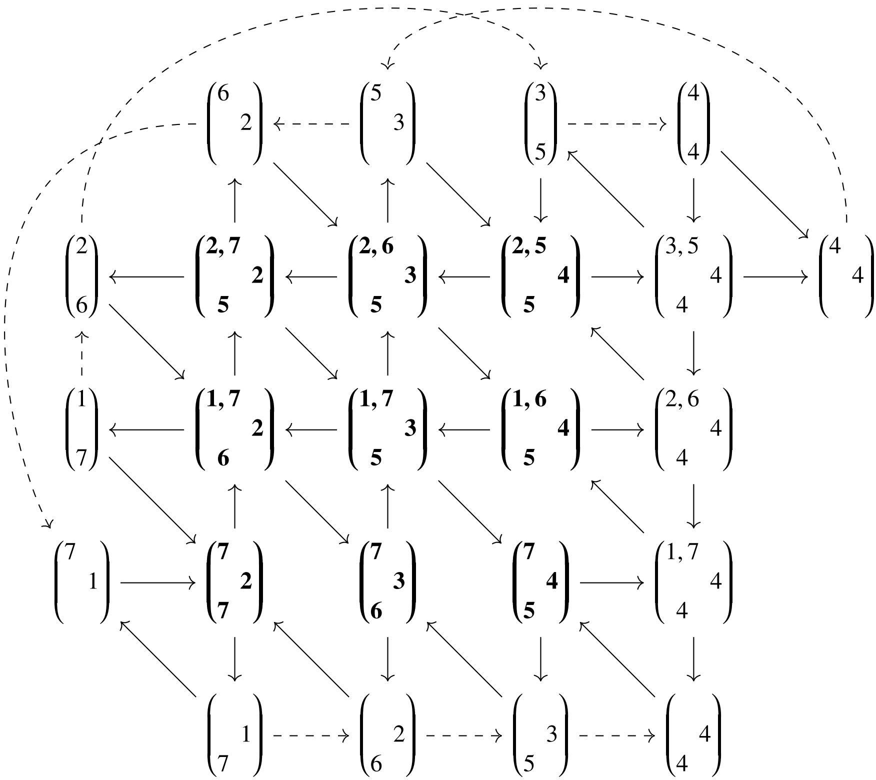

Figure 11. The quiver and the functions for $\mathrm{Conf}_3\,\mathcal{A}_{Sp_8}$ after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3.

when $i \geqslant j$, we assign the function $\begin{pmatrix} n-i & \\ & j \\ n+i-j & \end{pmatrix}$ to x_{ij}. Thus the function attached to x_{ij} transforms as follows:

$$\begin{pmatrix} n-i & \\ & j \\ n+i-j & \end{pmatrix} \to \begin{pmatrix} 2n-1, n-i-1 & \\ & j+1 \\ n+i-j+1 & \end{pmatrix} \to \begin{pmatrix} 2n-2, n-i-2 & \\ & j+2 \\ n+i-j+2 & \end{pmatrix} \to \cdots$$

$$\to \begin{pmatrix} n+i+1, 1 & \\ & n-i+j-1 \\ 2n-j-1 & \end{pmatrix} \to \begin{pmatrix} n+i & \\ & n-i+j \\ 2n-j & \end{pmatrix} = \begin{pmatrix} n-i & \\ & n+i-j \\ j & \end{pmatrix}$$

When $i < j$ and $i \neq 0$, we assign the function $\begin{pmatrix} n-i, 2n+i-j & \\ & j \\ n & \end{pmatrix}$ to x_{ij}. Thus the

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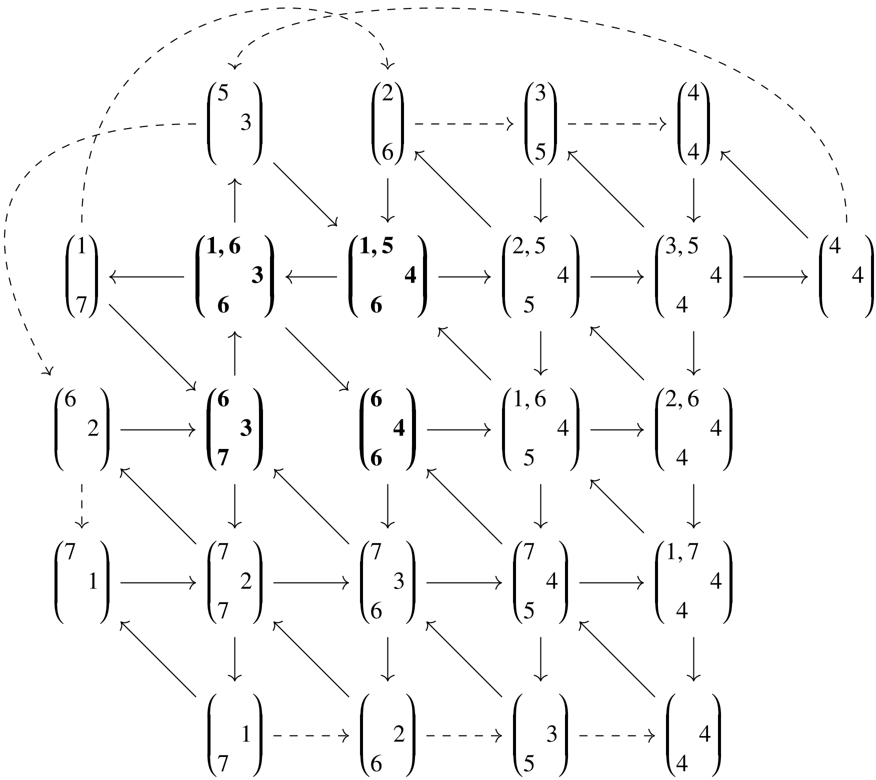

Figure 12a. The quiver and the functions for $\mathrm{Conf}_3 \mathcal{A}_{Sp_8}$ after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3, 1, 2.

function attached to x_{ij} transforms as follows:

$$\begin{aligned}
&\begin{pmatrix} n-i, 2n+i-j & \\ & j \\ n & \end{pmatrix} \to \begin{pmatrix} n-i-1, 2n+i-j-1 & \\ & j+1 \\ n+1 & \end{pmatrix} \\
&\to \begin{pmatrix} n-i-2, 2n+i-j-2 & \\ & j+2 \\ n+2 & \end{pmatrix} \to \cdots \\
&\to \begin{pmatrix} j-i+1, n+i+1 & \\ & n-1 \\ 2n-j-1 & \end{pmatrix} \to \begin{pmatrix} j-i, n+i & \\ & n \\ 2n-j & \end{pmatrix} = \begin{pmatrix} n-i, 2n+i-j & \\ & n \\ j & \end{pmatrix}.
\end{aligned}$$

We will give two approaches. One is conceptually transparent, but hard to carry out. The second is more direct.

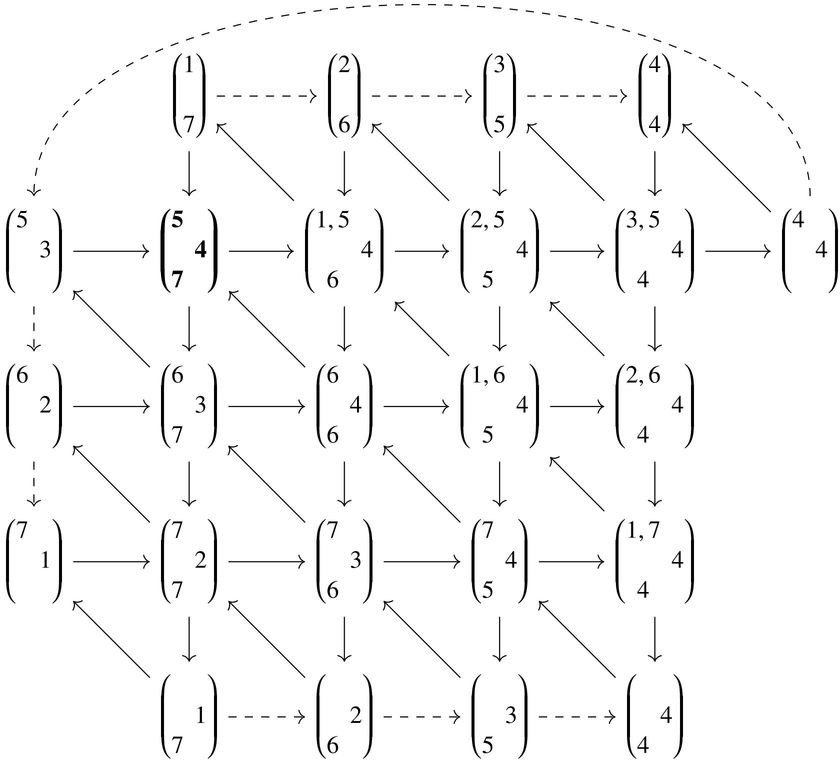

Figure 12b. The quiver and the functions for $\mathrm{Conf}_3 \mathcal{A}_{Sp_8}$ after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3, 1, 2, 1.

The first approach would be to use what we know about the cactus sequence. Recall that if we start with the standard cluster structure on $\mathrm{Conf}_3 \mathcal{A}_{SL_{2n}}$ and perform the mutations

$$x_{N-2,1,1}, x_{N-3,1,2}, \ldots, x_{1,1,N-2}$$

$$x_{N-3,2,1}, x_{N-4,2,2}, \ldots, x_{1,2,N-3}$$

$$x_{N-2,1,1}, x_{N-3,1,2}, \ldots, x_{2,1,N-3}$$

$$x_{N-4,3,1}, x_{N-5,3,2}, \ldots, x_{1,3,N-4}$$

$$x_{N-3,2,1}, x_{N-4,2,2}, \ldots, x_{2,2,N-4}$$

$$x_{N-2,1,1}, x_{N-3,1,2}, \ldots, x_{3,1,N-4}$$

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$$\dots$$

$$x_{n,n-1,1}, x_{n-1,n-1,2}, \dots, x_{1,n-1,n}$$

$$\dots$$

$$x_{N-3,2,1}, x_{N-4,2,2}, \dots, x_{n-2,2,n}$$

$$x_{N-2,1,1}, x_{N-3,1,2}, \dots, x_{n-1,1,n}$$

then if we identify the vertices x_{ijk} and x_{ikj}, we obtain the cluster algebra structure on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$. Recall that for $j \leqslant k$ and $j \geqslant n$, we have that the vertex x_{ijk} corresponds to the vertex $x_{n-i,k}$ in the quiver for $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$, while for $j \leqslant k$ and $j < n$, we have that the vertex x_{ijk} corresponds to the vertex $x_{n-k,n+k-j}$.

If we then apply the sequence of mutations (3.3)

$$x_{11}, x_{21}, x_{22}, x_{12}, x_{13}, x_{23}, x_{33}, x_{32}, x_{33}, \dots, x_{1,n-1}, \dots, x_{n-1,n-1}, \dots, x_{n-1,1},$$

$$x_{11}, x_{21}, x_{22}, x_{12}, \dots x_{1,n-2}, \dots, x_{n-2,n-2}, \dots, x_{n-2,1},$$

$$\dots,$$

$$x_{11}, x_{21}, x_{22}, x_{12},$$

$$x_{11}$$

on the cluster algebra for $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$, we can lift this sequence of mutations to the unfolded cluster algebra for $\mathrm{Conf}_3 \mathcal{A}_{SL_{2n}}$. Therefore, once we keep track of indices, we find that the cluster we end up with is exactly the result of applying the sequence of mutations

$$x_{N-2,1,1}, x_{N-3,2,1}, \dots, x_{1,N-2,1}$$

$$x_{N-3,1,2}, x_{N-4,2,2}, \dots, x_{1,N-3,2}$$

$$x_{N-2,1,1}, x_{N-3,2,1}, \dots, x_{1,N-2,1}$$

$$x_{N-4,1,3}, x_{N-5,2,3}, \dots, x_{1,N-4,3}$$

$$x_{N-3,1,2}, x_{N-4,2,2}, \dots, x_{1,N-3,2}$$

$$x_{N-2,1,1}, x_{N-3,2,1}, \dots, x_{1,N-2,1}$$

$$\dots$$

$$x_{n,1,n-1}, x_{n-1,2,n-1}, \dots, x_{1,n,n-1}$$

$$\dots$$

$$x_{N-3,1,2}, x_{N-4,2,2}, \dots, x_{1,N-3,2}$$

$$x_{N-2,1,1}, x_{N-3,2,1}, \ldots, x_{1,N-2,1}$$

to the original seed for $\mathrm{Conf}_3 \mathcal{A}_{SL_{2n}}$.

Again, we then identify the vertices x_{ijk} and x_{ikj} and look downstairs on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$. Then the roles of j and k are exactly reversed: For $k \leqslant j$ and $k \geqslant n$, we have that the vertex x_{ijk} corresponds to the vertex $x_{n-i,j}$ in the transposed quiver for $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$, while for $k \leqslant j$ and $l < n$, we have that the vertex x_{ijk} corresponds to the vertex $x_{n-j,n+j-k}$ in the transposed quiver.

Alternatively, and perhaps more simply, it is enough to observe that all that mutations in the sequence (3.3) reduce to one of the cluster transformations involved in the cactus sequence of mutations. □

Now consider the original seed for the cluster algebra structure on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$, where the quiver had vertices x_{ij}, y_k for $0 \leqslant i \leqslant n$, $1 \leqslant j, k \leqslant n$. Now consider the seed that has quiver with vertices x'_{ij}, y'_k that comes from exchanging the roles of the second and third principal flags. Then the sequence of mutations explained mutates the functions associated to x_{ij} into those associated to x'_{ij} for $i > 0$. The functions x_{0j} are equal to the functions y'_j, and the functions y_j are equal to the functions x'_{0j}. Moreover, the associated quiver on the vertices x_{ij}, y_k mutates into the corresponding quiver on the vertices x'_{ij}, y'_k. Thus, this sequence of mutations relates the two seeds that come from the transposition of the second and third flags.

3.4.2. The second transposition. Let us now give the sequence of mutations that realizes that S_3 symmetry $(A, B, C) \to (C, B, A)$.

The sequence of mutations is as follows:

$$\begin{gathered}
x_{n-1,n}, x_{n-2,n-1}, x_{n-2,n}, x_{n-3,n-2}, x_{n-3,n-1}, x_{n-3,n}, \ldots, x_{1,2}, \ldots, x_{1,n}, \\
x_{n-1,n}, x_{n-2,n-1}, x_{n-2,n}, x_{n-3,n-2}, x_{n-3,n-1}, x_{n-3,n}, \ldots, x_{2,3}, \ldots, x_{2,n}, \\
x_{n-1,n}, x_{n-2,n-1}, x_{n-2,n}, x_{n-3,n-2}, x_{n-3,n-1}, x_{n-3,n}, \ldots, x_{3,4}, \ldots, x_{3,n}, \\
\ldots x_{n-1,n}, x_{n-2,n-1}, x_{n-2,n}, x_{n-3,n-2}, x_{n-3,n-1}, x_{n-3,n}, \\
x_{n-1,n}, x_{n-2,n-1}, x_{n-2,n} \\
x_{n-1,n}.
\end{gathered} \tag{3.4}$$

The sequence can be thought of as follows: we only mutate those x_{ij} with $i < j$. At any step of the process, we mutate all x_{ij} in the kth row (the kth row consists of x_{ij} such that $i = k$) such that $i < j$. It will not matter in which order we mutate these x_{ij}. The sequence of rows that we mutate is

$$n-1, n-2, \ldots, 2, 1, n-1, n-2, \ldots, 2, n-1, \ldots, 3, \ldots, n-1, n-2, n-1.$$

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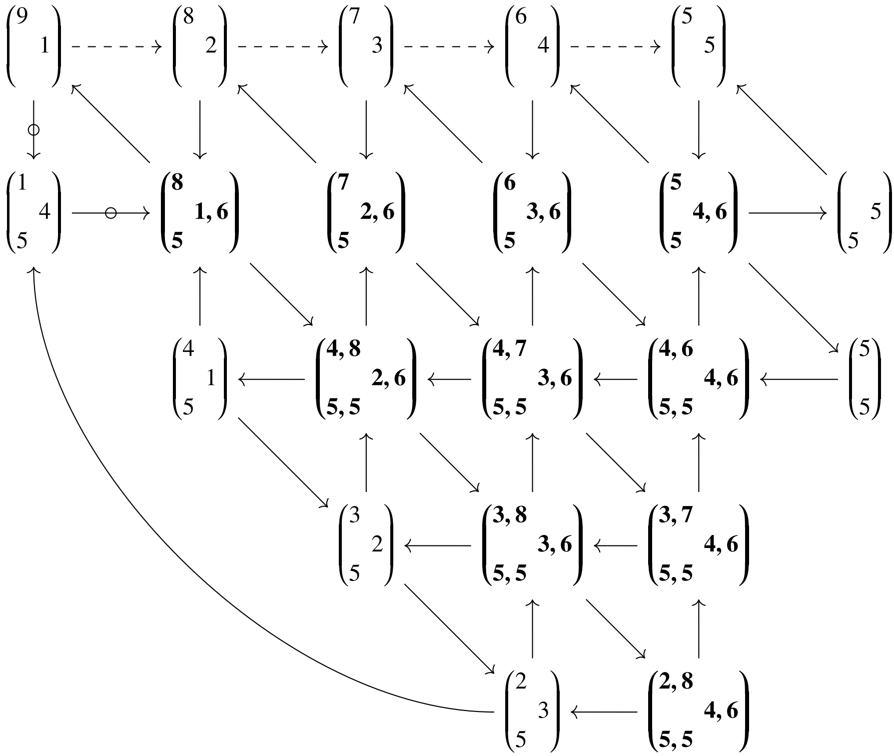

Figure 13. The quiver and functions for $\operatorname{Conf}_3 \mathcal{A}_{Sp_{10}}$ after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1.

In Figure 13, we depict how the quiver for $\operatorname{Conf}_3 \mathcal{A}_{Sp_{10}}$ changes after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1. We only depict those functions that are affected by this mutation sequence.

REMARK 3.9. The circle on several of the arrows below depicts how the quiver for $\operatorname{Conf}_3 \mathcal{A}_{Sp_{10}}$ lifts to $\operatorname{Conf}_3 \mathcal{A}_{SL_{10}}$. They are just a bookkeeping device which we do not need to be concerned about yet. We will elaborate upon this further when we analyze how the functions transform. From these diagrams the various quivers in the general case of $\operatorname{Conf}_3 \mathcal{A}_{Sp_{2n}}$ should be clear.

In Figure 14, we depict the state of the quiver after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1, 4, 3, 2; 4, 3, 2, 1, 4, 3, 2, 4, 3; and 4, 3, 2, 1, 4, 3, 2, 4, 3, 4.

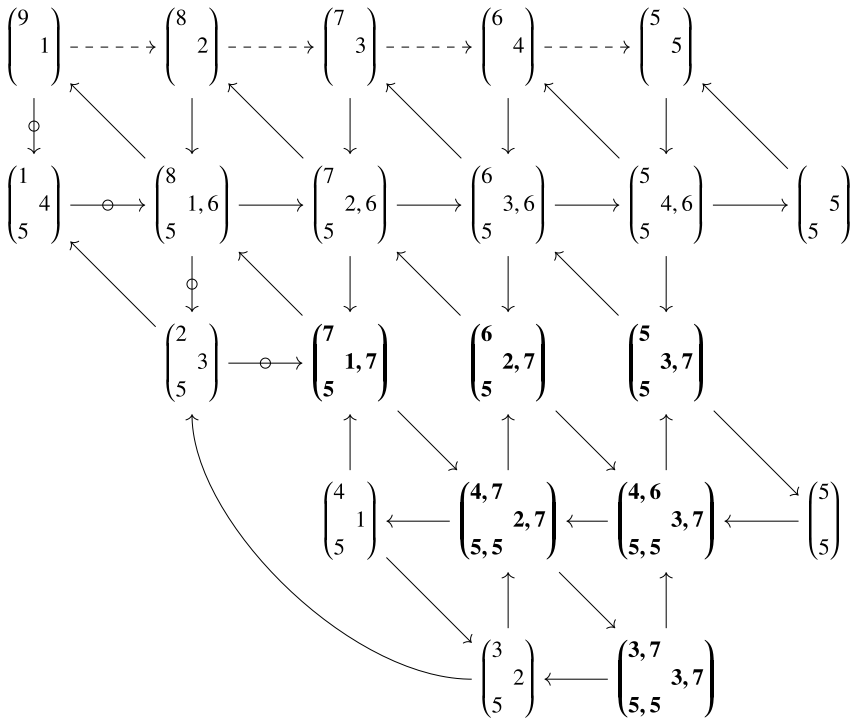

Figure 14a. The quiver and functions for $\mathrm{Conf}_3 \mathcal{A}_{Sp_{10}}$ after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1, 4, 3, 2.

Now we need to understand the various functions attached to the vertices of the quiver at different stages in the sequence of mutations. In order to do this, we need to define some new functions.

Let $N = 2n$. Let $0 \leqslant a, b, c, d \leqslant N$ such that $a, c \leqslant n$ and $b, d \geqslant n$. Moreover, suppose that $a + b + c + d = 4n = 2N$. Then we would like to define a function that we will call

$$\begin{pmatrix} a, b & & \\ & & c, d \\ n, n & & \end{pmatrix}.$$

This is a function on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ that is pulled back from a function on $\mathrm{Conf}_3 \mathcal{A}_{SL_{2n}}$. The function on $\mathrm{Conf}_3 \mathcal{A}_{SL_N}$ is given by an invariant in the space

$$[V_{\omega_a+\omega_b} \otimes V_{2\omega_n} \otimes V_{\omega_c+\omega_d}]^{SL_N}.$$

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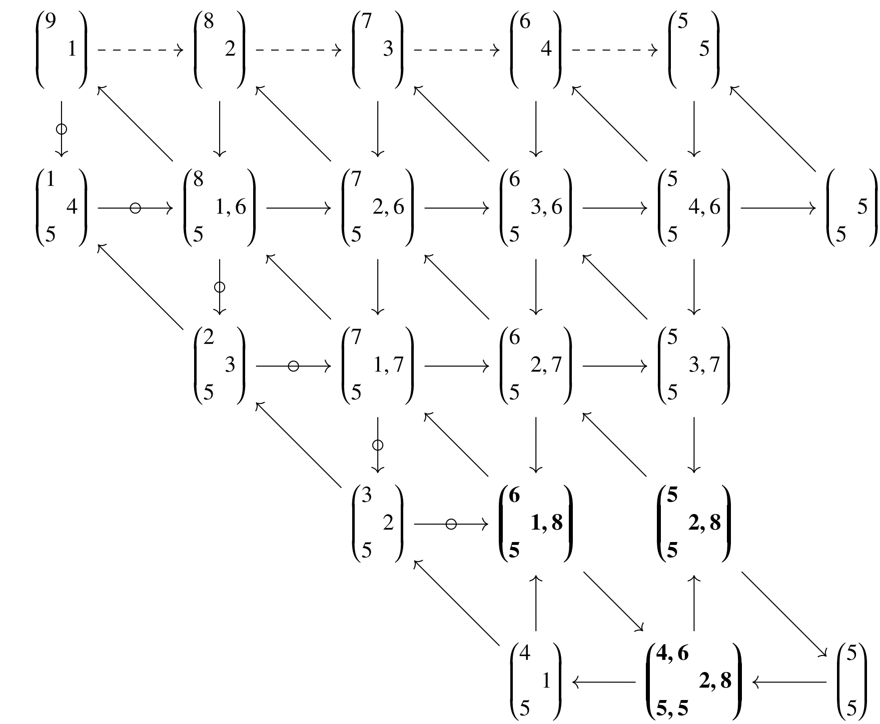

Figure 14b. The quiver and functions for $\mathrm{Conf}_3 \mathcal{A}_{Sp_{10}}$ after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1, 4, 3, 2, 4, 3.

It turns out that this is a one-dimensional vector space. We pick out the function given by the web in Figure 15:

We are now ready to analyze how the functions in our cluster algebra transform under the sequence of mutations (3.4).

THEOREM 3.10. *The sequence of mutations* (3.4) *realizes the* S_3 *symmetry* $(A, B, C) \to (C, B, A)$.

Proof. If $i < j$, then x_{ij} is mutated a total of i times. Recall that when $i < j$, we assign the function $\begin{pmatrix} n-i, 2n+i-j & \\ & j \\ n & \end{pmatrix}$ to x_{ij}. The function attached to x_{ij} transforms as follows:

$$\begin{pmatrix} n-i, 2n+i-j & \\ & j \\ n & \end{pmatrix} \to \begin{pmatrix} n-i+1, 2n+i-j-1 & \\ & j-1, n+1 \\ n, n & \end{pmatrix}$$

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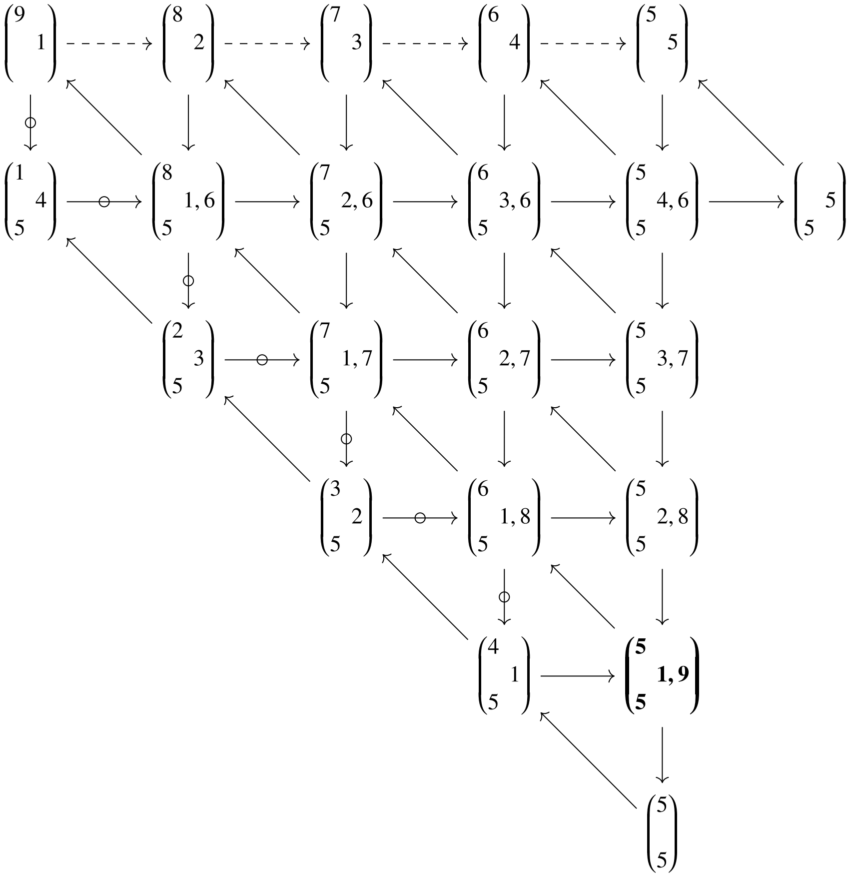

Figure 14c. The quiver and functions for $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{10}}$ after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1, 4, 3, 2, 4, 3, 4.

$$
\begin{aligned}
&\to \begin{pmatrix} n-i+2,\, 2n+i-j-2 & \\ & j-2,\, n+2 \\ n,\, n & \end{pmatrix} \\
&\to \cdots \to \begin{pmatrix} n-1,\, 2n-j+1 & \\ & j-i+1,\, n+i-1 \\ n,\, n & \end{pmatrix} \\
&\to \begin{pmatrix} 2n-j & \\ & j-i,\, n+i \\ n & \end{pmatrix} = \begin{pmatrix} j & \\ & n-i,\, 2n+i-j \\ n & \end{pmatrix}.
\end{aligned}
$$

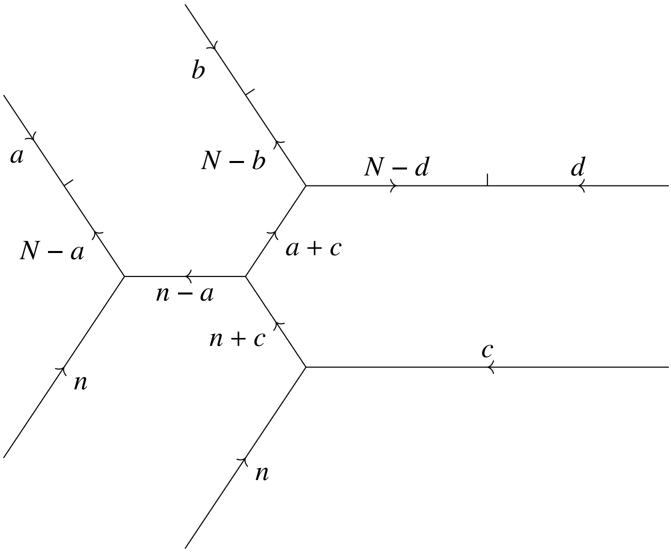

Figure 15. Web for the function $\begin{pmatrix} a,b & \\ & c,d \\ n,n & \end{pmatrix}$ where $a+b+c+d=2N$.

The first transformation can be seen as the composite of two steps,

$$\begin{pmatrix} n-i,2n+i-j & \\ & j \\ n & \end{pmatrix} \sim \begin{pmatrix} n-i,2n+i-j & \\ & j,n \\ n,n & \end{pmatrix}$$
$$\to \begin{pmatrix} n-i+1,2n+i-j-1 & \\ & j-1,n+1 \\ n,n & \end{pmatrix},$$

while the last transformation can also be seen as the composite of two steps,

$$\begin{pmatrix} n-1,2n-j+1 & \\ & j-i+1,n+i-1 \\ n,n & \end{pmatrix} \to \begin{pmatrix} n,2n-j & \\ & j-i,n+i \\ n,n & \end{pmatrix}$$
$$\sim \begin{pmatrix} 2n-j & \\ & j-i,n+i \\ n & \end{pmatrix}.$$

Then with each transformation, two of the parameters increase by one, and two decrease by one.

We have already described the quivers at the various stages of mutation. We must then check that the functions above satisfy the identities of the associated cluster transformations.

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This is the first sequence of mutations where we have to mutate the white vertices x_{in} (y_n is also a white vertex, but it does not get mutated). For a black vertex x_{ij}, recall that the formula for mutation says that

$$\begin{aligned}&(\text{old function attached to } x_{ij})(\text{new function attached to } x_{ij})\\ &\quad= \prod(\text{functions attached to incoming arrows})\\ &\qquad+ \prod(\text{functions attached to outgoing arrows}).\end{aligned}$$

However, for white vertices, we have

$$\begin{aligned}&(\text{old function attached to } x_{ij})(\text{new function attached to } x_{ij})\\ &\quad= \prod(\text{incoming arrows from white vertices})\\ &\qquad(\text{incoming arrows from black vertices})^2\\ &\qquad+ \prod(\text{outgoing arrows from white vertices})\\ &\qquad(\text{outgoing arrows from black vertices})^2.\end{aligned}$$

In other words, arrows going from black to white vertices are weighted by 2 when calculating the mutation of a white vertex.

In verifying the cluster identities that we need, we will actually be computing functions on $\text{Conf}_3\,\mathcal{A}_{SL_N}$. Thus we will use the arrows with a circle on them as a bookkeeping device. Circled arrows only occur between two black vertices. If a black vertex x has an incoming or outgoing circled arrow from another black vertex x', this means that in computing the mutation of x, we should use the dual of the function attached to x'.

The reason for this is as follows. We can lift the quiver Q for $\text{Conf}_3\,\mathcal{A}_{Sp_N}$ to a quiver $\tilde{Q}$ for $\text{Conf}_3\,\mathcal{A}_{SL_N}$ using the following rule. For each vertex black vertex x, there are two lifts of this vertex in $\tilde{Q}$, call them x^* and x^{**}. Then for every regular arrow between two black vertices x and x' in the quiver Q, we lift it to the corresponding arrow between x^* and x'^* and between x^{**} and x'^{**}. On the other hand, for every arrow with a circle between two black vertices x and x' in the quiver Q, we lift it to the corresponding arrow between x^{**} and x'^* and between x^* and x'^{**}.

Now suppose x is a black vertex and x' is a white vertex. x has lifts x^* and x^{**}, while x' has just one lift x'^*. Then an arrow between x and x' in the quiver Q lifts to the corresponding arrows between x^* and x'^* and between x^{**} and x'^*. Hence, when computing the mutation of x', we use the product of the function attached to x and the dual of this function.

With these rules in mind, we are ready to compute mutations.

We need the following facts:

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- Let $1 \leqslant a, b, c, d \leqslant N$, and $a+b+c+d = 2N$.

$$\begin{pmatrix} a,b & \\ & c,d \\ n,n & \end{pmatrix}\begin{pmatrix} a+1,b-1 & \\ & c-1,d+1 \\ n,n & \end{pmatrix}$$

$$= \begin{pmatrix} a,b & \\ & c-1,d+1 \\ n,n & \end{pmatrix}\begin{pmatrix} a+1,b-1 & \\ & c,d \\ n,n & \end{pmatrix} + \begin{pmatrix} a+1,b & \\ & c-1,d \\ n,n & \end{pmatrix}\begin{pmatrix} a,b-1 & \\ & c,d+1 \\ n,n & \end{pmatrix}.$$

- If $a+c = n$ and $b+d = 3n$,

$$\begin{pmatrix} a,b & \\ & c,d \\ n,n & \end{pmatrix} = \begin{pmatrix} a & \\ & c \\ n & \end{pmatrix}\begin{pmatrix} b & \\ & d \\ n & \end{pmatrix}.$$

- If $a = n$,

$$\begin{pmatrix} n,b & \\ & c,d \\ n,n & \end{pmatrix} = \begin{pmatrix} n \\ \\ n \end{pmatrix}\begin{pmatrix} b & \\ & c,d \\ n & \end{pmatrix}.$$

  Similarly, we have

$$\begin{pmatrix} a,n & \\ & c,d \\ n,n & \end{pmatrix} = \begin{pmatrix} n \\ \\ n \end{pmatrix}\begin{pmatrix} a & \\ & c,d \\ n & \end{pmatrix},$$

$$\begin{pmatrix} a,b & \\ & n,d \\ n,n & \end{pmatrix} = \begin{pmatrix} & \\ & n \\ n & \end{pmatrix}\begin{pmatrix} a,b & \\ & d \\ n & \end{pmatrix},$$

$$\begin{pmatrix} a,b & \\ & c,n \\ n,n & \end{pmatrix} = \begin{pmatrix} & \\ & n \\ n & \end{pmatrix}\begin{pmatrix} a,b & \\ & c \\ n & \end{pmatrix}.$$

- We will need the duality identities of (3.1), and also the following duality identity:

$$\begin{pmatrix} a,b & \\ & c,d \\ n,n & \end{pmatrix} = \begin{pmatrix} N-b, N-a & \\ & N-d, N-c \\ n,n & \end{pmatrix}.$$

All identities but the first follow directly from the definitions. The first can be reduced to the octahedron recurrence.

Most of the cluster mutations are given by the first identity. All other cluster mutations are degenerations of this identity, and are obtained from this one by applying the other three identities. □

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3.4.3. The third transposition: an alternative approach. In this section, we describe the sequence of mutations that realizes that S_3 symmetry $(A, B, C) \to (B, A, C)$. This approach has the advantage that it is more conceptual and requires less computation. It is also independent of type, and thus could replace the sections on the second transposition in the later sections of this paper, as one only needs two transpositions to generate S_3.

On the other hand, it will not be as explicit, and depends on results of [**BFZ**], while our approach in the previous two sections was explicit and not dependent on previous work. Also, the sequence of mutations that one obtains from this approach is quite inefficient, and the number of mutations required to realize the third transposition is approximately twice as many as the number of mutations needed to realize either of the other two transpositions.

Recall that there is a birational map

$$i : (b, h) \in B^- \times H \to (U^-, \overline{w_0} h U^-, b \cdot \overline{w_0} U^-) \in \mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}.$$

We have exhibited all the cluster variables we deal with as tensor invariants inside spaces such as

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^G.$$

Changing the value of h in the above map then scales the corresponding cluster variable by $\mu(h)$. It is therefore easy to reduce the calculation of any mutation sequence from the image of $B^- \times H$ to the image of B^-.

Let us briefly explain what is needed to pin down the H action. For an integer r, define $r^+ = \max(0, r)$ and $r^- = \min(0, r)$. For any weight $\lambda = \sum r_i \omega_i$, we can define $\lambda^+ = \sum r_i^+ \omega_i$ and $\lambda^- = \sum r_i^- \omega_i$. We then need that the generalized minor $\Delta_{u_l \omega_{i_l}, \omega_{i_l}}$ lies in the invariant space

$$[V_\lambda \otimes V_\mu \otimes V_{\omega_i}]^G$$

where $\lambda = -w_0(u\omega_i)^+$ and $\mu = -(u\omega_i)^-$. We explain how to prove this formula in [**LL**].

Thus any identity on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ can be proved by pulling back to B^- under the map

$$i : b \in B^- \to (U^-, \overline{w_0} U^-, b \cdot \overline{w_0} U^-) \in \mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}.$$

Let us start with the labels of the vertices of the quiver for $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ as in Figure 4: x_{ij} and y_k, where $0 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant n$. The image $i(B^-)$ is precisely the set where all the cluster variables attached to the frozen vertices y_k are equal to 1. Moreover, the variables attached to x_{ij} are all given by the generalized minors $\Delta_{u_{ij}\omega_j, \omega_j}$ for $i \neq 0$ as calculated in Section 3.2, and the

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variable attached to x_{0j} is $\Delta_{\omega_j,\omega_j}$. They are the generalized minors corresponding to the reduced word $(s_n s_{n-1} \cdots s_1)^n$.

Now, in [**BFZ**], the authors construct a cluster on B^- corresponding to each reduced word $w_0 = s_{i_1} \cdots s_{i_K}$. Let $u_l = s_{i_1} \cdots s_{i_l}$. Then the cluster variables are given by the generalized minors $\Delta_{\omega_j,\omega_j}$ for all j and $\Delta_{u_l\omega_{i_l},\omega_{i_l}}$. Moreover, they give a sequence of mutations relating any two clusters coming from two different reduced words.

Now perform the sequence of mutations that transforms from our original cluster, associated to the reduced word $(s_n s_{n-1} \cdots s_1)^n$, to the cluster associated with the reduced word $(s_1 s_2 \cdots s_n)^n$. Then the cluster variables in this new cluster are given by generalized minors for this new reduced word. Computations as in Section 3.2 will tell us that these functions are given by tensor invariants and that, moreover, they are the tensor invariants that we expect from the third transposition.

Finally, we should check that the quiver we obtain after the above sequence of mutations is the quiver coming from the third transposition. This is clear if we ignore the vertices y_k. Notice that in Figure 4, the vertices y_k are attached to vertical edges. The vertical edges are in bijection with letters in the reduced word $(s_n s_{n-1} \cdots s_1)^n$. It is well known that each letter in a reduced word for w_0 is attached to a positive root: if we consider the partial words that start at some letter and go rightwards to the end of the word, at each stage a new root flips from positive to negative, and that is the root attached to that letter. Then the y_k are attached to those positive roots that are simple. This is easy to check in the initial cluster, and it is also easy to check that it persists under changes of reduced words. Thus it also holds for the reduced word $(s_1 s_2 \cdots s_n)^n$, and this gives us that the y_k are attached to the rest of the cluster as we would expect from the third transposition. See [**LL**] for more details on how the vertices y_k are attached to the quiver for different reduced words.

3.5. The sequence of mutations for a flip. Recall that to construct the cluster structure (a quiver plus the set of functions attached to the vertices) on $\mathrm{Conf}_4 \mathcal{A}_{Sp_{2n}}$, we choose a triangulation of the 4-gon, as well as one of the six cluster structures on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ for each of the resulting triangles. The previous section showed how to mutate between the six cluster structures on each triangle. In this section, we will give a sequence of mutations that relates two of the clusters coming from different triangulations of the 4-gon. Combined with the previous section, this allows us to connect by mutations all 72 different clusters we have constructed for $\mathrm{Conf}_4 \mathcal{A}_{Sp_{2n}}$ (for each of the two triangulations, we have $6 \cdot 6$ different clusters). This will allow us to connect by mutations all the clusters that we have constructed on $\mathrm{Conf}_m \mathcal{A}_{Sp_{2n}}$.

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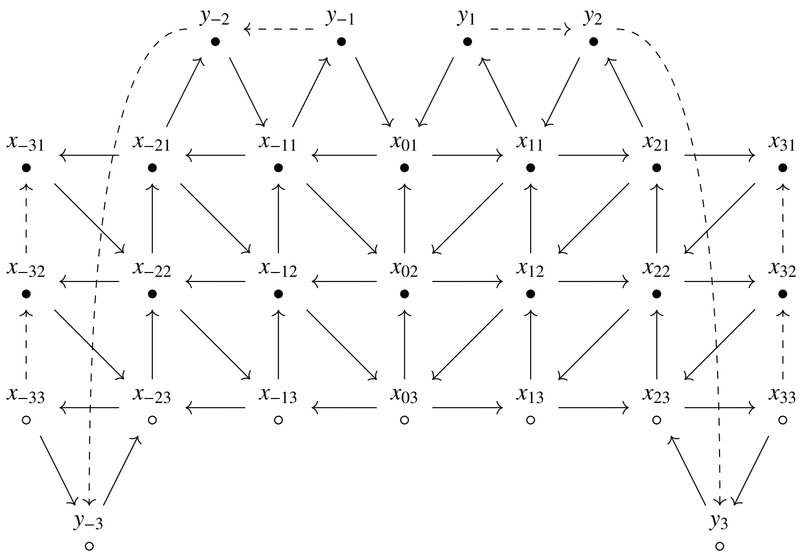

Figure 16. The quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Sp_6}$. The associated functions are depicted in Figure 20a.

Given a configuration $(A, B, C, D) \in \mathrm{Conf}_4 \mathcal{A}_{Sp_{2n}}$, we will give a sequence of mutations that relates a cluster coming from the triangulation ABC, ACD to a cluster coming from the triangulation ABD, BCD.

We will need to label the quiver for $\mathrm{Conf}_4 \mathcal{A}_{Sp_{2n}}$ with vertices x_{ij}, y_k, with $-n \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and $1 \leqslant |k| \leqslant n$. The quiver we will start with is as in Figure 16, pictured for Sp_6.

The functions attached to these vertices are as follows. Let $N = 2n$. The edge functions are

$$\begin{pmatrix} k & \\ & N-k \\ & \end{pmatrix} \longleftrightarrow y_k, \quad \text{for } k > 0;$$

$$\begin{pmatrix} & \\ |k| & \\ & N-|k| \end{pmatrix} \longleftrightarrow y_k, \quad \text{for } k < 0;$$

$$\begin{pmatrix} & j \\ N-j & \\ & \end{pmatrix} \longleftrightarrow x_{-n,j};$$

$$\begin{pmatrix} & \\ & j \\ N-j & \end{pmatrix} \longleftrightarrow x_{nj};$$

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$$\begin{pmatrix} & j & \\ & & \\ N-j & & \end{pmatrix} \longleftrightarrow x_{0j}.$$

The face functions in the triangle where $i > 0$ are

$$\begin{pmatrix} i+j & & \\ & & N-i \\ N-j & & \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } 0 < i < n, i+j \leqslant n;$$

$$\begin{pmatrix} & n & \\ & & j+i-n, N-i \\ N-j & & \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } 0 < i < n, i+j > n;$$

while the face functions in the triangle where $i < 0$ are

$$\begin{pmatrix} & j & \\ |i| & & \\ & & N-|i|-j \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } -n < i < 0, |i|+j \leqslant n;$$

$$\begin{pmatrix} & & j \\ |i|, N+n-|i|-j & & \\ & & n \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } -n < i < 0, |i|+j > n.$$

REMARK 3.11. Note that our labeling of the vertices is somewhat different from before. The vertices labeled x_{ij} correspond to the vertices labeled $x_{n-|i|,j}$ in $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$.

First, let us define the functions above. Note that there are natural maps

$$p_1, p_2, p_3, p_4 : \mathrm{Conf}_4 \mathcal{A}_{Sp_{2n}} \to \mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$$

that map a configuration (A, B, C, D) to (B, C, D), (A, C, D), (A, B, D), (A, B, C), respectively. Pulling back functions from $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ allows us to define functions on $\mathrm{Conf}_4 \mathcal{A}_{Sp_{2n}}$. For example,

$$p_3^* \begin{pmatrix} n & & \\ & n-i, 2n+i-j & \\ j & & \end{pmatrix} =: \begin{pmatrix} & n & \\ j & & n-i, 2n+i-j \\ & & \end{pmatrix}.$$

Similarly, we can pull back functions from various maps

$$\mathrm{Conf}_4 \mathcal{A}_{Sp_{2n}} \to \mathrm{Conf}_2 \mathcal{A}_{Sp_{2n}}$$

to define functions such as

$$\begin{pmatrix} j & \\ & N-j \end{pmatrix}.$$

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There is also a map

$$T : \operatorname{Conf}_4 \mathcal{A} \to \operatorname{Conf}_4 \mathcal{A}$$

which sends

$$(A, B, C, D) \to (s_G \cdot D, A, B, C)$$

which allows us to define, for example

$$T^*\left(\begin{array}{ccc} & n & \\ j & & n-i, 2n+i-j \\ & & \end{array}\right) =: \left(\begin{array}{ccc} & j & \\ & & n \\ & n-i, 2n+i-j & \end{array}\right).$$

The forgetful maps and twist maps, combined with the functions described below, will furnish all the functions necessary for the computation of the flip mutation sequence.

More interestingly, we will have to use some functions which depend on all four flags. Let $N = 2n$. Let $0 \leqslant a, b, c, d \leqslant N$ such that $a + b + c + d = 4n = 2N$ and $b + c \leqslant N$. Then we would like to define a function that we will call

$$\left(\begin{array}{ccc} & a & \\ b & \diagup & d \\ & c & \end{array}\right).$$

Note that our notation uses a new symbol, '$\diagup$'. This is because the construction does not exhibit cyclic symmetry, that is,

$$T^*\left(\begin{array}{ccc} & a & \\ b & \diagup & d \\ & c & \end{array}\right) \neq \left(\begin{array}{ccc} & b & \\ c & \diagup & a \\ & d & \end{array}\right).$$

Instead, we use the notation

$$T^*\left(\begin{array}{ccc} & a & \\ b & \diagup & d \\ & c & \end{array}\right) =: \left(\begin{array}{ccc} & b & \\ c & \diagdown & a \\ & d & \end{array}\right).$$

We can also define

$$(T^2)^*\left(\begin{array}{ccc} & a & \\ b & \diagup & d \\ & c & \end{array}\right) =: \left(\begin{array}{ccc} & c & \\ d & \diagup & b \\ & a & \end{array}\right).$$

The function $\left(\begin{smallmatrix} & a & \\ b & \diagup & d \\ & c & \end{smallmatrix}\right)$ on $\operatorname{Conf}_4 \mathcal{A}_{Sp_{2n}}$ is pulled back from a function on $\operatorname{Conf}_4 \mathcal{A}_{SL_N}$. The function on $\operatorname{Conf}_4 \mathcal{A}_{SL_N}$ is given by an invariant in the space

$$[V_{\omega_a} \otimes V_{\omega_b} \otimes V_{\omega_c} \otimes V_{\omega_d}]^{SL_N}.$$

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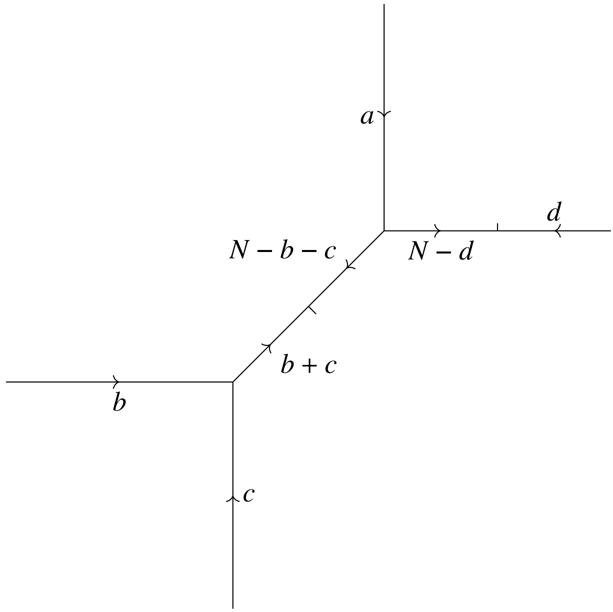

Figure 17. Web for the function $\left(b \overset{a}{\underset{c}{\diagup}} d\right)$, where $a+b+c+d=2N$.

It turns out that this is not always a one-dimensional vector space, so we will have to proceed with some care. We pick out the function given by the web in Figure 17:

This is a function on $\operatorname{Conf}_4 \mathcal{A}_{SL_N}$. Pulling back gives a function on $\operatorname{Conf}_4 \mathcal{A}_{Sp_{2n}}$.

We now define a second type of function on $\operatorname{Conf}_4 \mathcal{A}_{Sp_{2n}}$. Let $0 \leqslant a, b, c, d \leqslant N$ such that $a+b+c+d=5n$, and $c \leqslant d$. Then we would like to define a function that we will call

$$\left(a \overset{n}{\underset{b}{\diagup}} c, d\right).$$

The function $\left(a \overset{n}{\underset{b}{\diagup}} c,d\right)$ on $\operatorname{Conf}_4 \mathcal{A}_{Sp_{2n}}$ is pulled back from a function on $\operatorname{Conf}_4 \mathcal{A}_{SL_{2n}}$. It is given by an invariant in the space

$$[V_{\omega_n} \otimes V_{\omega_a} \otimes V_{\omega_b} \otimes V_{\omega_c+\omega_d}]^{SL_N}$$

picked out by the web in Figure 18a:

Using the twist map T, we can also define the functions $\left(b \overset{a}{\underset{c,d}{\diagdown}} n\right)$, $\left(c,d \overset{b}{\underset{n}{\diagup}} a\right)$, and $\left(n \overset{c,d}{\underset{a}{\diagdown}} b\right)$.

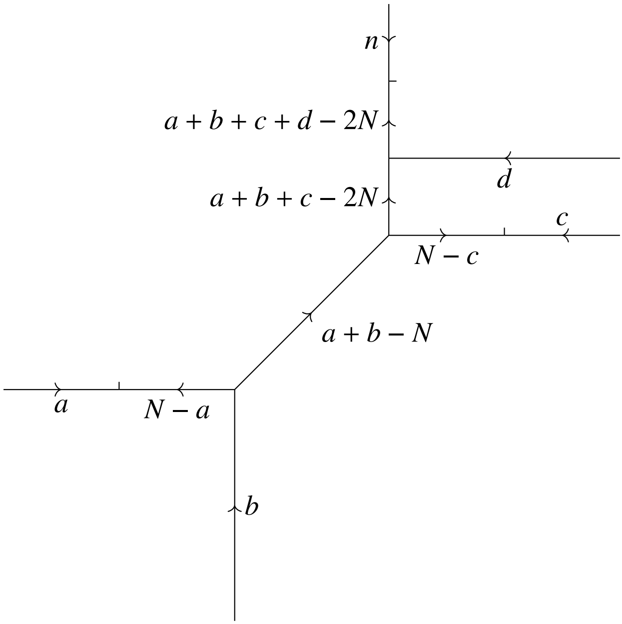

Figure 18a. Web for the function $\left(a \overset{n}{\underset{b}{\diagup}} c,d\right)$, where $a+b+c+d=2N+n$.

Using duality, there is also a function $\left(a \overset{n}{\underset{b}{\diagup}} c,d\right)$ on $\mathrm{Conf}_4\,\mathcal{A}_{Sp_{2n}}$ for $0 \leqslant a, b, c, d \leqslant N$, $a+b+c+d=N+n$, and $c \leqslant d$.

The function $\left(a \overset{n}{\underset{b}{\diagup}} c,d\right)$ on $\mathrm{Conf}_4\,\mathcal{A}_{Sp_{2n}}$ is pulled back from a function on $\mathrm{Conf}_4\,\mathcal{A}_{SL_{2n}}$. The function on $\mathrm{Conf}_4\,\mathcal{A}_{SL_N}$ is given by an invariant in the space

$$[V_{\omega_n} \otimes V_{\omega_a} \otimes V_{\omega_b} \otimes V_{\omega_c+\omega_d}]^{SL_N}.$$

This vector space is generally multi-dimensional. To pick out the correct invariant, we use the web in Figure 18b:

Using the twist map T, we can also define the functions $\left(b \overset{a}{\underset{c,d}{\diagdown}} n\right)$, $\left(c,d \overset{b}{\underset{n}{\diagup}} a\right)$, and $\left(n \overset{c,d}{\underset{a}{\diagdown}} b\right)$.

We will need to define one more type of function to do our calculations. Let $N=2n$. Let $0 \leqslant a, b, c, d \leqslant N$ such that $a+b+c+d=4n=2N$, $a \leqslant n \leqslant b$ and $c \leqslant n \leqslant d$. Then we would like to define a function that we will call

$$\left(a, b \overset{n}{\underset{n}{\diagup}} c, d\right).$$

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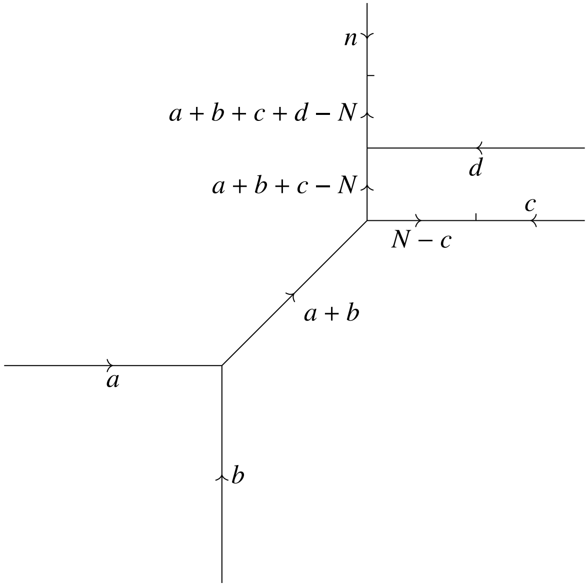

Figure 18b. Web for the function $\left(a \overset{n}{\underset{b}{\diagup}} c,d\right)$, where $a+b+c+d=N+n$.

The function $\left(a,b \overset{n}{\underset{n}{\diagup}} c,d\right)$ on $\operatorname{Conf}_4 \mathcal{A}_{Sp_{2n}}$ is pulled back from a function on $\operatorname{Conf}_4 \mathcal{A}_{SL_{2n}}$. The function on $\operatorname{Conf}_4 \mathcal{A}_{SL_N}$ is given by an invariant in the space

$$[V_{\omega_n} \otimes V_{\omega_a+\omega_b} \otimes V_{\omega_n} \otimes V_{\omega_c+\omega_d}]^{SL_N}.$$

This vector space is generally multi-dimensional. To pick out the correct invariant, we use the web in Figure 19:

Now we give the sequence of mutations realizing the flip of a triangulation. The sequence of mutations leaves $x_{-n,j}$, x_{nj}, y_k untouched as they are frozen variables. Hence we only mutate x_{ij} for $-n \leqslant i \leqslant n$. We now describe the sequence of mutations. The sequence of mutations will have $3n-2$ stages. At the rth step, we mutate all vertices such that

$$|i|+j \leqslant r,$$

$$j-|i|+2n-2 \geqslant r,$$

$$|i|+j \equiv r \mod 2.$$

Note that the first inequality is empty for $r \geqslant 2n-1$, while the second inequality

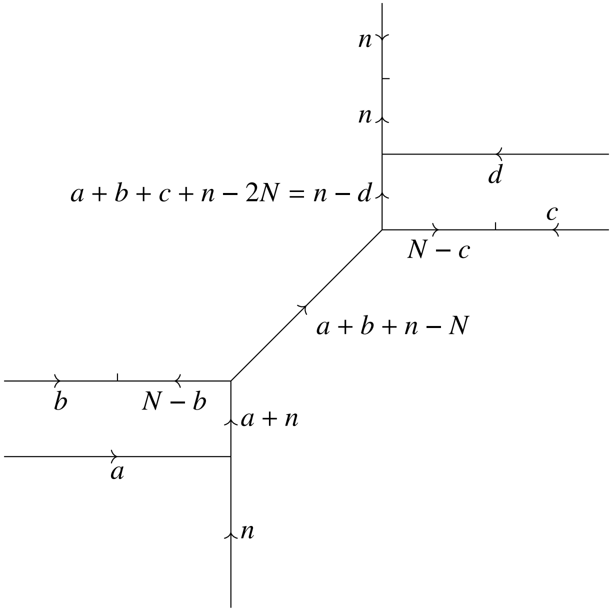

Figure 19. Web for the function $\left(a,b \overset{n}{\underset{n}{\diagup}} c,d\right)$, where $a+b+c+d=2N$.

is empty for $r \leqslant n$. For example, for Sp_6, the sequence of mutations is

$$\begin{gathered} x_{01}, \\ x_{-1,1}, x_{02}, x_{11}, \\ x_{-2,1}, x_{-1,2}, x_{01}, x_{03}, x_{12}, x_{21}, \\ x_{-2,2}, x_{-1,1}, x_{-1,3}, x_{02}, x_{11}, x_{13}, x_{22}, \\ x_{-2,3}, x_{-1,2}, x_{01}, x_{03}, x_{12}, x_{23}, \\ x_{-1,3}, x_{02}, x_{13}, \\ x_{03}. \end{gathered} \tag{3.5}$$

In Figure 20, we depict how the quiver for $\operatorname{Conf}_4 \mathcal{A}_{Sp_6}$ changes after each of the seven stages of mutation.

The analogue in the general case of $\operatorname{Conf}_4 \mathcal{A}_{Sp_{2n}}$ should be clear. Note that there are no circled arrows in the diagram, so that lifting from this sequence of mutations from $\operatorname{Conf}_4 \mathcal{A}_{Sp_{2n}}$ to $\operatorname{Conf}_4 \mathcal{A}_{SL_{2n}}$ is straightforward.

We now have the main theorem of this section:

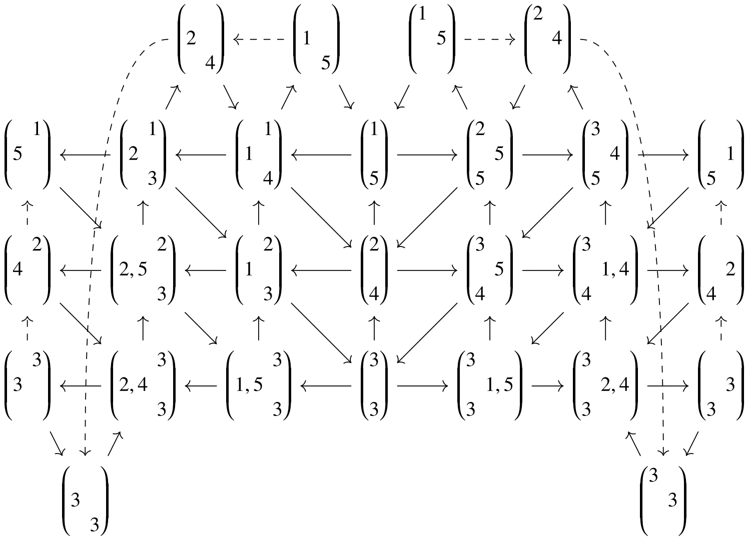

Figure 20a. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4\,\mathcal{A}_{Sp_6}$.

THEOREM 3.12. *The sequence of mutations* (3.5) *realizes the flip of triangulations on* $\mathrm{Conf}_4\,\mathcal{A}_{Sp_{2n}}$.

Proof. We will track how each cluster variable x_{ij} mutates. We first analyze the situation when $i > 0$. The vertex x_{ij} is mutated a total of $n - i$ times. There are four cases.

- When $i + j < n$ and $i < j$, the function attached to x_{ij} mutates in three stages, consisting of $n - i - j$, i, and $j - i$ mutations, respectively:

(1)

$$\begin{pmatrix} i+j & & \\ & & N-i \\ N-j & & \end{pmatrix} \to \begin{pmatrix} & i+j+1 & & \\ 1 & / & & N-i-1 \\ & N-j-1 & & \end{pmatrix}$$

$$\to \begin{pmatrix} & i+j+2 & & \\ 2 & / & & N-i-2 \\ & N-j-2 & & \end{pmatrix} \to \cdots \to \begin{pmatrix} & & n & \\ n-i-j & & / & n+j \\ & & n+i & \end{pmatrix}$$

(2)

$$\begin{pmatrix} & & n & \\ n-i-j & & / & n+j \\ & & n+i & \end{pmatrix} = \begin{pmatrix} & & n & \\ n-i-j & & / & 0, n+j \\ & & n+i & \end{pmatrix}$$

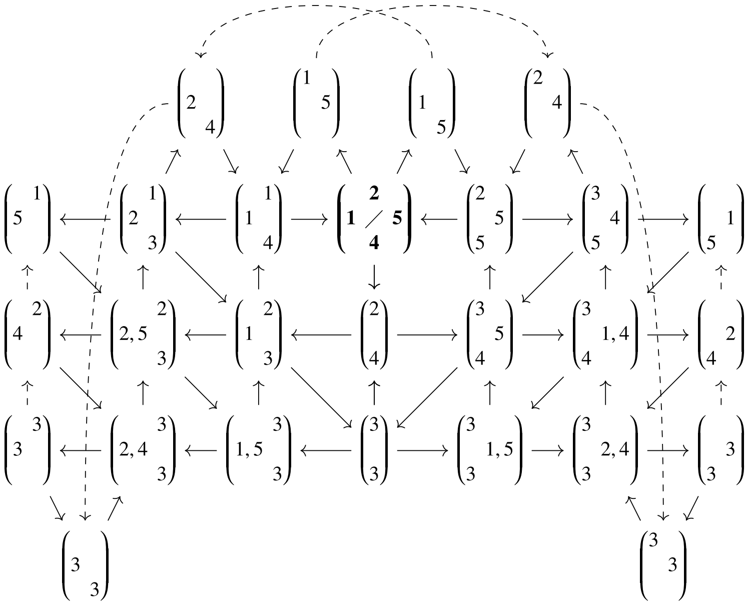

Figure 20b. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Sp_6}$ after the first stage of mutation.

$$
\rightarrow \begin{pmatrix} n-i-j+1 & \overset{n}{\underset{n+i-1}{\diagup}} & 1, n+j-1 \end{pmatrix}
$$

$$
\rightarrow \begin{pmatrix} n-i-j+2 & \overset{n}{\underset{n+i-2}{\diagup}} & 2, n+j-2 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} n-j & \overset{n}{\underset{n}{\diagup}} & i, n+j-i \end{pmatrix}
$$

(3)

$$
\begin{pmatrix} n-j & \overset{n}{\underset{n}{\diagup}} & i, n+j-i \end{pmatrix} = \begin{pmatrix} n-j, N & \overset{n}{\underset{n}{\diagup}} & i, n+j-i \end{pmatrix}
$$

$$
\rightarrow \begin{pmatrix} n-j+1, N-1 & \overset{n}{\underset{n}{\diagup}} & i+1, n+j-i-1 \end{pmatrix}
$$

$$
\rightarrow \begin{pmatrix} n-j+2, N-2 & \overset{n}{\underset{n}{\diagup}} & i+2, n+j-i-2 \end{pmatrix}
$$

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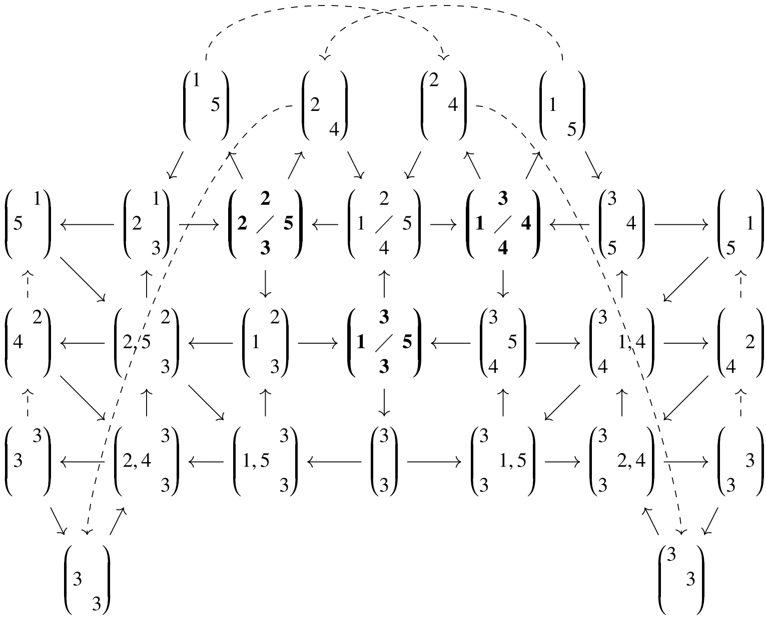

Figure 20c. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Sp_6}$ after the second stage of mutation.

$$\to \cdots \to \left[\begin{pmatrix} & n & \\ n-i, N-j+i & \diagup & j, n \\ & n & \end{pmatrix}\right] \begin{pmatrix} & & \\ n-i, N-j+i & & j \\ & & n \end{pmatrix}.$$

- When $i+j \geqslant n$ and $i < j$, the function attached to x_{ij} mutates in two stages, consisting of $n-j$, and $j-i$ mutations, respectively:

(1)

$$\begin{pmatrix} n & & \\ & & j+i-n, N-i \\ N-j & & \end{pmatrix} \to \begin{pmatrix} & n & \\ 1 & \diagup & j+i-n+1, N-i-1 \\ & N-j-1 & \end{pmatrix}$$

$$\to \begin{pmatrix} & n & \\ 2 & \diagup & j+i-n+2, N-i-2 \\ & N-j-2 & \end{pmatrix} \to \cdots \to \begin{pmatrix} & n & \\ n-j & \diagup & i, n-i+j \\ & n & \end{pmatrix}$$

(2)

$$\begin{pmatrix} & n & \\ n-j & \diagup & i, n-i+j \\ & n & \end{pmatrix} \left[\begin{pmatrix} & n & \\ n-j, N & \diagup & i, n-i+j \\ & n & \end{pmatrix}\right]$$

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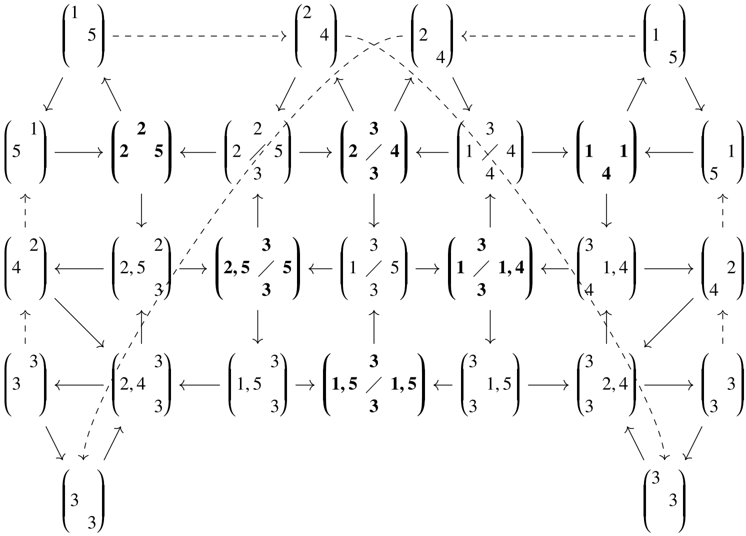

Figure 20d. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4\,\mathcal{A}_{Sp_6}$ after the third stage of mutation.

$$\rightarrow \begin{pmatrix} & n & \\ n-j+1, N-1 & \diagup & i+1, n-i+j-1 \\ & n & \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} & n & \\ n-j+2, N-2 & \diagup & i+2, n-i+j-2 \\ & n & \end{pmatrix}$$

$$\rightarrow \cdots \rightarrow \left[\begin{pmatrix} & n & \\ n-i, N-j+i & \diagup & j, n \\ & n & \end{pmatrix}\right] \begin{pmatrix} n-i, N-j+i & & j \\ & n & \end{pmatrix}.$$

- When $i+j<n$ and $i \geqslant j$, the function attached to x_{ij} mutates in two stages, consisting of $n-i-j$, and j mutations, respectively:

(1)

$$\begin{pmatrix} i+j & & \\ & & N-i \\ N-j & & \end{pmatrix} \rightarrow \begin{pmatrix} & i+j+1 & \\ 1 & & N-i-1 \\ & N-j-1 & \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} & i+j+2 & \\ 2 & & N-i-2 \\ & N-j-2 & \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} & n & \\ n-i-j & & n+j \\ & n+i & \end{pmatrix}$$

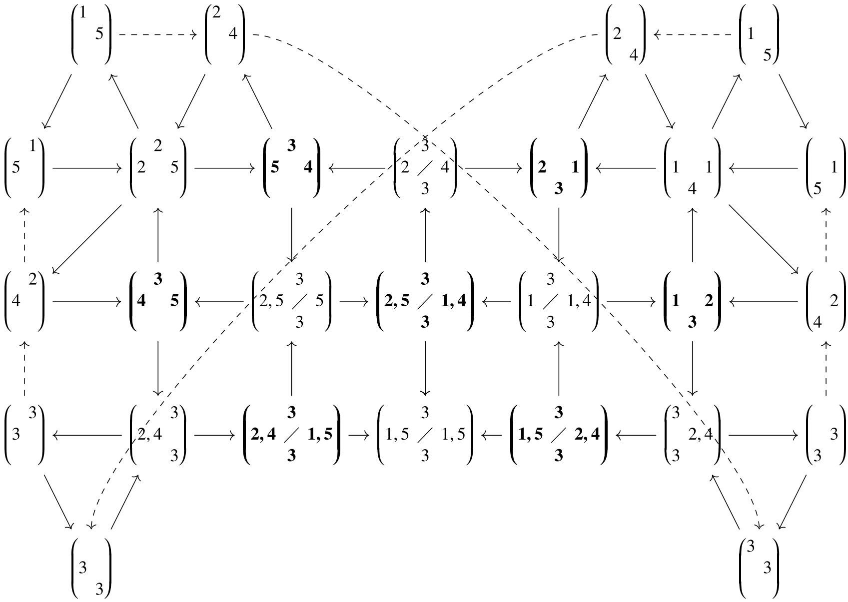

Figure 20e. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4\,\mathcal{A}_{Sp_6}$ after the fourth stage of mutation.

(2)

$$\begin{pmatrix} & n & \\ n-i-j & & n+j \\ & n+i & \end{pmatrix} \to \begin{pmatrix} & n & \\ n-i-j+1 & \diagup & 1, n+j-1 \\ & n+i-1 & \end{pmatrix}$$

$$\to \begin{pmatrix} & n & \\ n-i-j+2 & \diagup & 2, n+j-2 \\ & n+i-2 & \end{pmatrix}$$

$$\to \cdots \to \left[\begin{pmatrix} & n & \\ n-i & \diagup & j, n \\ & n+i-j & \end{pmatrix}\right] \begin{pmatrix} & & \\ n-i & & j \\ & n+i-j & \end{pmatrix}$$

- When $i+j \geqslant n$ and $i \geqslant j$, the function attached to x_{ij} mutates in one stage consisting of $n-i$ mutations:

$$\begin{pmatrix} & n & \\ & & j+i-n, N-i \\ N-j & & \end{pmatrix} \to \begin{pmatrix} & n & \\ 1 & \diagup & j+i-n+1, N-i-1 \\ & N-j-1 & \end{pmatrix}$$

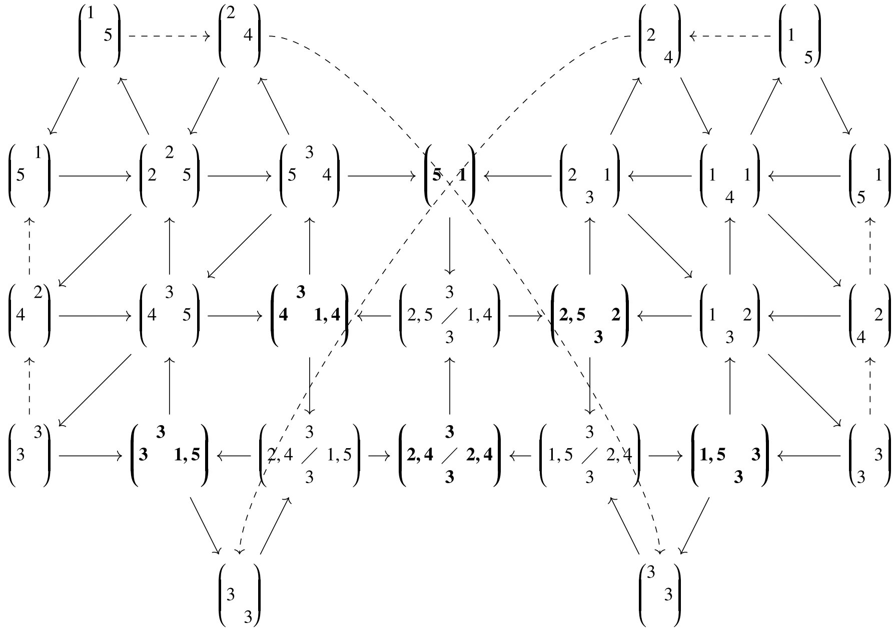

Figure 20f. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4\,\mathcal{A}_{Sp_6}$ after the fifth stage of mutation.

$$\to \begin{pmatrix} & n & & \\ 2 & \diagup & & j+i-n+2, N-i-2 \\ & N-j-2 & & \end{pmatrix}$$

$$\to \cdots \to \left[\begin{pmatrix} & n & \\ n-i & \diagup & j, n \\ & n+i-j & \end{pmatrix}\right] \begin{pmatrix} & & \\ n-i & & j \\ & n+i-j & \end{pmatrix}.$$

The mutation sequence when $i \leqslant 0$ is completely parallel, so we do not include it.

In all these sequences, for each mutation, two parameters increase, and two decrease. Within a stage, the same parameters increase or decrease. The only exception is that sometimes after the last mutation, one removes the factor $\left({}^{n}\;{}_{n}\right)$ (or $\left({}^{n}\;{}_{n}\right)$ when $i \leqslant 0$). The expressions in square brackets indicate the functions before removing these factors.

To prove that these are the functions occurring in the mutation sequence (3.5), we use a handful of identities in conjunction, as in previous proofs of this type. Here are the identities we use:

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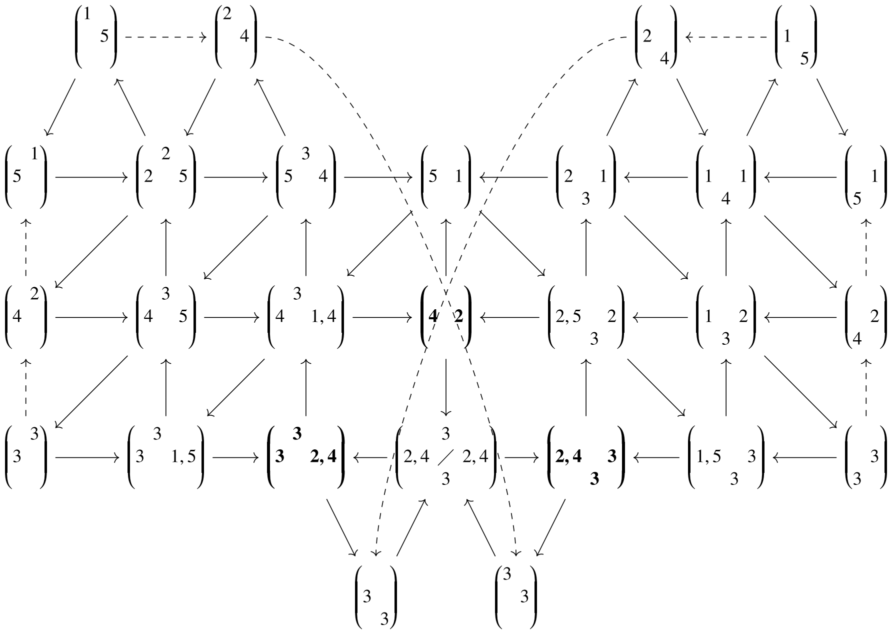

Figure 20g. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4\,\mathcal{A}_{Sp_6}$ after the sixth stage of mutation.

- Let $0 \leqslant a, b, c, d \leqslant N$, and $a + b + c + d = 2N$.

$$\begin{pmatrix} & a & \\ b & \diagup & d \\ & c & \end{pmatrix}\begin{pmatrix} & a+1 & \\ b+1 & \diagup & d-1 \\ & c-1 & \end{pmatrix}$$

$$= \begin{pmatrix} & a & \\ b+1 & \diagup & d \\ & c-1 & \end{pmatrix}\begin{pmatrix} & a+1 & \\ b & \diagup & d-1 \\ & c & \end{pmatrix} + \begin{pmatrix} & a+1 & \\ b & \diagup & d \\ & c-1 & \end{pmatrix}\begin{pmatrix} & a & \\ b+1 & \diagup & d-1 \\ & c & \end{pmatrix}.$$

- Let $0 \leqslant a, b, c, d \leqslant N$, and $a + b + c + d = 3n$.

$$\begin{pmatrix} & n & \\ a & \diagup & c, d \\ & b & \end{pmatrix}\begin{pmatrix} & n & \\ a+1 & \diagup & c+1, d-1 \\ & b-1 & \end{pmatrix}$$

$$= \begin{pmatrix} & n & \\ a+1 & \diagup & c, d \\ & b-1 & \end{pmatrix}\begin{pmatrix} & n & \\ a & \diagup & c+1, d-1 \\ & b & \end{pmatrix} + \begin{pmatrix} & n & \\ a+1 & \diagup & c, d-1 \\ & b & \end{pmatrix}\begin{pmatrix} & n & \\ a & \diagup & c+1, d \\ & b-1 & \end{pmatrix}.$$

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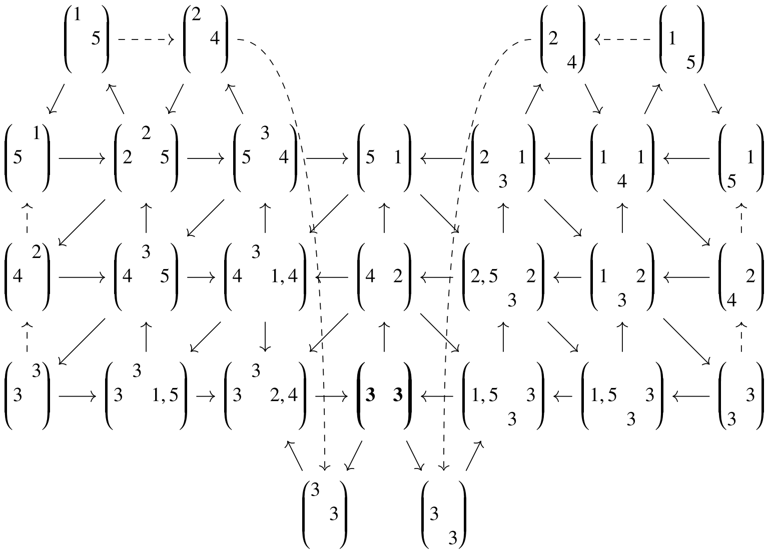

Figure 20h. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4\,\mathcal{A}_{Sp_6}$ after the seventh and last stage of mutation.

There is also a dual identity when $a+b+c+d=5n$ that we use when $i<0$:

$$\left(\begin{array}{ccc} & n & \\ a & \diagup & c, d \\ & b & \end{array}\right)\left(\begin{array}{ccc} & n & \\ a-1 & \diagup & c+1, d-1 \\ & b+1 & \end{array}\right)$$

$$= \left(\begin{array}{ccc} & n & \\ a & \diagup & c+1, d-1 \\ & b & \end{array}\right)\left(\begin{array}{ccc} & n & \\ a-1 & \diagup & c, d \\ & b+1 & \end{array}\right)+\left(\begin{array}{ccc} & n & \\ a-1 & \diagup & c+1, d \\ & b & \end{array}\right)\left(\begin{array}{ccc} & n & \\ a & \diagup & c, d-1 \\ & b+1 & \end{array}\right).$$

- Let $0\leqslant a, b, c, d\leqslant N$, and $a+b+c+d=4n$.

$$\left(\begin{array}{ccc} & n & \\ a, b & \diagup & c, d \\ & n & \end{array}\right)\left(\begin{array}{ccc} & n & \\ a+1, b-1 & \diagup & c+1, d-1 \\ & n & \end{array}\right)$$

$$= \left(\begin{array}{ccc} & n & \\ a+1, b-1 & \diagup & c, d \\ & n & \end{array}\right)\left(\begin{array}{ccc} & n & \\ a, b & \diagup & c+1, d-1 \\ & n & \end{array}\right)$$

$$+ \left(\begin{array}{ccc} & n & \\ a+1, b & \diagup & c, d-1 \\ & n & \end{array}\right)\left(\begin{array}{ccc} & n & \\ a, b-1 & \diagup & c+1, d \\ & n & \end{array}\right).$$

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- Let $0 \leqslant a, b, c, d \leqslant N$ such that $a + b + c + d = 4n = 2N$, $a \leqslant b$ and $c \leqslant d$. If $a + d = b + c = N$,

$$\begin{pmatrix} & a & \\ b & \diagup & d \\ & c & \end{pmatrix} = \begin{pmatrix} b & \\ & c \end{pmatrix}\begin{pmatrix} a & \\ & d \end{pmatrix}.$$

- Let $0 \leqslant a, b, c, d \leqslant N$ such that $a + b + c + d = 4n = 2N$, $a \leqslant b$ and $c \leqslant d$. If a, b, c, or $d = n$,

$$\begin{pmatrix} & n & \\ n, b & \diagup & c, d \\ & n & \end{pmatrix} = \begin{pmatrix} & n & \\ b & & c, d \end{pmatrix}\begin{pmatrix} n & \\ & n \end{pmatrix},$$

$$\begin{pmatrix} & n & \\ a, n & \diagup & c, d \\ & n & \end{pmatrix} = \begin{pmatrix} & n & \\ a & & c, d \end{pmatrix}\begin{pmatrix} n & \\ & n \end{pmatrix},$$

$$\begin{pmatrix} & n & \\ a, b & \diagup & n, d \\ & n & \end{pmatrix} = \begin{pmatrix} a, b & & d \\ & n & \end{pmatrix}\begin{pmatrix} n & \\ & n \end{pmatrix},$$

$$\begin{pmatrix} & n & \\ a, b & \diagup & c, n \\ & n & \end{pmatrix} = \begin{pmatrix} a, b & & c \\ & n & \end{pmatrix}\begin{pmatrix} n & \\ & n \end{pmatrix}.$$

- Let $0 \leqslant a, b, c, d \leqslant N$ such that $a + b + c + d = 4n = 2N$, $a \leqslant b$ and $c \leqslant d$. When $a = 0$, $b = N$, $c = 0$ or $d = N$, we have

$$\begin{pmatrix} & n & \\ 0, b & \diagup & c, d \\ & n & \end{pmatrix} =: \begin{pmatrix} & n & \\ b & \diagup & c, d \\ & n & \end{pmatrix},$$

$$\begin{pmatrix} & n & \\ a, N & \diagup & c, d \\ & n & \end{pmatrix} =: \begin{pmatrix} & n & \\ a & \diagup & c, d \\ & n & \end{pmatrix},$$

$$\begin{pmatrix} & n & \\ a, b & \diagup & 0, d \\ & n & \end{pmatrix} =: \begin{pmatrix} & n & \\ a, b & \diagup & d \\ & n & \end{pmatrix},$$

$$\begin{pmatrix} & n & \\ a, b & \diagup & c, N \\ & n & \end{pmatrix} =: \begin{pmatrix} & n & \\ a, b & \diagup & c \\ & n & \end{pmatrix}.$$

If $a = 0$ and $d = N$, we will have

$$\begin{pmatrix} & n & \\ 0, b & \diagup & c, N \\ & n & \end{pmatrix} = \begin{pmatrix} & n & \\ b & \diagup & c \\ & n & \end{pmatrix}.$$

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A similar equality holds when $b = N, c = 0$. If $a = 0$, $b = N$, $c = 0$ and $d = N$, we will have that

$$\begin{pmatrix} & n & \\ 0, N & \diagup & 0, N \\ & n & \end{pmatrix} = \begin{pmatrix} & n & \\ 0 & \diagup & 0 \\ & n & \end{pmatrix}.$$

The first three sets of identities are the most important. They are variations on the octahedron recurrence. When $i + j < n$ and $i < j$, the three stages use the first, second and third set of identities, respectively. When $i + j \geqslant n$ and $i < j$, the two stages use the second and third set of identities, respectively. When $i + j < n$ and $i \geqslant j$, the two stages use the first and second set of identities, respectively. When $i + j \geqslant n$ and $i \geqslant j$, the one stage uses only the second set of identities.

The last three sets of identities are used to give degenerate versions of the previous three sets of identities. □

3.5.1. An alternative proof of the sequence of mutations for a flip. In this section, we sketch a more conceptual proof for the sequence of mutations in Equation (3.5). The proof given in this section is less computational, but relies on [**BFZ**]. One also needs to keep track of some gradings that can be read off from the sequence of Figures 20a–h. These gradings are dealt with in more detail in [**LL**]. The proof strategy will be similar to Section 3.4.3.

First, recall the double Bruhat cell

$$G^{u,v} := B^+uB^+ \cap B^-vB^-.$$

In [**BFZ**], the authors construct a cluster structure $G^{u,v}$. In fact, they give a cluster for $G^{u,v}$ for every reduced word for $(u, v) \in W \times W$. A reduced word for (u, v) is a shuffle of a reduced word for u and a reduced word for v. The reduced word for u will be in the letters $-1, -2, \dots, -n$, while the word for v will be in the letters $1, 2, \dots, n$. The reflections s_{-i} generate the first copy of W, while the reflections s_i generate the second copy of W.

Consider the natural map from G^{w_0,w_0} to $\mathrm{Conf}_4 \mathcal{A}_G$ given by the formula

$$i : g \in G^{w_0,w_0} \to (U^-, \overline{w_0}U^-, g \cdot \overline{w_0}U^-, g \cdot U^-) \in \mathrm{Conf}_4 \mathcal{A}_G.$$

This is an injective map. The cluster variables on $\mathrm{Conf}_4 \mathcal{A}_G$ pull back to give a subset of the cluster variables on G^{w_0,w_0}. More precisely, in any particular cluster, the edge variables for the edges A_1A_2 and A_3A_4 take the value 1 on the image $i(G^{w_0,w_0})$, while the remaining cluster variables pull back to give a cluster on G^{w_0,w_0}.

For the purposes of comparison, it turns out to be easier to consider a cyclic shift of i, which we call j:

$$j : g \in G^{w_0,w_0} \to (s_G \cdot g \cdot U^-, U^-, \overline{w_0}U^-, g \cdot \overline{w_0}U^-) \in \mathrm{Conf}_4 \mathcal{A}_G.$$

Let $(A, B, C, D) \in \mathrm{Conf}_4 \mathcal{A}_G$. Then the edge variables for the edges corresponding to AD and BC are 1 on the image of j.

Let us specialize to $G = Sp_{2n}$. In the notation of Figure 16, the cluster algebra structure on $\mathrm{Conf}_4 \mathcal{A}_{Sp_{2n}}$ has vertices x_{ij} and y_k. The vertices y_k, $k > 0$ are associated to the edge AD, while the vertices y_k for $k < 0$ are associated to the edge BC. The corresponding cluster variables are all equal to 1 on the image of j. Then the remaining cluster variables, which are attached to the x_{ij}, are generalized minors for the double Bruhat cell G^{w_0,w_0}. Moreover, they are the generalized minors associated to the reduced word $(s_{-n} \cdots s_{-1})^n (s_n \cdots s_1)^n$ for (w_0, w_0).

There is an H^4 action on $\mathrm{Conf}_4 \mathcal{A}_{Sp_{2n}}$, with one copy of H acting on each principal flag. By acting on the flags A and C, one can always arrange that the edge variables associated to the vertices y_k are 1. In other words, any point in $\mathrm{Conf}_4 \mathcal{A}_{Sp_{2n}}$ can be moved into the image of j by using the H action. Therefore, if we understand the H-action, we can reduce computation of the flip sequence of mutations to the corresponding sequence on G^{w_0,w_0}.

To understand the H-action, we just need to know the tensor invariant space corresponding to any cluster variable. Recall that the functions we constructed lie in invariant spaces of the form

$$[V_\lambda \otimes V_\mu \otimes V_\nu \otimes V_\kappa]^G.$$

The invariants spaces can be easily read off from the expression of cluster variables as webs in Figures 20a–h. In this approach we can avoid checking that all the identities in the previous section hold: we only need to know that they respect the four H-gradings. Alternatively, we can use the following result from [**LL**]:

PROPOSITION 3.13. *For any reduced word for (w_0, w_0), we have a corresponding cluster for G^{w_0,w_0}, as constructed in* [**BFZ**]. *These clusters can be compatibly extended to* $\mathrm{Conf}_4 \mathcal{A}_G$. *Each reduced minor of the form $\Delta_{u\omega_i,v\omega_i}$ on G^{w_0,w_0} extends to a function on* $\mathrm{Conf}_4 \mathcal{A}_G$ *lying in the invariant space*

$$[V_\lambda \otimes V_\mu \otimes V_\nu \otimes V_\kappa]^G,$$

where

$$-w_0\lambda = (u_l\omega_i)^+,$$

$$\mu = -(u_l\omega_i)^-,$$

$$w_0 v = (v_l \omega_i^*)^-,$$
$$\kappa = (v_l \omega_i^*)^+.$$

Either way, knowing the H-action on the various cluster variables allows us to reduce the computation of the sequence of mutations for a flip to G^{w_0,w_0}. We then use the result of [BFZ] that states that any two reduced words are related by a sequence of cluster mutations. To perform the flip, we must perform the sequence of mutations that takes us from the reduced word $(s_{-n}\cdots s_{-1})^n(s_1\cdots s_n)^n$ to the reduced word $(s_1\cdots s_n)^n(s_{-n}\cdots s_{-1})^n$. For example, Figures 20a–h correspond to the reduced words

$$s_{-3}s_{-2}s_{-1}s_{-3}s_{-2}s_{-1}s_{-3}s_{-2}s_{-1}s_1s_2s_3s_1s_2s_3s_1s_2s_3$$

$$s_{-3}s_{-2}s_{-1}s_{-3}s_{-2}s_{-1}s_{-3}s_{-2}s_1s_{-1}s_2s_3s_1s_2s_3s_1s_2s_3$$

$$s_{-3}s_{-2}s_{-1}s_{-3}s_{-2}s_1s_{-3}s_2s_{-1}s_1s_{-2}s_3s_{-1}s_2s_3s_1s_2s_3$$

$$s_{-3}s_{-2}s_1s_{-3}s_2s_{-1}s_3s_{-2}s_1s_{-1}s_2s_{-3}s_1s_{-2}s_3s_{-1}s_2s_3$$

$$s_1s_2s_{-3}s_1s_{-2}s_3s_{-1}s_2s_{-3}s_3s_{-2}s_1s_{-3}s_2s_{-1}s_3s_{-2}s_{-1}$$

$$s_1s_2s_3s_1s_2s_{-3}s_1s_{-2}s_3s_{-3}s_2s_{-1}s_3s_{-2}s_{-1}s_{-3}s_{-2}s_{-1}$$

$$s_1s_2s_3s_1s_2s_3s_1s_2s_{-3}s_3s_{-2}s_{-1}s_{-3}s_{-2}s_{-1}s_{-3}s_{-2}s_{-1}$$

$$s_1s_2s_3s_1s_2s_3s_1s_2s_3s_{-3}s_{-2}s_{-1}s_{-3}s_{-2}s_{-1}s_{-3}s_{-2}s_{-1}.$$

We checked in Section 3.2 that our initial cluster consisted of functions on $\mathrm{Conf}_4\,\mathcal{A}_{Sp_{2n}}$ that coincided with generalized minors for the reduced word $(s_{-n}\cdots s_{-1})^n(s_1\cdots s_n)^n$. Then we perform the sequence of mutations taking us to the cluster associated to the reduced word $(s_1\cdots s_n)^n(s_{-n}\cdots s_{-1})^n$. By [BFZ], the functions are again given by a set of generalized minors. To complete the proof, we use the same calculations as in Section 3.2 to calculate that these generalized minors coincide with the functions on $\mathrm{Conf}_4\,\mathcal{A}_{Sp_{2n}}$ which we described by tensor invariants.

The approach of the previous section was straightforward and explicit, though computational. We hope this section clarifies the conceptual underpinnings of the previous one.

4. The cluster algebra structure on $\mathrm{Conf}_m\,G/U$ for $G = Spin_{2n+1}$

We now define the cluster algebra structure on $\mathrm{Conf}_m\,G/U$ when $G = Spin_{2n+1}$. The story will be parallel to the previous case when $G = Sp_{2n}$. The similarities are striking, and reflect the Langlands duality between the seeds

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as predicted in [FG2]. There is no relationship that we know of between the functions on $\operatorname{Conf}_m G/U$ and $\operatorname{Conf}_m G^\vee/U^\vee$, so we will describe the functions explicitly. On the other hand, the seeds as well as the mutation sequences realizing S_3-symmetries and flips are Langlands dual to each other. We discuss this further in Section 4.2.

As in the case of Sp_{2n}, we will utilize what we understand about functions on $\operatorname{Conf}_m \mathcal{A}_{SL_{2n+1}}$ in order to study $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+1}}$.

Recall that $Spin_{2n+1}$ is the double cover of the group SO_{2n+1}, which is the subgroup of SL_{2n+1} preserving a symmetric quadratic form. We take the quadratic form given in the standard basis $e_1, \ldots, e_{2n+1}$ by

$$\langle e_i, e_{2n+2-i} \rangle = (-1)^{i-1}$$

and $\langle e_i, e_j \rangle = 0$ otherwise.

REMARK 4.1. Here the signs are chosen so that the embedding is compatible with the positive structures on $Spin_{2n+1}$ and SL_{2n+1}. Note that the signature of the quadratic form is $(n+1, n)$, so that taking real points gives the split real form of SO_{2n+1}.

The maps

$$Spin_{2n+1} \twoheadrightarrow SO_{2n+1} \hookrightarrow SL_{2n+1}$$

induce maps

$$\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+1}} \to \operatorname{Conf}_m \mathcal{A}_{SO_{2n+1}} \to \operatorname{Conf}_m \mathcal{A}_{SL_{2n+1}}.$$

Let us describe these maps concretely. The variety $\mathcal{A}_{SO_{2n+1}}$ parameterizes chains of isotropic vector spaces

$$V_1 \subset V_2 \subset \cdots \subset V_n \subset V$$

inside the $2n+1$-dimensional standard representation V, where $\dim V_i = i$, and where each V_i is equipped with a volume form.

Equivalently, a point of $\mathcal{A}_{SO_{2n+1}}$ is given by a sequence of vectors

$$v_1, v_2, \ldots, v_n,$$

where

$$V_i := \langle v_1, \ldots, v_i \rangle$$

is isotropic, and where v_i is only determined up to adding linear combinations of v_j for $j < i$.

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The volume form on V_i is then $v_1 \wedge \cdots \wedge v_i$.

From the sequence of vectors $v_1, \ldots, v_n$, we can complete to a basis $v_1, v_2, \ldots, v_{2n+1}$, where $\langle v_i, v_{2n+2-i} \rangle = (-1)^{i-1}$, and $\langle v_i, v_j \rangle = 0$ otherwise. Equivalently, the quadratic form induces an isomorphism $< -, - >: V \to V^*$. At the same time, there are perfect pairings

$$\bigwedge^k V \times \bigwedge^k V^* \to F$$

$$\bigwedge^{2n+1-k} V \times \bigwedge^k V \to F$$

that induce an isomorphism

$$\bigwedge^{2n+1-k} V \simeq \bigwedge^k V^*.$$

Composing this with the inverse of the isomorphism

$$\langle -, - \rangle : \bigwedge^k V \to \bigwedge^k V^*$$

gives an isomorphism

$$\bigwedge^{2n+1-k} V \simeq \bigwedge^k V^* \simeq \bigwedge^k V.$$

Then $v_{n+1}, \ldots, v_{2n+1}$ are chosen so that this isomorphism takes $v_1 \wedge \cdots \wedge v_k$ to $v_1 \wedge \cdots \wedge v_{2n+1-k}$.

Then $v_1, v_2, \ldots, v_{2n+1}$ determines a point of $\mathcal{A}_{SL_{2n+1}}$, as $\mathcal{A}_{SL_{2n+1}}$ parameterizes chains of vector subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_{2n+1} = V$$

along with volume forms $v_1 \wedge \cdots \wedge v_i$, $1 \leqslant i \leqslant 2n+1$.

From the embedding

$$\mathcal{A}_{SO_{2n+1}} \hookrightarrow \mathcal{A}_{SL_{2n+1}},$$

one naturally gets an embedding $\operatorname{Conf}_m \mathcal{A}_{SO_{2n+1}} \hookrightarrow \operatorname{Conf}_m \mathcal{A}_{SL_{2n+1}}$. We can then pull back functions from $\operatorname{Conf}_m \mathcal{A}_{SL_{2n+1}}$ to get functions on $\operatorname{Conf}_m \mathcal{A}_{SO_{2n+1}}$. However, we are ultimately interested in functions on $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+1}}$.

The functions on $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+1}}$ that we will use to define the cluster structure on $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+1}}$ will be invariants of tensor products of representations of $Spin_{2n+1}$. For $m = 3$, they will lie inside

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^G$$

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where λ, μ, ν are elements of the dominant cone inside the weight lattice. In general, not all such functions will come from pulling back functions on $\mathrm{Conf}_m \mathcal{A}_{SL_{2n+1}}$.

However, suppose that

$$f \in [V_\lambda \otimes V_\mu \otimes V_\nu]^G \subset \mathcal{O}(\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+1}}).$$

Then

$$f^2 \in [V_{2\lambda} \otimes V_{2\mu} \otimes V_{2\nu}]^G \subset \mathcal{O}(\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+1}}).$$

However, because $2\lambda, 2\mu, 2\nu$ are dominant weights for SO_{2n+1}, f^2 may be viewed as a function on $\mathrm{Conf}_m \mathcal{A}_{SO_{2n+1}}$. This function is then a pull-back of a function on $\mathrm{Conf}_m \mathcal{A}_{SL_{2n+1}}$. Therefore functions on $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+1}}$ which are tensor invariants are either the pull-backs of functions on $\mathrm{Conf}_m \mathcal{A}_{SL_{2n+1}}$ or square roots of such functions. The square root here corresponds to the fact that $Spin_{2n+1}$ is a double cover of SO_{2n+1}. The choice of the branch of the square root that we take is determined by the positive structure on $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+1}}$: if f is a positive function on $\mathrm{Conf}_m \mathcal{A}_{SL_{2n+1}}$ such that its square root is a function on $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+1}}$, there is a unique choice of $\sqrt{f}$ that is positive on $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+1}}$. That is the square root that we will always take.

4.1. Construction of the seed. We are now ready to construct the seed for the cluster structure on $\mathrm{Conf}_m \mathcal{A}$ when $G = Spin_{2n+1}$. Throughout this section, $G = Spin_{2n+1}$ unless otherwise noted.

Recall that $Spin_{2n+1}$ is associated to the type B Dynkin diagram:

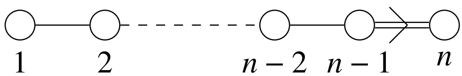

Figure 21. B_n Dynkin diagram.

The nodes of the diagram correspond to $n-1$ long roots, numbered $1, 2, \ldots, n-1$, and one short root, which is numbered n. To describe the cluster structure on $\mathrm{Conf}_3 \mathcal{A}$, we need to give the following data: the set I parameterizing vertices, the functions on $\mathrm{Conf}_3 \mathcal{A}$ corresponding to each vertex, and the B-matrix for this seed.

The B-matrix is encoded via a quiver which consists of $n^2 + 2n$ vertices, of which $n+2$ are black, while the remaining vertices are white. There are n edge functions for each edge of the triangle, and $n^2 - n$ face functions. There is one black vertex for each edge.

In Figure 22, we see the quiver for $Spin_7$. The generalization for other values of n should be clear.

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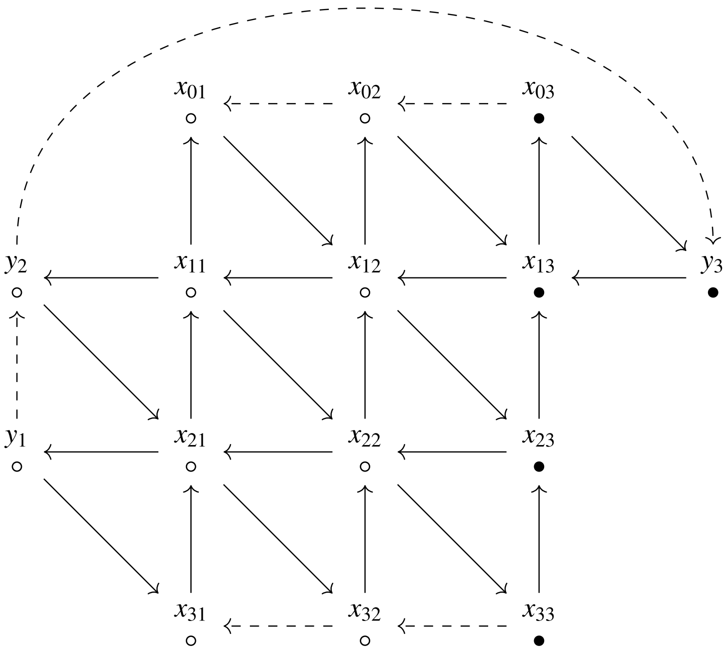

Figure 22. Quiver encoding the cluster structure for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_7}$.

Let us now recall some facts about the representation theory of $Spin_{2n+1}$. The fundamental representations of $Spin_{2n+1}$ are labeled by the fundamental weights $\omega_1, \ldots, \omega_n$. $Spin_{2n+1}$ has a standard $2n + 1$-dimensional representation V. Let $\langle -, - \rangle$ be the orthogonal pairing. Then for $i < n$ the representation V_{ω_i} corresponding to ω_i is precisely $\bigwedge^i V$. The representation V_{ω_n} is the *spin representation* of $Spin_{2n+1}$. The representation $V_{2\omega_n}$ is isomorphic to $\bigwedge^n V$.

We now say which functions are attached to the vertices of the quiver. Recall the functions defined via the webs from Figures 1, 2, and 6. It turns out to be easier to describe the functions attached to the white vertices and the square of the functions attached to the black vertices. In other words, if the function f_{ij} is attached to vertex x_{ij}, it is sometimes more convenient to consider the function $f_{ij}^{2d_{ij}}$. Thus we break down the description of the functions attached to x_{ij} into steps:

(1) For $k < n$, assign the function $\left(\begin{smallmatrix} k \\ 2n+1-k \end{smallmatrix}\right)$ to y_k.

(2) When $i \geqslant j$, assign the function $\left(\begin{smallmatrix} & n-i & \\ & & j \\ n+1+i-j & & \end{smallmatrix}\right)$ to x_{ij}.

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(3) When $i < j$ and $i \neq 0$, we assign the function $\binom{n-i,2n+1+i-j \qquad j}{n+1}$ to x_{ij}.

(4) When $i = 0$, we assign the function $\binom{2n+1-j \qquad j}{}$ to x_{ij}.

(5) Now take the square root of the functions assigned to x_{in} and y_n.

This completely describes the cluster structure on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$. The fact that we can take the square roots of the functions assigned to x_{in} and y_n and get functions that are well-defined on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$ follows from the computations of the next section. Note that the cluster structure is not symmetric with respect to the three flags. Performing various S_3 symmetries, we obtain six different possible cluster structures on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$. These six structures are related by sequences of mutations that we describe in the next section. Below, in Figure 23, we depict the standard cluster described above.

To get the functions for clusters related by S_3 symmetries, we permute the arguments in our notation for the function. For example, rotating the function $\binom{n-i,2n+1+i-j \qquad j}{n+1}$ gives the function $\binom{j}{n-i,2n+1+i-j \qquad n+1}$, while transposing the first two arguments gives $\binom{n+1}{n-i,2n+1+i-j \qquad j}$.

In the next section, we discuss Langlands duality, which will give us a framework for relating, and explaining the similarities between, the cluster algebra structures on $\mathrm{Conf}_m\,\mathcal{A}_G$ for $G = Sp_{2n}$ and $Spin_{2n+1}$.

4.2. Langlands duality. In this section, we make some remarks related to Langlands duality. Note that the quivers for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$ and $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$ are the same, except that white and black vertices switch colors. Moreover, if we mutate corresponding vertices in the quivers for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$ and $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$, the relationship persists: we still get the same quivers, but with colors reversed. This is because the seeds we have constructed for the cluster algebras on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$ and $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$ are Langlands dual seeds:

DEFINITION 4.2 [**FG2**]. Two seeds that have the same set of vertices I and the same set of frozen vertices I_0 are said to be *Langlands dual* if they have B-matrices b_{ij} and $b_{ij}^\vee$ and multipliers d_i and $d_i^\vee$ where

$$d_i = (d_i^\vee)^{-1} D,$$

$$b_{ij} d_j = -b_{ij}^\vee d_j^\vee,$$

for some rational number D.

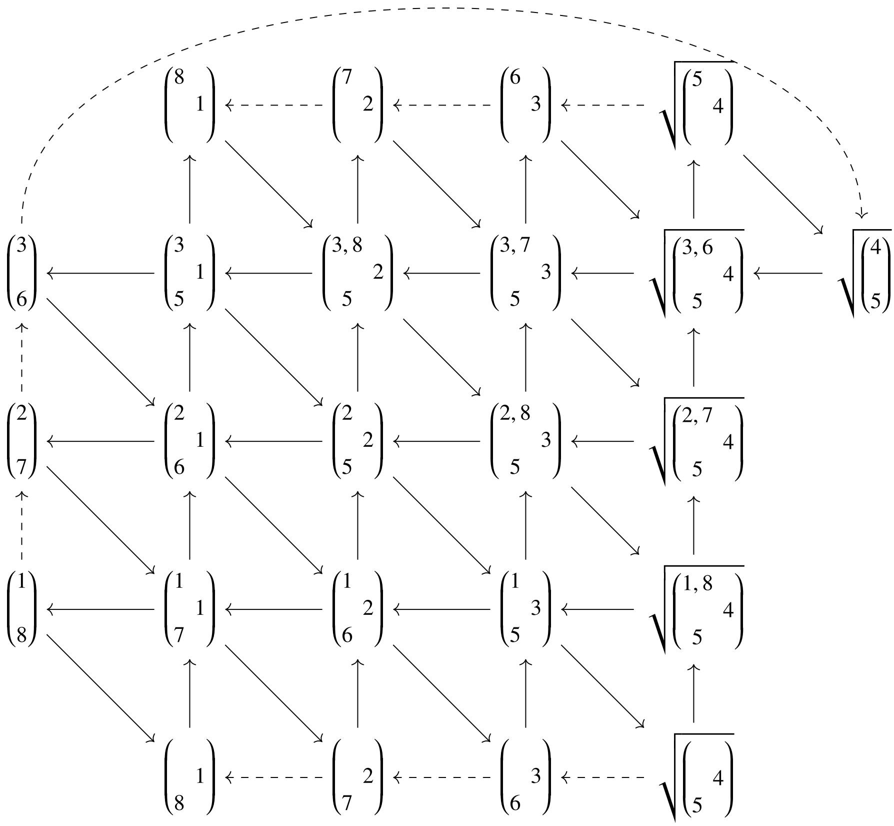

Figure 23. One cluster structure for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_9}$.

REMARK 4.3. Note that the multipliers d_i for a cluster algebra are determined only up to simultaneous scaling by a rational number. Conventions sometimes differ on how to specify the values for the d_i. This is one reason the rational number D appears in the above definition.

REMARK 4.4. The cluster algebras for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$ and $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$ as we have defined them satisfy

$$b_{ij}d_j = b^{\vee}_{ij}d^{\vee}_j,$$

without the negative sign. Negating the B-matrix does not change the cluster algebra, and for that reason a B-matrix and its negative are often considered to be equivalent. The reason that [FG2] include a negative sign in their definition is to make the quantization of the 'modular double' cleaner. We will ignore this issue for most of this paper.

In fact, the seeds for $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+1}}$ and $\mathrm{Conf}_m \mathcal{A}_{Sp_{2n}}$ are Langlands dual; the property is preserved under amalgamation.

When two seeds are Langlands dual, there is a close relationship between the resulting cluster algebras. Suppose that (I, I_0, b_{ij}, d_i) and $(I, I_0, b_{ij}^\vee, d_i^\vee)$ are Langlands dual seeds. Let the cluster variables for the initial seeds be $x_1, \ldots, x_n$ and $x_1^\vee, \ldots, x_n^\vee$, respectively. These cluster variables are naturally in bijection. Then if we mutate x_k to obtain the new cluster variable x_k', we can do the same to $x_k^\vee$ to get $(x_k')^\vee$ and then match x_k' and $(x_k')^\vee$. Continuing in this manner, one conjecturally gets a bijection between all the cluster variables for the Langlands dual seeds. Let us make an observation:

OBSERVATION 4.5. Suppose that we have a cluster variable

$$f \in [V_\lambda \otimes V_\mu \otimes V_\nu]^{Sp_{2n}} \subset \mathcal{O}(\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}).$$

Then if $f^\vee$ is the dual cluster variable to f, then

$$f^\vee \in [V_\lambda \otimes V_\mu \otimes V_\nu]^{Spin_{2n+1}} \subset \mathcal{O}(\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}})$$

if f is associated to a black vertex and $f^\vee$ is associated to a white vertex, while

$$f^\vee \in [V_{\lambda/2} \otimes V_{\mu/2} \otimes V_{\nu/2}]^{Spin_{2n+1}} \subset \mathcal{O}(\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}})$$

if f is associated to a white vertex and $f^\vee$ is associated to a black vertex.

This is clearly true in the initial cluster, and as long as all the cluster variables are functions on $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ and $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$, and not just rational functions on those spaces (as we expect, but do not know how to prove), it is easy to check that the above observation remains true under mutation. Certainly, in all the clusters we consider in this paper this will be the case, as is borne out in the computations that follow.

This gives us the following principle which will underlie the computations of the S_3 symmetries on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$ and the flip on $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+1}}$:

OBSERVATION 4.6. We can compute the formulas for the cluster variables on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$ and $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+1}}$ that appear at various stages of mutation in the following way: start with the formula for the corresponding cluster variable on $\mathrm{Conf}_m \mathcal{A}_{Sp_{2n}}$. Replace every instance of 'a' where $1 \leqslant a \leqslant n-1$ by 'a,' and replace every instance of '$2n - a$' where $1 \leqslant a \leqslant n-1$ by '$2n + 1 -$

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a.' Every instance of 'n' should be replaced by either 'n' or '$n+1$,' depending on the context. Take a square root if the cluster variable corresponds to a black vertex for $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+1}}$. One then obtains the formula for the cluster variable on $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+1}}$.

All the formulas we derive for $Spin_{2n+1}$ will follow this principle.

4.3. Reduced words. We now relate the cluster structure on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$ given in the previous section to Berenstein, Fomin and Zelevinsky's cluster structure on B, the Borel of the group G [BFZ].

We can consider the map

$$i : b \in B^- \to (U^-, \overline{w_0}U^-, b \cdot \overline{w_0}U^-) \in \mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}.$$

PROPOSITION 4.7. *The cluster algebra constructed above on* $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$, *when restricted to the image of* i, *coincides with the cluster algebra structure given in* [BFZ] *on* $G^{w_0,e} \subset B^-$.

Proof. We choose a reduced word for w_0. In the numbering of the nodes of the Dynkin diagram given above for $Spin_{2n+1}$, we choose the reduced word expression

$$w_0 = (s_n s_{n-1} \cdots s_2 s_1)^n.$$

Here our convention is that the above word corresponds to the string $i_1, i_2, \ldots, i_{n-1}, i_n$ repeated n times.

In our situation, we are interested in such minors when $u, v = e$, or when $v = e$ and $u = u_{ij} = (s_n s_{n-1} \cdots s_2 s_1)^{i-1} s_n s_{n-1} \cdots s_j$ for $1 \leqslant i \leqslant n$ and $n \geqslant j \geqslant 1$.

Then the cluster functions on B^- given in [BFZ] are $\Delta_{\omega_i,\omega_i}$ for $1 \leqslant i \leqslant n$ (these are the functions associated to $u, v = e$), and, for $1 \leqslant i \leqslant n$ and $j = 1, 2, \ldots, n-1, n$,

$$\Delta_{u_{ij}\omega_j,\omega_j},$$

which are the functions associated to $v = e$ and $u = u_{ij} = (s_n s_{n-1} \cdots s_2 s_1)^i s_n s_{n-1} \cdots s_j$. Note that u_{ij} is the subword of u that stops on the ith iteration of s_j.

We have the following claims:

(1) When $i = 0$, the function we assigned to x_{ij} $\left(\binom{2n+1-j}{j}\right.$ for $j < n$ and $\left.\sqrt{\binom{n+1}{n}} \text{ for } j = n\right)$ is precisely $\Delta_{\omega_j,\omega_j}$.

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(2) When $i \geqslant j \neq n$, the function we assigned to x_{ij}, $\begin{pmatrix} n-i & \\ & j \\ n+1+i-j & \end{pmatrix}$, is precisely $\Delta_{u_{ij}\omega_j,\omega_j}$.

When $i = j = n$, the function we assigned to x_{nn}, $\sqrt{\begin{pmatrix} & n \\ n+1 & \end{pmatrix}}$, is precisely $\Delta_{u_{nn}\omega_n,\omega_n}$.

(3) When $i < j < n$, the function we assigned to x_{ij}, $\begin{pmatrix} n-i, 2n+1+i-j & \\ & j \\ n+1 & \end{pmatrix}$, is precisely $\Delta_{u_{ij}\omega_j,\omega_j}$. When $i < j = n$, the function we assigned to x_{in}, $\sqrt{\begin{pmatrix} n-i, n+1+i & \\ & n \\ n+1 & \end{pmatrix}}$, is precisely $\Delta_{u_{in}\omega_n,\omega_n}$.

Thus, in all cases the function assigned to x_{ij} is precisely $\Delta_{u_{ij}\omega_j,\omega_j}$.
The proof of these claims is a straightforward calculation. □

Finally, note that we have the following equalities of functions coming:

$$\begin{gathered}
\begin{pmatrix} k \\ \\ 2n+1-k \end{pmatrix} = \begin{pmatrix} 2n+1-k \\ \\ k \end{pmatrix} \\
\begin{pmatrix} n-i & \\ & j \\ n+1+i-j & \end{pmatrix} = \begin{pmatrix} n+1+i & \\ & 2n+1-j \\ n-i+j & \end{pmatrix} \\
\begin{pmatrix} n-i, 2n+1+i-j & \\ & j \\ n+1 & \end{pmatrix} = \begin{pmatrix} n+1+i, j-i & \\ & 2n+1-j \\ n & \end{pmatrix}.
\end{gathered} \tag{4.1}$$

These equalities arise from duality with respect to the quadratic form, which induces an isomorphism between $\bigwedge^i V$ and $\bigwedge^{2n+1-i} V$.

4.4. Relationship with cactus transformations. As in the case of Sp_{2n}, the cactus sequence can be used to construct the cluster algebra structure on $\operatorname{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$. The material of this section is not relevant to the remainder of the paper. We include this material for the following reasons: for the sake of completeness; to emphasize the parallels between Sp_{2n} and $Spin_{2n+1}$; to motivate some computations that we perform later; and to point out a new phenomenon similar to folding of cluster seeds that deserves study in its own right.

Recall the cactus sequence of mutations on $\operatorname{Conf}_3 \mathcal{A}_{SL_N}$. We now specialize to $N = 2n+1$. The cactus sequence will allow us to give an alternative way to construct the cluster algebra structure for $\operatorname{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$. We perform the following sequence of mutations on $\operatorname{Conf}_3 \mathcal{A}_{SL_{2n+1}}$:

$$x_{N-2,1,1}, x_{N-3,1,2}, \ldots, x_{1,1,N-2}$$

$$x_{N-3,2,1}, x_{N-4,2,2}, \dots, x_{1,2,N-3}$$
$$x_{N-2,1,1}, x_{N-3,1,2}, \dots, x_{2,1,N-3}$$
$$x_{N-4,3,1}, x_{N-5,3,2}, \dots, x_{1,3,N-4}$$
$$x_{N-3,2,1}, x_{N-4,2,2}, \dots, x_{2,2,N-4}$$
$$x_{N-2,1,1}, x_{N-3,1,2}, \dots, x_{3,1,N-4}$$
$$\dots$$
$$x_{n+1,n-1,1}, x_{n,n-1,2}, \dots, x_{1,n-1,n+1}$$
$$x_{n+2,n-2,1}, x_{n,n-1,2}, \dots, x_{2,n-2,n+1}$$
$$\dots$$
$$x_{N-2,1,1}, x_{N-3,1,2}, \dots, x_{n-1,1,n+1}.$$

One can picture the above sequence as consisting of $n-1$ stages, where each stage consists of mutating all vertices lying in a parallelogram.

Additionally, we perform the following sequence of mutations, which are ‘half’ of the mutations in the next stage.

$$x_{n,n,1}, x_{n-1,n,2}, \dots, x_{2,n,n-1}$$
$$x_{n+1,n-1,1}, x_{n,n-1,2}, \dots, x_{4,n-1,n-2}$$
$$\dots$$
$$x_{N-4,3,1}, x_{N-5,3,2}$$
$$x_{N-3,2,1}$$

The result will be that we get the quiver pictured in Figure 24 for $Spin_7$.

The function attached to x_{ijk} (here we take $i, j, k > 0$ to exclude frozen vertices) will be

- $\begin{pmatrix} i & \\ & k \\ j & \end{pmatrix}$ if $j \geqslant n+1$;
- $\begin{pmatrix} n+j, i+j-n-1 & \\ & k+n+1-j \\ n+1 & \end{pmatrix}$ if $k < j < n+1$ and $k < n+1$;
- $\begin{pmatrix} n+1+j, i+j-n & \\ & k+n-j \\ n & \end{pmatrix}$ if $k \leqslant j < n+1$;
- $(j+k, i+j; k+i)$ if $k \geqslant n$.

Thus we see that the functions attached to x_{ijk} and x_{ikj} pull back to the same function on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$ via the map $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}} \to \mathrm{Conf}_3 \mathcal{A}_{SL_{2n+1}}$. Here we use the identities (3.1). In fact, for $j > k$ and $j \geqslant n+1$, we have that the vertex x_{ijk} corresponds to the vertex $x_{n-i,k}$ in the quiver for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$, while for $j > k$ and $j \leqslant n$, we have that the vertex x_{ijk} corresponds to the vertex $x_{k,n+1+k-j}$.

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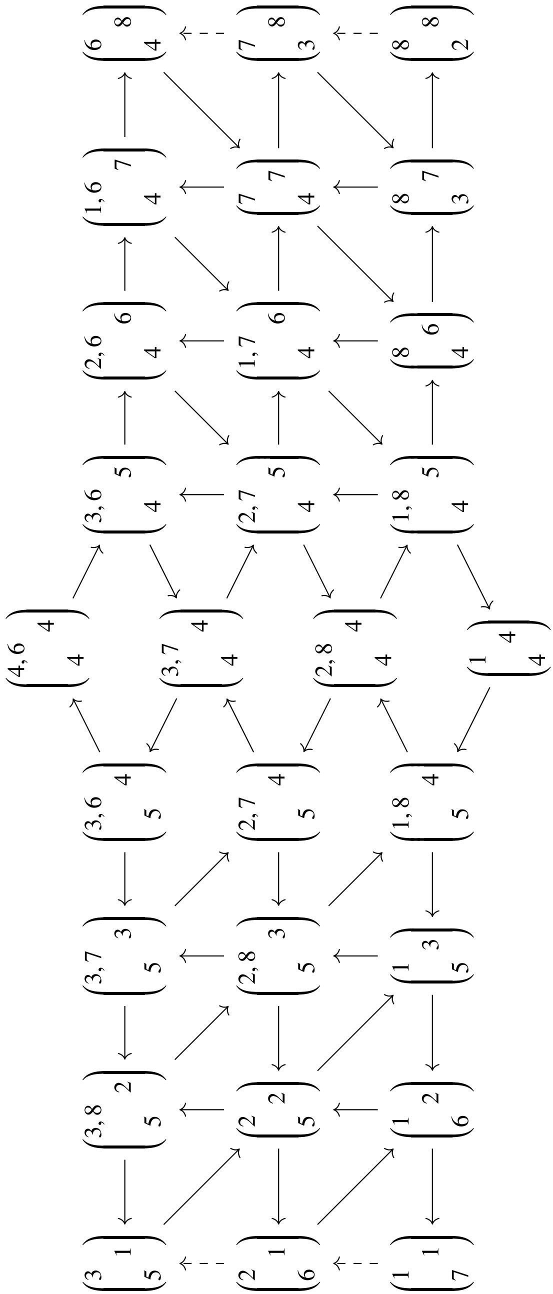

Figure 24. The functions and quiver for the cluster $\mathrm{Conf}_3\,\mathcal{A}_{SL_9}$ which folds to give a cluster for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_9}$.

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The vertices where $j = k$ do not correspond to vertices for the quiver for the cluster algebra on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$. However, something interesting happens. Note that the function attached to $x_{2n+1-2j,j,j}$ is $\left(\begin{smallmatrix} n+1+j,n+1-j & \\ & n \\ n & \end{smallmatrix}\right)$. Mutating this vertex gives the function $\left(\begin{smallmatrix} n+j,n-j & \\ & n+1 \\ n+1 & \end{smallmatrix}\right)$. However, note that

$$\begin{pmatrix} n+1+j, n+1-j & \\ & n \\ n & \end{pmatrix} = \begin{pmatrix} n+j, n-j & \\ & n+1 \\ n+1 & \end{pmatrix}$$

using the identities (4.1). Using the identity for the cluster transformation we get that

$$\begin{aligned}
\begin{pmatrix} n+1+j, n+1-j & \\ & n \\ n & \end{pmatrix}^2 &= \begin{pmatrix} n+1+j, n+1-j & \\ & n \\ n & \end{pmatrix}\begin{pmatrix} n+j, n-j & \\ & n+1 \\ n+1 & \end{pmatrix} \\
&= \begin{pmatrix} n+1+j, n-j & \\ & n \\ n+1 & \end{pmatrix}\begin{pmatrix} n+j, n+1-j & \\ & n+1 \\ n & \end{pmatrix} \\
&\quad + \begin{pmatrix} n+1+j, n-j & \\ & n+1 \\ n & \end{pmatrix}\begin{pmatrix} n+j, n+1-j & \\ & n \\ n+1 & \end{pmatrix} \\
&= 2\begin{pmatrix} n+1+j, n-j & \\ & n \\ n+1 & \end{pmatrix}\begin{pmatrix} n+j, n+1-j & \\ & n+1 \\ n & \end{pmatrix}.
\end{aligned}$$

(In some cases we will get a degenerate version of this identity.)

Thus, although the functions $\left(\begin{smallmatrix} n+1+j,n+1-j & \\ & n \\ n & \end{smallmatrix}\right)$ attached to $x_{2n+1-2j,j,j}$ do not get used as cluster functions on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$, we can use the above identity to express them in terms cluster functions. This identity will become useful to us later.

The result of this is that we can find the cluster functions on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$ as follows: start with the standard cluster algebra on $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n+1}}$, which has functions attached to vertices x_{ijk}. Perform series of mutations given above, which is a subsequence of the cactus sequence. Discard the functions $\left(\begin{smallmatrix} n+1+j,n+1-j & \\ & n \\ n & \end{smallmatrix}\right)$ attached to $x_{2n+1-2j,j,j}$. Identify the following pairs of vertices, which have functions which are equal when restricted to the image of $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$: $x_{i,2n-i,0}$ and $x_{2n-i,i,0}$; $x_{i,0,2n-i}$ and $x_{2n-i,0,i}$; $x_{0,j,2n-j}$ and $x_{0,2n-j,j}$; x_{ijk} and x_{ikj} for $i, j, k > 0$ and $j \neq k$. Now take the square roots of the appropriate functions (those that correspond to $x_{i,n}$ or y_n). The only problem is that it is unclear to us how to read off the correct B-matrix/quiver. This phenomenon, which is reminiscent of folding of cluster algebras, deserves to be investigated more generally.

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More invariantly, there is an outer automorphism of SL_{2n+1} that has SO_{2n+1} as its fixed locus. This gives an involution of $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n+1}}$ (and more generally $\mathrm{Conf}_m\,\mathcal{A}_{SL_{2n+1}}$) that has $\mathrm{Conf}_3\,\mathcal{A}_{SO_{2n+1}}$ (respectively, $\mathrm{Conf}_m\,\mathcal{A}_{SO_{2n+1}}$) as its fixed locus. It turns out that the cluster algebra structure on $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$ is preserved by this involution. Moreover, there is a particular seed, constructed above, that is *almost* preserved by this involution: only the functions attached to $x_{2n+1-2j,j,j}$ change.

It is also clarifying to step back and motivate the sequence of mutations realizing the cactus sequence above. As explained in Section 3.2, the cluster structure on $\mathrm{Conf}_3\,\mathcal{A}$ comes from a reduced word for the longest element w_0 in the Weyl group of G. The initial seed for $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n+1}}$ is built using the reduced word

$$s_1 s_2 \ldots s_{2n} s_1 s_2 \ldots s_{2n-1} \ldots s_1 s_2 s_3 s_1 s_2 s_1.$$

Here $s_1, \ldots, s_{2n}$ are the generators of the Weyl group for $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n+1}}$. It is known how to use cluster transformations to pass between the clusters that are associated to different reduced words [**BFZ**]. The cactus sequence transforms between the cluster above and the cluster associated to the reduced word

$$s_{2n} s_{2n-1} \ldots s_1 s_{2n} s_{2n-1} \ldots s_2 \ldots s_{2n} s_{2n-1} s_{2n-2} s_{2n} s_{2n-1} s_{2n}.$$

The subsequence given above transforms the initial cluster into the cluster associated with the reduced word

$$(s_n s_{n+1} s_n s_{n-1} s_{n+2} s_{n-2} s_{n+3} \ldots s_1 s_{2n})^n.$$

Now let $s'_1, s'_2, \ldots, s'_n$ be the generators of the Weyl group of $\mathrm{Conf}_3\,\mathcal{A}_{Sp_{2n}}$. There is an injection from the Weyl group of $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$ to the Weyl group of $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n+1}}$ that takes

$$s'_n \to s_n s_{n+1} s_n,$$

$$s'_i \to s_i s_{2n+1-i}.$$

Thus we see that the folding-like phenomenon that relates the cluster structures on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$ and $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n+1}}$ has an incarnation on the level of Weyl groups.

4.5. The sequences of mutations realizing S_3 symmetries. We have described six different cluster structures on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$. We would now like to give sequences of mutations relating these six clusters to show that they are actually all clusters in the same cluster algebra.

As in the case of Sp_{2n}, we will realize the S_3 symmetries on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$ by exhibiting sequences of mutations that realize two different transpositions in the

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group S_3. In fact, the sequences of mutations used to realize these transpositions are the same as for Sp_{2n}: the mutation sequences are Langlands dual to each other. As before, the identities used in the mutation sequence of one of these transpositions are exactly those of the cactus sequence. The other transposition requires a different analysis.

4.5.1. The first transposition. Let $(A, B, C) \in \mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$ be a triple of flags. The sequence of mutations that realizes that S_3 symmetry $(A, B, C) \to (A, C, B)$ is the same as (3.3):

$$\begin{gathered} x_{11}, x_{21}, x_{22}, x_{12}, x_{13}, x_{23}, x_{33}, x_{32}, x_{31}, \ldots, x_{1,n-1}, \ldots, x_{n-1,n-1}, \ldots, x_{n-1,1}, \\ x_{11}, x_{21}, x_{22}, x_{12}, \ldots x_{1,n-2}, \ldots, x_{n-2,n-2}, \ldots, x_{n-2,1}, \\ \ldots, \\ x_{11}, x_{21}, x_{22}, x_{12}, \\ x_{11}. \end{gathered} \tag{4.2}$$

The sequence can be thought of as follows: at any step of the process, we mutate all x_{ij} such that $\max(i, j)$ is constant. It will not matter in which order we mutate these x_{ij} because the vertices we mutate have no arrows between them. So we first mutate the x_{ij} such that $\max(i, j) = 1$, then the x_{ij} such that $\max(i, j) = 2$, then the x_{ij} such that $\max(i, j) = 3$, and so on. The sequence of maximums that we use is

$$1, 2, 3, \ldots, n-1, 1, 2, \ldots, n-2, \ldots, 1, 2, 3, 1, 2, 1.$$

The evolution of the quiver for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$ is just as in the case for $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$, as pictured in Figures 11 and 12, with the only difference being that black and white vertices switch colors.

In Figure 25, we depict how the quiver for $\mathrm{Conf}_3 \mathcal{A}_{Spin_9}$ changes after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3.

In Figure 26, we depict the state of the quiver after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3, 1, 2; and 1, 2, 3, 1, 2, 1.

From these diagrams the various quivers in the general case of $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$ should be clear.

THEOREM 4.8. *The sequence of mutations* (4.2) *realizes the* S_3 *symmetry* $(A, B, C) \to (A, C, B)$ *on* $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$.

Proof. The proof is identical to the case where $G = Sp_{2n}$. We track how all the cluster variables mutate under this sequence of mutations.

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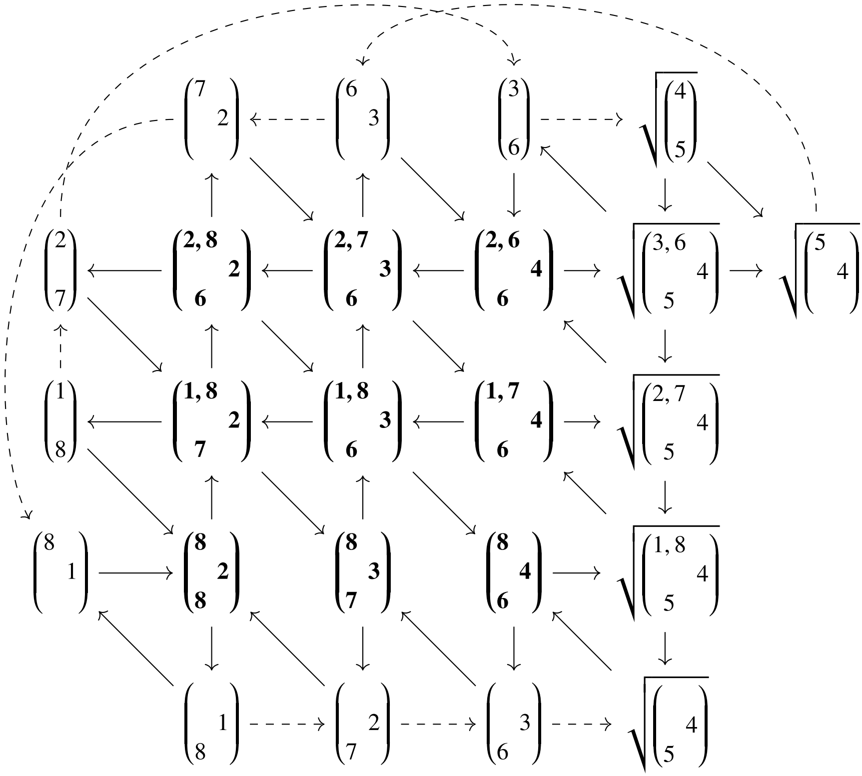

Figure 25. The quiver and the functions for $\mathrm{Conf}_3 \mathcal{A}_{Spin_9}$ after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3.

If $\max(i, j) = k$, then x_{ij} is mutated a total of $n-k$ times. Recall that when $i \geqslant j$, we assign the function $\begin{pmatrix} & n-i & \\ & & j \\ n+1+i-j & & \end{pmatrix}$ to x_{ij}. Thus the function attached to x_{ij} transforms as follows:

$$\begin{pmatrix} & n-i & \\ & & j \\ n+1+i-j & & \end{pmatrix} \to \begin{pmatrix} 2n, n-i-1 & & \\ & & j+1 \\ n+i-j+2 & & \end{pmatrix} \to \begin{pmatrix} 2n-1, n-i-2 & & \\ & & j+2 \\ & n+i-j+3 & \end{pmatrix} \to \cdots$$

$$\to \begin{pmatrix} n+i+2, 1 & & \\ & & n-i+j-1 \\ & 2n-j & \end{pmatrix} \to \begin{pmatrix} & n+i+1 & \\ & & n-i+j \\ 2n+1-j & & \end{pmatrix} = \begin{pmatrix} n-i & & \\ & & n+1+i-j \\ & j & \end{pmatrix}$$

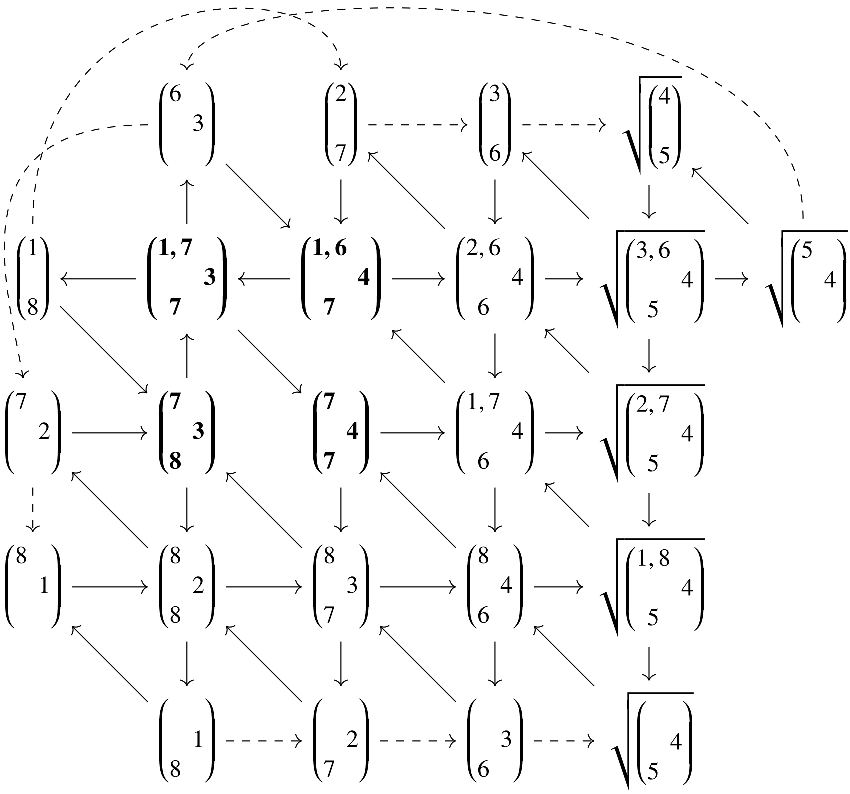

Figure 26a. The quiver and the functions for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_9}$ after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3, 1, 2.

When $i < j$ and $i \neq 0$, we assign the function $\begin{pmatrix} n-i, 2n+1+i-j & \\ & j \\ n+1 & \end{pmatrix}$ to x_{ij}. Thus the function attached to x_{ij} transforms as follows:

$$\begin{aligned} &\begin{pmatrix} n-i, 2n+1+i-j & \\ & j \\ n+1 & \end{pmatrix} \to \begin{pmatrix} n-i-1, 2n+i-j & \\ & j+1 \\ n+2 & \end{pmatrix} \\ &\to \begin{pmatrix} n-i-2, 2n+i-j-1 & \\ & j+2 \\ n+3 & \end{pmatrix} \to \cdots \\ &\to \begin{pmatrix} j-i+1, n+i+2 & \\ & n-1 \\ 2n-j & \end{pmatrix} \to \begin{pmatrix} j-i, n+i+1 & \\ & n \\ 2n+1-j & \end{pmatrix} \end{aligned}$$

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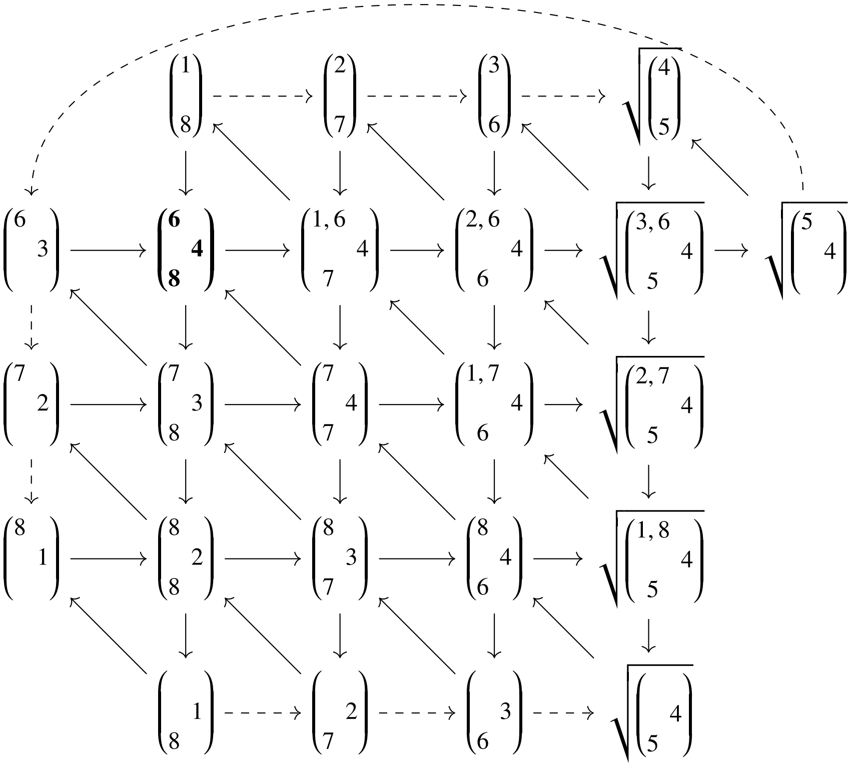

Figure 26b. The quiver and the functions for $\mathrm{Conf}_3 \mathcal{A}_{Spin_9}$ after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3, 1, 2, 1.

$$= \begin{pmatrix} n-i, 2n+1+i-j & \\ & n+1 \\ j & \end{pmatrix}$$

The identities we need to use are exactly those appearing in the cactus sequence. □

The above sequence of mutations takes us from one seed for the cluster algebra structure on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$ to another seed where the roles of the second and third principal flags have been reversed. Thus, we have realized the first of the transpositions necessary to construct all the S_3 symmetries of $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$.

4.5.2. The second transposition. Let us now give the sequence of mutations that realizes that S_3 symmetry $(A, B, C) \to (C, B, A)$.

The sequence of mutations is as in the case of Sp_{2n}, (3.4):

$$\begin{gathered} x_{n-1,n},\, x_{n-2,n-1},\, x_{n-2,n},\, x_{n-3,n-2},\, x_{n-3,n-1},\, x_{n-3,n},\, \ldots,\, x_{1,2},\, \ldots,\, x_{1,n}, \\ x_{n-1,n},\, x_{n-2,n-1},\, x_{n-2,n},\, x_{n-3,n-2},\, x_{n-3,n-1},\, x_{n-3,n},\, \ldots,\, x_{2,3},\, \ldots,\, x_{2,n}, \\ x_{n-1,n},\, x_{n-2,n-1},\, x_{n-2,n},\, x_{n-3,n-2},\, x_{n-3,n-1},\, x_{n-3,n},\, \ldots,\, x_{3,4},\, \ldots,\, x_{3,n}, \\ \ldots x_{n-1,n},\, x_{n-2,n-1},\, x_{n-2,n},\, x_{n-3,n-2},\, x_{n-3,n-1},\, x_{n-3,n}, \\ x_{n-1,n},\, x_{n-2,n-1},\, x_{n-2,n} \\ x_{n-1,n}. \end{gathered} \tag{4.3}$$

The sequence can be thought of as follows: we only mutate those x_{ij} with $i < j$. At any step of the process, we mutate all x_{ij} in the kth row (the kth row consists of x_{ij} such that $i = k$) such that $i < j$. It will not matter in which order we mutate these x_{ij}. The sequence of rows that we mutate is

$$n-1, n-2, \ldots, 2, 1, n-1, n-2, \ldots, 2, n-1, \ldots, 3, \ldots, n-1, n-2, n-1.$$

As in the previous transposition, the evolution of the quiver for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$ is just as in the case for $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$, as pictured in Figures 15 and 16, with the only difference being that black and white vertices switch colors.

In Figure 27, we depict how the quiver for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{11}}$ changes after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1.

In Figure 28, we depict the state of the quiver after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1; 4, 3, 2, 1, 4, 3, 2; 4, 3, 2, 1, 4, 3, 2, 4, 3; and 4, 3, 2, 1, 4, 3, 2, 4, 3, 4.

The circles on several of the arrows are a bookkeeping device that tell us how to lift a function from $\mathrm{Conf}_3 \mathcal{A}_{Spin_{11}}$ to $\mathrm{Conf}_3 \mathcal{A}_{SL_{11}}$ in the way that is most convenient for our computations. From these diagrams the various quivers in the general case of $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$ should be clear.

Recall the functions defined via Figure 15. They will appear when we perform the sequence of mutations above. In particular, we make use of functions of the form

$$\begin{pmatrix} n-i, n+1+i & & \\ & & n-j, n+1+j \\ n, n+1 & & \end{pmatrix}.$$

These functions are invariants (unique up to scale) of the tensor product

$$[V_{2\omega_{n-i}} \otimes V_{4\omega_n} \otimes V_{2\omega_{n-j}}]^{Spin_{2n+1}}.$$

These functions will have square roots which are invariants (again, unique up to scale) of the tensor product

$$[V_{\omega_{n-i}} \otimes V_{2\omega_n} \otimes V_{\omega_{n-j}}]^{Spin_{2n+1}}.$$

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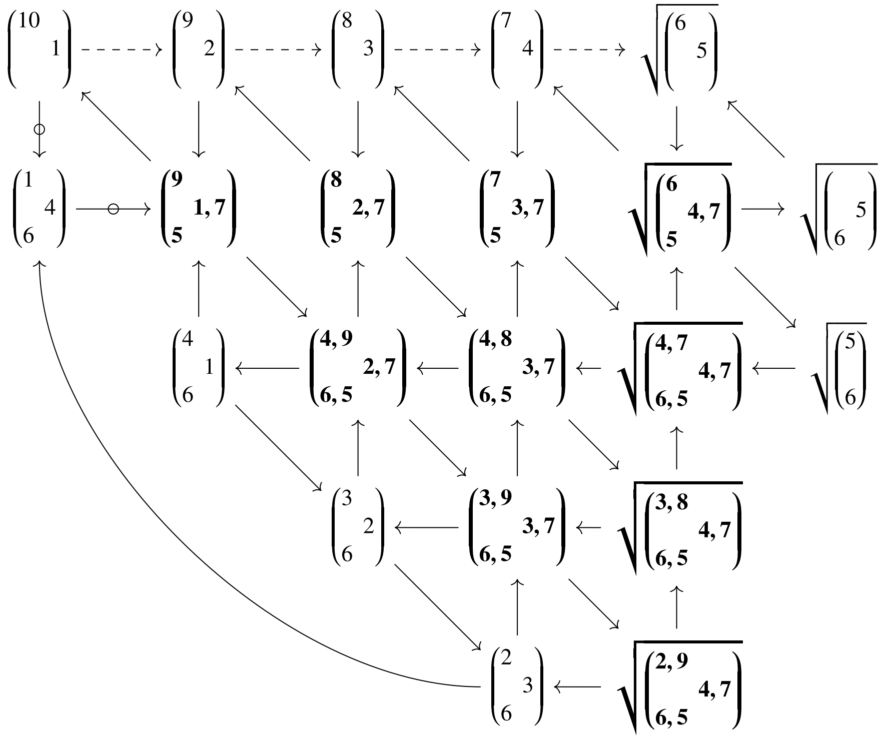

Figure 27. The quiver and functions for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{11}}$ after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1.

Thus

$$\sqrt{\begin{pmatrix} n-i, n+1+i & & \\ & & n-j, n+1+j \\ & n, n+1 & \end{pmatrix}}$$

is a well-defined function on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$.

Note that in the above sequence of mutations, x_{ij} is mutated i times if $i < j$. We can now state the main theorem of this section.

THEOREM 4.9. *The sequence of mutations* (4.3) *realizes the* S_3*-symmetry* $(A, B, C) \to (C, B, A)$ *for* $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$.

Proof. We will track how all the cluster variables mutate. If $i < j$, then x_{ij} is mutated a total of i times. Recall that when $i < j$, we assign either the function

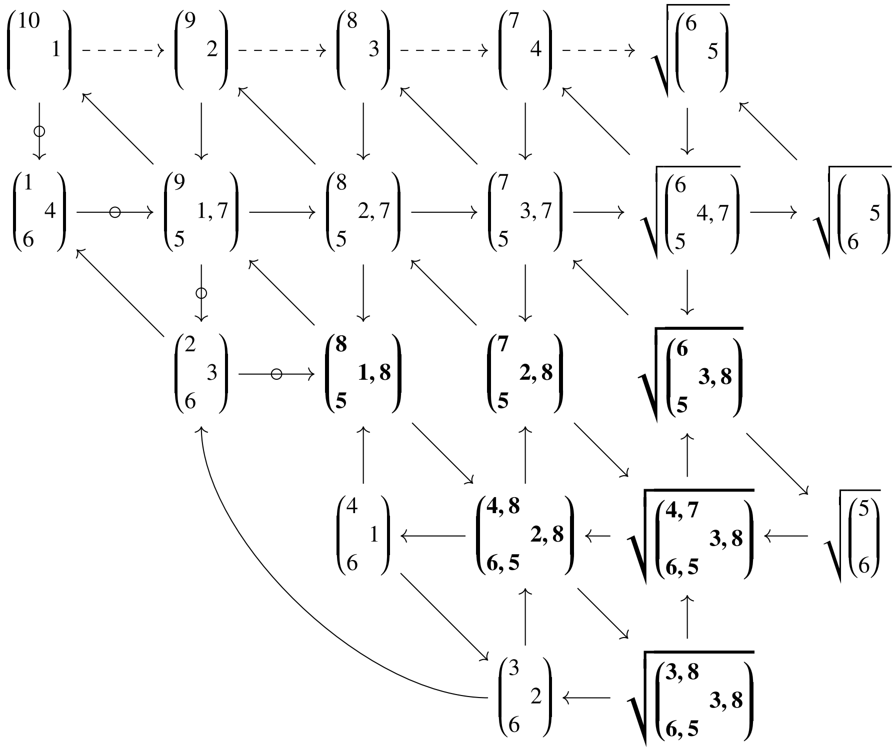

Figure 28a. The quiver and functions for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{11}}$ after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1, 4, 3, 2.

$\begin{pmatrix} n-i, 2n+1+i-j & \\ & j \\ n & \end{pmatrix}$ or its square root to x_{ij} depending on whether $j < n$ or $j = n$. For $j < n$, the function attached to x_{ij} transforms as follows:

$$\begin{pmatrix} n-i, 2n+1+i-j & \\ & j \\ n+1 & \end{pmatrix} \to \begin{pmatrix} n-i+1, 2n+i-j & \\ & j-1, n+2 \\ n, n+1 & \end{pmatrix}$$

$$\to \begin{pmatrix} n-i+2, 2n+i-j-1 & \\ & j-2, n+3 \\ n, n+1 & \end{pmatrix}$$

$$\to \cdots \to \begin{pmatrix} n-1, 2n+2-j & \\ & j-i+1, n+i \\ n, n+1 & \end{pmatrix}$$

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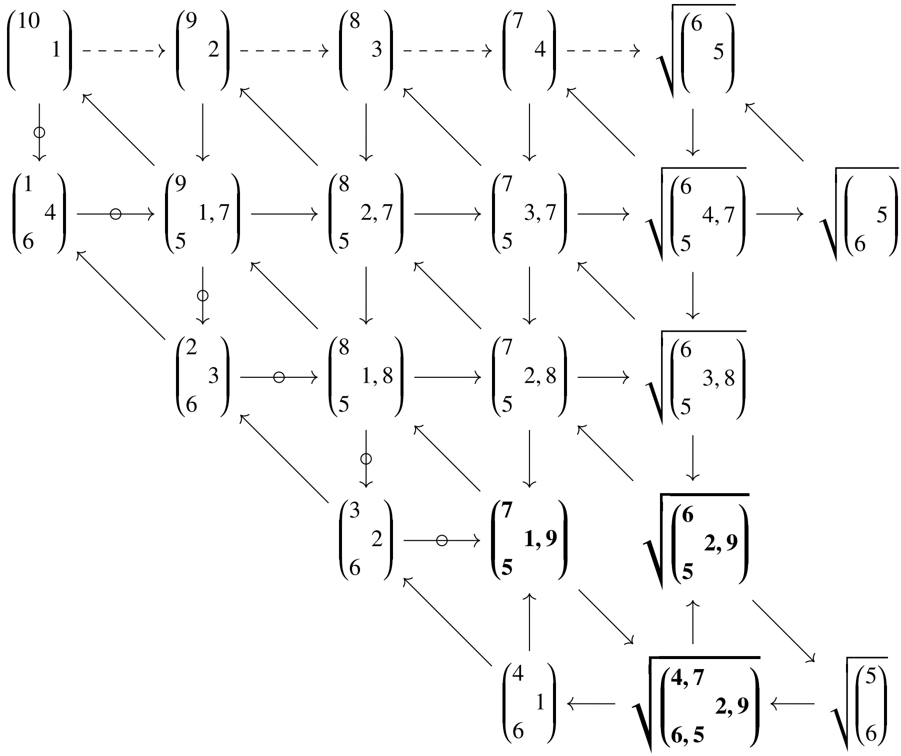

Figure 28b. The quiver and functions for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{11}}$ after performing after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1, 4, 3, 2, 4, 3.

$$\to \begin{pmatrix} 2n+1-j & \\ & j-i, n+i+1 \\ n & \end{pmatrix} = \begin{pmatrix} j & \\ & n-i, 2n+1+i-j \\ n+1 & \end{pmatrix}$$

For $j = n$, we use the same formulas, but take the square roots of all the resulting functions.

We have already described the quivers at the various stages of mutation. We must then check that the functions above satisfy the identities of the associated cluster transformations.

This is the first sequence of mutations where we have to mutate the black vertices x_{in} (y_n is also black, but it does not get mutated). Proving that the black vertices mutate as expected will be our most difficult task.

In verifying the cluster identities that we need, we will actually be computing functions on $\mathrm{Conf}_3 \mathcal{A}_{SL_N}$. Thus we will use the arrows with a circle on them as

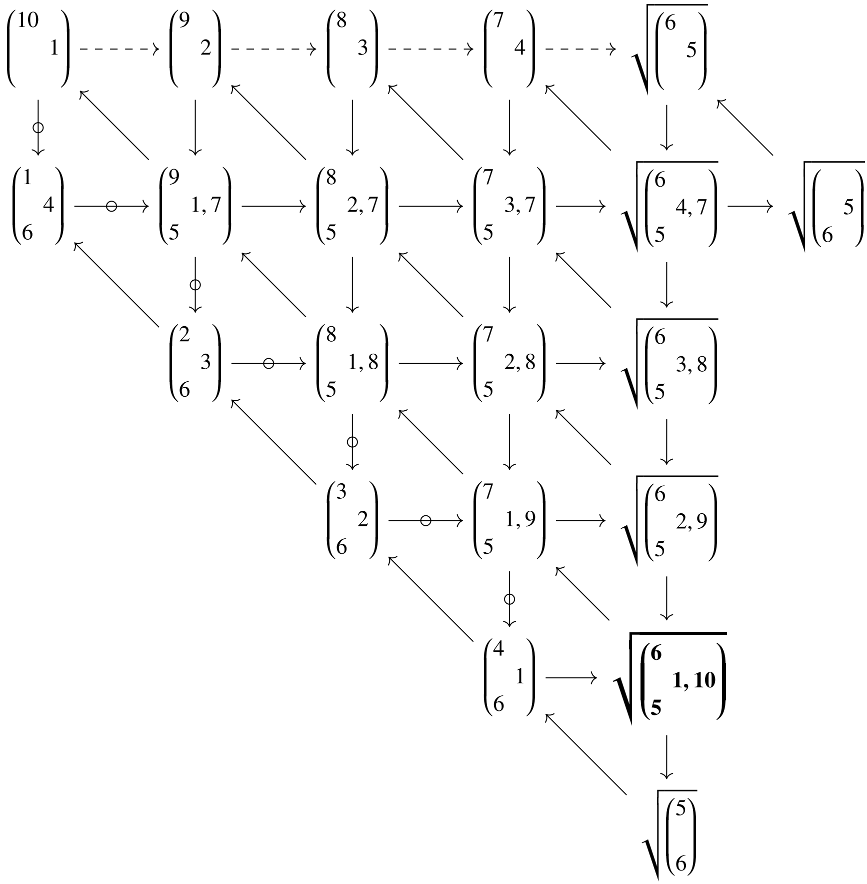

Figure 28c. The quiver and functions for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{11}}$ after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1, 4, 3, 2, 4, 3, 4.

a bookkeeping device. Circled arrows only occur between two white vertices. If a white vertex x has an incoming or outgoing circled arrow from another white vertex x', this means that in computing the mutation of x, we should use the dual of the function attached to x'. With this in mind, mutation of white vertices will be much like the case of Sp_{2n}.

The mutation of black vertices requires some more work. We will also make use of functions of the form

$$\begin{pmatrix} a,b & & \\ & & c,d \\ x,y & & \end{pmatrix}$$

where $x = n$ or $n+1$, $y = n$ or $n+1$, and $a+b+x+y+c+d = 3N$. These functions are defined just as in Figure 15.

The most general identity we will need to show has the form

$$\sqrt{\begin{pmatrix} n-i, n+1+i & \\ & n-j, n+1+j \\ n+1, n & \end{pmatrix}}\sqrt{\begin{pmatrix} n+1-i, n+i & \\ & n-1-j, n+2+j \\ n+1, n & \end{pmatrix}}$$
$$= \sqrt{\begin{pmatrix} n-i, n+1+i & \\ & n-1-j, n+2+j \\ n+1, n & \end{pmatrix}}\sqrt{\begin{pmatrix} n+1-i, n+i & \\ & n-j, n+1+j \\ n+1, n & \end{pmatrix}}$$
$$+ \begin{pmatrix} n+1-i, n+1+i & \\ & n-1-j, n+1-j \\ n+1, n & \end{pmatrix}.$$

Note that by duality (4.1),

$$\begin{pmatrix} n-i, n+i & \\ & n-j, n+2-j \\ n+1, n & \end{pmatrix} = \begin{pmatrix} n+1-i, n+1+i & \\ & n-1-j, n+1-j \\ n, n+1 & \end{pmatrix},$$

which explains the seeming asymmetry of the last term.

The general identity above follows directly from the following identities. Let $i, j < n$. Then

- $$\begin{pmatrix} n-i, n+1+i & \\ & n-j, n+1+j \\ n+1, n & \end{pmatrix}\begin{pmatrix} n+1-i, n+i & \\ & n-1-j, n+1+j \\ n+1, n+1 & \end{pmatrix}$$
$$= \begin{pmatrix} n+1-i, n+i & \\ & n-j, n+1+j \\ n+1, n & \end{pmatrix}\begin{pmatrix} n-i, n+1+i & \\ & n-1-j, n+1+j \\ n+1, n+1 & \end{pmatrix}$$
$$+ \begin{pmatrix} n+1-i, n+1+i & \\ & n-1-j, n+1-j \\ n+1, n & \end{pmatrix}\begin{pmatrix} n-i, n+i & \\ & n-j, n+1+j \\ n+1, n+1 & \end{pmatrix},$$

- $$\begin{pmatrix} a, N-a & \\ & b-1, N-b \\ n+1, n+1 & \end{pmatrix} = \sqrt{2\begin{pmatrix} a, N-a & \\ & b-1, N-b+1 \\ n+1, n & \end{pmatrix}\begin{pmatrix} a, N-a & \\ & b, N-b \\ n+1, n & \end{pmatrix}}.$$

The first identity is a relative of the octahedron recurrence, which we treated previously. □

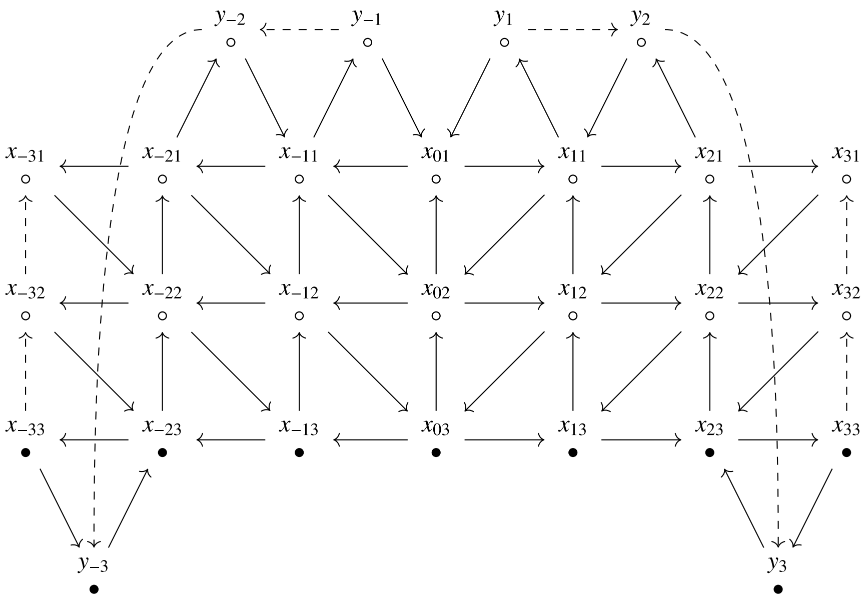

Figure 29. The quiver for the cluster algebra on $\mathrm{Conf}_4\,\mathcal{A}_{Spin_7}$. The associated functions are pictured in Figure 30a.

4.6. The sequence of mutations for a flip. In this section, we will give a sequence of mutations that relates two of the clusters coming from different triangulations of the 4-gon.

Given a configuration $(A, B, C, D) \in \mathrm{Conf}_4\,\mathcal{A}_{Spin_{2n+1}}$, we will give a sequence of mutations that relates a cluster coming from the triangulation ABC, ACD to a cluster coming from the triangulation ABD, BCD.

We will need to label the quiver for $\mathrm{Conf}_4\,\mathcal{A}_{Spin_{2n+1}}$ with vertices x_{ij}, y_k, with $-n \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and $1 \leqslant |k| \leqslant n$. The quiver we will start with is as in Figure 29, pictured for $Spin_7$.

We will describe the functions attached to these vertices in two steps. First we will assign some functions to each vertex. Then we will take the square root of the functions assigned to the black vertices.

Let $N = 2n + 1$. First make an assignment of functions to vertices as follows:

$$\begin{pmatrix} k & \\ & N-k \end{pmatrix} \longleftrightarrow y_k, \quad \text{for } k > 0;$$

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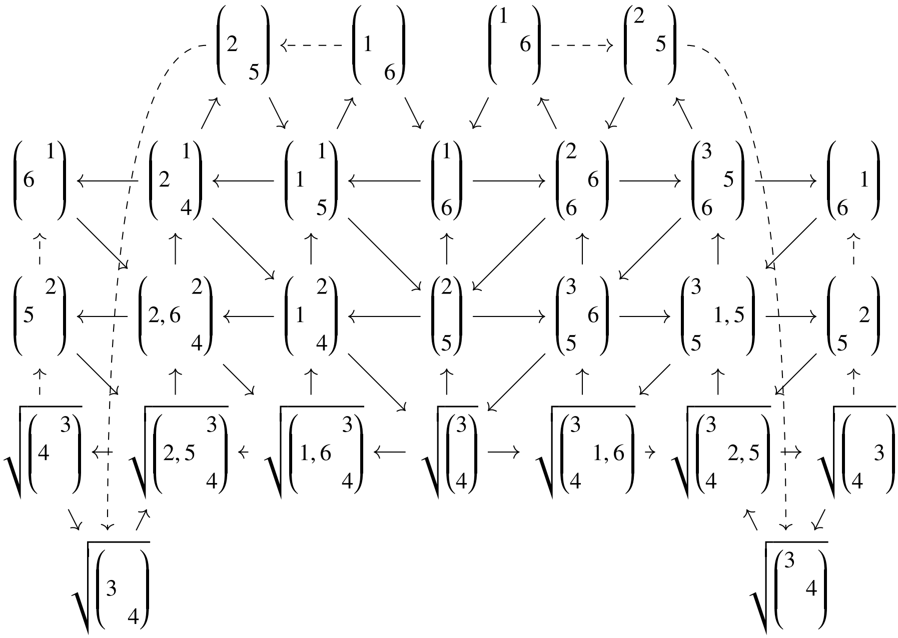

Figure 30a. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_7}$.

$$\begin{pmatrix} |k| & \\ & N-|k| \end{pmatrix} \longleftrightarrow y_k, \quad \text{for } k < 0;$$

$$\begin{pmatrix} & j \\ N-j & \\ & \end{pmatrix} \longleftrightarrow x_{-n,j};$$

$$\begin{pmatrix} & \\ & j \\ N-j & \end{pmatrix} \longleftrightarrow x_{nj};$$

$$\begin{pmatrix} j \\ \\ N-j \end{pmatrix} \longleftrightarrow x_{0j}.$$

The face functions in the triangle where $i > 0$ are

$$\begin{pmatrix} i+j & \\ & N-i \\ N-j & \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } 0 < i < n, i+j \leqslant n;$$

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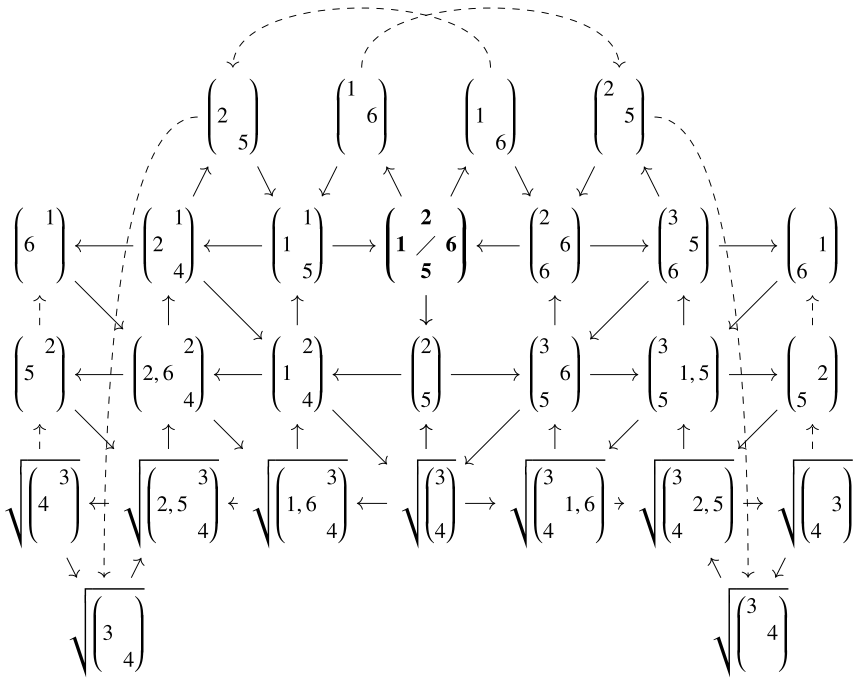

Figure 30b. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_7}$ after the first stage of mutation.

$$\begin{pmatrix} n & & \\ & j+i-n, N-i & \\ N-j & & \end{pmatrix} \longleftrightarrow x_{ij}, \qquad \text{for } 0 < i < n, i+j > n;$$

while the face functions in the triangle where $i < 0$ are

$$\begin{pmatrix} & j \\ |i| & \\ & N-|i|-j \end{pmatrix} \longleftrightarrow x_{ij}, \qquad \text{for } -n < i < 0, |i|+j \leqslant n;$$

$$\begin{pmatrix} & j \\ |i|, N+n-|i|-j & \\ & n+1 \end{pmatrix} \longleftrightarrow x_{ij}, \qquad \text{for } -n < i < 0, |i|+j > n.$$

REMARK 4.10. Note that our labeling of the vertices is somewhat different from before. The vertices labeled x_{ij} correspond to the vertices labeled $x_{n-|i|,j}$ in $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$.

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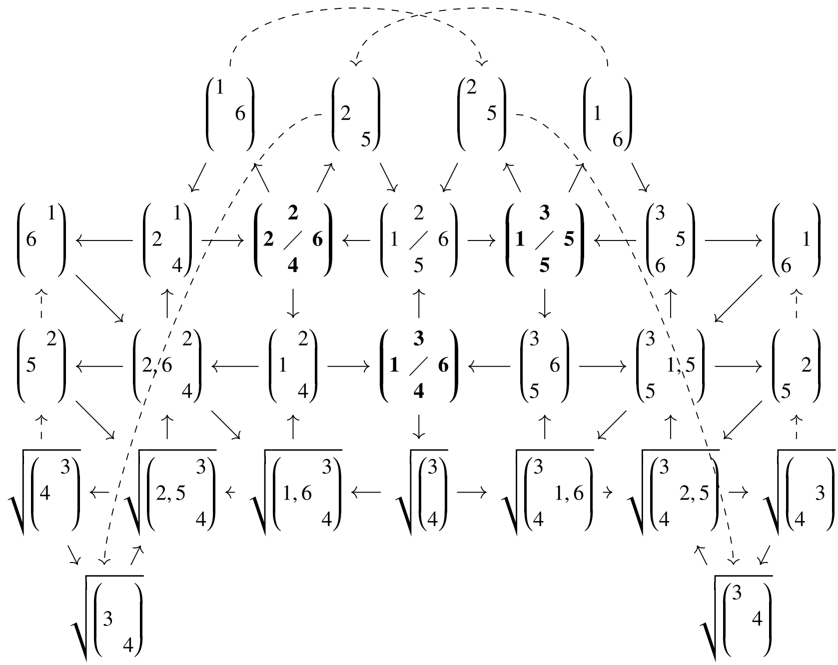

Figure 30c. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_7}$ after the second stage of mutation.

Now, take the square root of the functions assigned to x_{in}, y_n. For example, when $i > 0$ we assign

$$\sqrt{\begin{pmatrix} n & \\ & i, N-i \\ n+1 & \end{pmatrix}} \longleftrightarrow x_{in}$$

and for $i < 0$, we assign

$$\sqrt{\begin{pmatrix} & n \\ |i|, N-|i| & \\ & n+1 \end{pmatrix}} \longleftrightarrow x_{in}.$$

As in the case of $G = Sp_{2n}$, we will have to use some functions which depend on all four flags. Let $N = 2n+1$. Let $0 \leqslant a, b, c, d \leqslant N$ such that $a+b+c+d =$

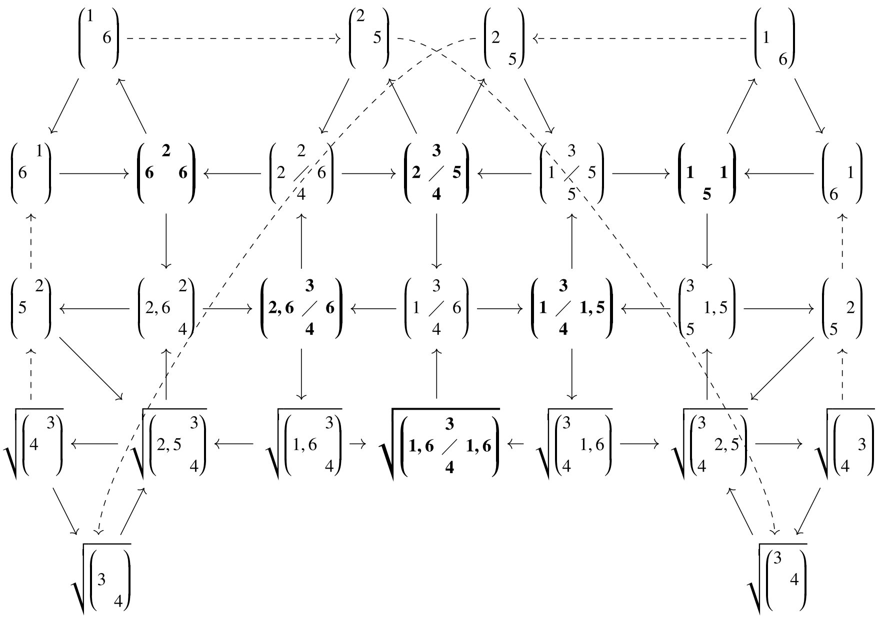

Figure 30d. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_7}$ after the third stage of mutation.

$4n+2 = 2N$ and $b+c \leqslant N$. Then as before we define the function

$$\begin{pmatrix} & a & \\ b & \diagup & d \\ & c & \end{pmatrix}$$

using the web in Figure 17.

Similarly, we use the notation

$$T^* \begin{pmatrix} & a & \\ b & \diagup & d \\ & c & \end{pmatrix} =: \begin{pmatrix} & b & \\ c & \diagdown & a \\ & d & \end{pmatrix}.$$

We now define a second type of function on $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+1}}$. If $a+b+c+d = 2N+n$, we define the function

$$\begin{pmatrix} & n+1 & \\ a & \diagup & c, d \\ & b & \end{pmatrix}.$$

It is given by the invariant picked out by a web similar to the one in Figure 18a.

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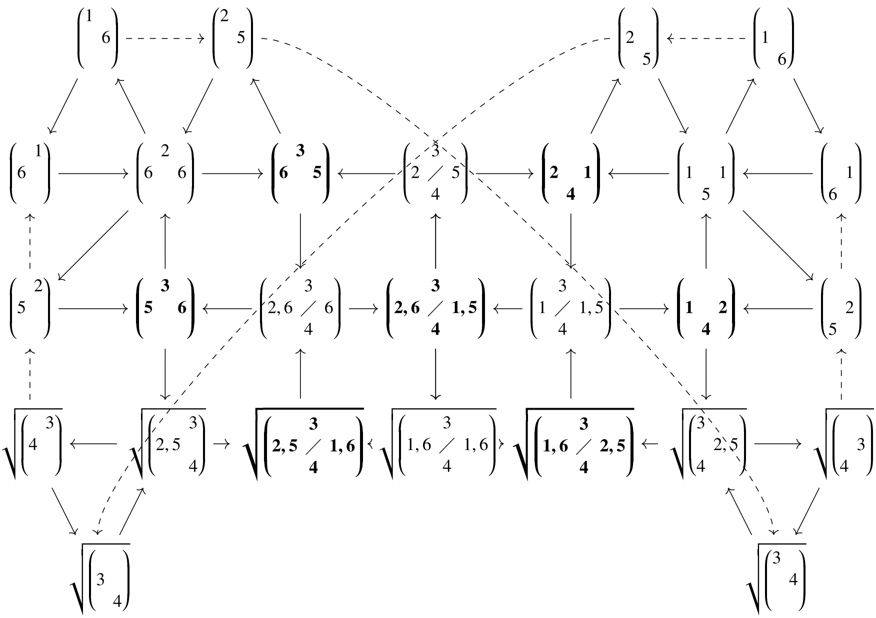

Figure 30e. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_7}$ after the fourth stage of mutation.

Using duality, there is also a function $\left(\begin{smallmatrix} & n & \\ a & \diagup & c,d \\ & b & \end{smallmatrix}\right)$ on $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+1}}$ for $0 \leqslant a, b, c, d \leqslant N$, $a+b+c+d = 3n+2 = N+n+1$, and $c \leqslant d$. This function is defined using the same web as in Figure 18b.

Using the twist map T, we can also define the functions $\left(\begin{smallmatrix} & a & \\ b & \diagdown & n \\ & c,d & \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & b & \\ c,d & \diagup & a \\ & n & \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} & c,d & \\ n & \diagdown & b \\ & a & \end{smallmatrix}\right)$.

We will need to define one more type of function to do our calculations. Let $0 \leqslant a, b, c, d \leqslant N$ such that $a+b+c+d = 4n+2 = 2N$, $a \leqslant n \leqslant b$ and $c \leqslant n \leqslant d$. Then we would like to define a function that we will call

$$\begin{pmatrix} & n & \\ a, b & \diagup & c, d \\ & n+1 & \end{pmatrix}.$$

We will make less frequent use of the functions

$$\begin{pmatrix} & n+1 & \\ a, b & \diagup & c, d \\ & n & \end{pmatrix}, \begin{pmatrix} & n & \\ a, b & \diagup & c, d \\ & n & \end{pmatrix}, \begin{pmatrix} & n+1 & \\ a, b & \diagup & c, d \\ & n+1 & \end{pmatrix}.$$

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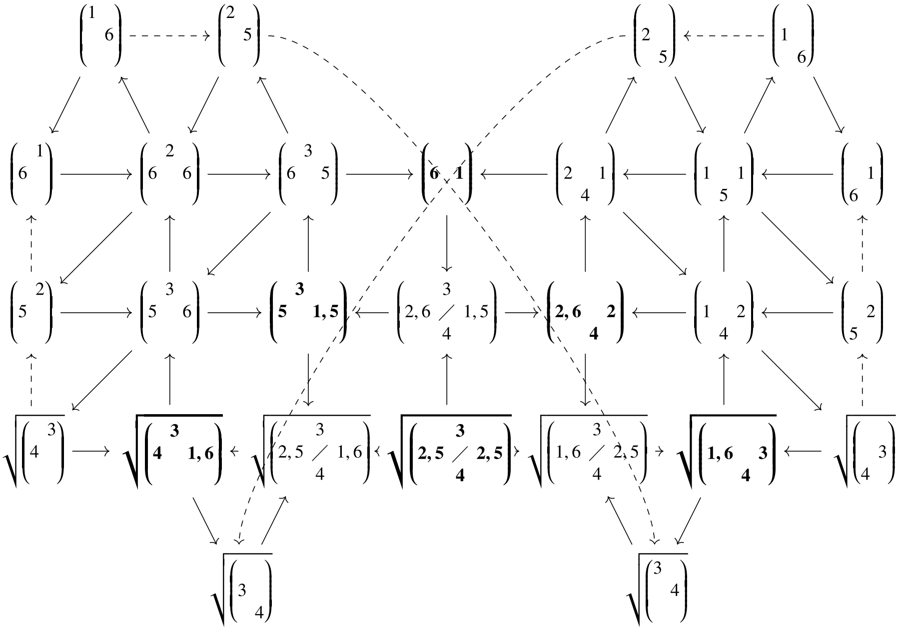

Figure 30f. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4\,\mathcal{A}_{Spin_7}$ after the fifth stage of mutation.

Here $a+b+c+d=2N$, $2N+1$ or $2N-1$, respectively. These functions are defined by webs similar to the one in Figure 19.

Note that

$$\left(a,b \quad \overset{n+1}{\underset{n}{\diagup}} \quad c,d\right) = (T^2)^*\left(c,d \quad \overset{n}{\underset{n+1}{\diagup}} \quad a,b\right).$$

Finally, note that if $a+b=c+d=N$, then

$$\sqrt{\left(a,b \quad \overset{n}{\underset{n+1}{\diagup}} \quad c,d\right)}$$

is a well-defined function on $\mathrm{Conf}_4\,\mathcal{A}_{Spin_{2n+1}}$. This is because the representations V_{ω_a} and V_{ω_b} of SL_N give the same representations of $Spin_N$, and V_{ω_n} and V_{ω_n+1}, as representations of $Spin_N$, have twice the weight of the spin representation.

Now we give the sequence of mutations realizing the flip of a triangulation. The sequence of mutations is exactly as in the case for Sp_{2n}. The sequence of mutations leaves $x_{-n,j}$, x_{nj}, y_k untouched as they are frozen variables. Hence we

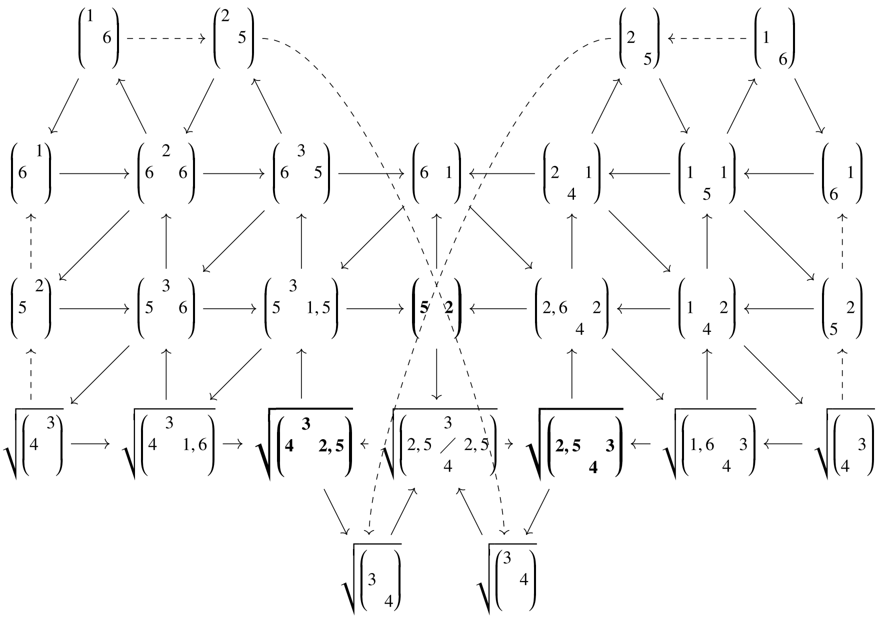

Figure 30g. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_7}$ after the sixth stage of mutation.

only mutate x_{ij} for $-n \leqslant i \leqslant n$. We now describe the sequence of mutations. The sequence of mutations will have $3n-2$ stages. At the rth step, we mutate all vertices such that

$$|i| + j \leqslant r,$$
$$j - |i| + 2n - 2 \geqslant r,$$
$$|i| + j \equiv r \mod 2.$$

Note that the first inequality is empty for $r \geqslant 2n-1$, while the second inequality is empty for $r \leqslant n$. For example, for $Spin_7$, the sequence of mutations is

$$\begin{gathered} x_{01}, \\ x_{-1,1}, x_{02}, x_{11}, \\ x_{-2,1}, x_{-1,2}, x_{01}, x_{03}, x_{12}, x_{21}, \\ x_{-2,2}, x_{-1,1}, x_{-1,3}, x_{02}, x_{11}, x_{13}, x_{22}, \\ x_{-2,3}, x_{-1,2}, x_{01}, x_{03}, x_{12}, x_{23}, \\ x_{-1,3}, x_{02}, x_{13}, \\ x_{03}. \end{gathered} \tag{4.4}$$

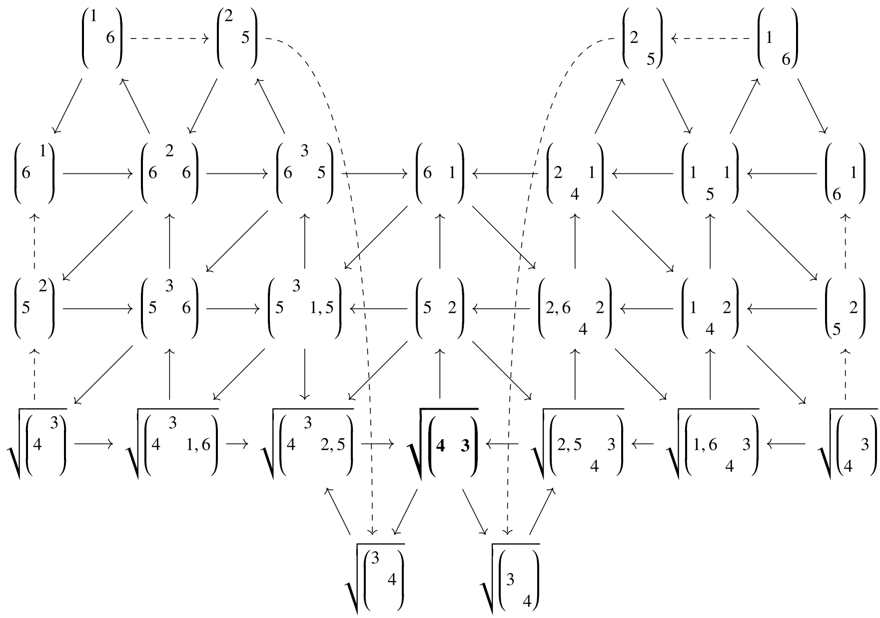

Figure 30h. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4\,\mathcal{A}_{Spin_7}$ after the seventh and last stage of mutation.

See Section 3.5.1 for the motivation behind this sequence. In Figure 30, we depict how the quiver for $\mathrm{Conf}_4\,\mathcal{A}_{Spin_7}$ changes after each of the seven stages of mutation.

The analogue for $\mathrm{Conf}_4\,\mathcal{A}_{Spin_{2n+1}}$ should be clear.

We now have the main theorem of this section:

THEOREM 4.11. *The sequence of mutations for a flip on* $\mathrm{Conf}_4\,\mathcal{A}_{Spin_{2n+1}}$ *yield a sequence of functions as depicted in Figure* 30.

Proof. The proofs are much like before. The main novelty occurs when $j = n$. Here we are mutating black vertices. Here we will need to derive some new identities. The general mutation identity when $j = n$ has the following form:

$$\sqrt{\begin{pmatrix} & n & \\ a, N-a & \diagup & b, N-b \\ & n+1 & \end{pmatrix}}\sqrt{\begin{pmatrix} & n & \\ a+1, N-a-1 & \diagup & b+1, N-b-1 \\ & n+1 & \end{pmatrix}}$$

$$= \sqrt{\left(a+1, N-a-1 \underset{n+1}{\overset{n}{\diagup}} b, N-b\right)}\sqrt{\left(a, N-a \underset{n+1}{\overset{n}{\diagup}} b+1, N-b-1\right)}$$
$$+ \left(a+1, N-a \underset{n+1}{\overset{n}{\diagup}} b, N-b-1\right).$$

The above identity in turn follows from the following identities:

-
$$\left(a, N-a \underset{n+1}{\overset{n}{\diagup}} b, N-b\right)\left(a+1, N-a \underset{n}{\overset{n}{\diagup}} b+1, N-b-1\right)$$
$$= \left(a+1, N-a \underset{n}{\overset{n}{\diagup}} b, N-b\right)\left(a, N-a \underset{n+1}{\overset{n}{\diagup}} b+1, N-b-1\right)$$
$$+ \left(a+1, N-a \underset{n+1}{\overset{n}{\diagup}} b, N-b-1\right)\left(a, N-a \underset{n}{\overset{n}{\diagup}} b+1, N-b\right).$$

-
$$\left(a+1, N-a \underset{n}{\overset{n}{\diagup}} b+1, N-b-1\right)$$
$$= \sqrt{2\left(a, N-a \underset{n+1}{\overset{n}{\diagup}} b+1, N-b-1\right)\left(a+1, N-a-1 \underset{n+1}{\overset{n}{\diagup}} b+1, N-b-1\right)}.$$

-
$$\left(a+1, N-a \underset{n}{\overset{n}{\diagup}} b, N-b\right)$$
$$= \sqrt{2\left(a, N-a \underset{n+1}{\overset{n}{\diagup}} b, N-b\right)\left(a+1, N-a-1 \underset{n+1}{\overset{n}{\diagup}} b, N-b\right)}.$$

-
$$\left(a, N-a \underset{n}{\overset{n}{\diagup}} b+1, N-b\right)$$

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$$= \sqrt{2\left(a, N-a \underset{n+1}{\overset{n}{\diagup}} b+1, N-b-1\right)\left(a, N-a \underset{n+1}{\overset{n}{\diagup}} b, N-b\right)}.$$

□

5. The cluster algebra structure on $\mathrm{Conf}_m\, G/U$ for $G = Spin_{2n}$

We now define the cluster algebra structure on $\mathrm{Conf}_m\, G/U$ when $G = Spin_{2n}$. In fact, to emphasize the parallels with the case of $Spin_{2n+1}$, we will let $G = Spin_{2n+2}$. When $G = Spin_{2n+2}$, the cluster algebra structure, along with the mutations realizing S_3 symmetries and the flip of triangulation, will be an unfolding of the same structures for $G = Spin_{2n+1}$.

We will utilize what we understand about functions on $\mathrm{Conf}_m\, \mathcal{A}_{SL_{2n+2}}$ in order to study $\mathrm{Conf}_m\, \mathcal{A}_{Spin_{2n+2}}$. However, because the Dynkin diagram of $Spin_{2n+2}$ is not obtained from SL_{2n+2} by folding, as was the case for Sp_{2n} and $Spin_{2n+1}$, there will be additional complications, particularly regarding signs.

Recall that $Spin_{2n+2}$ is the double cover of the group SO_{2n+2}, which is the subgroup of SL_{2n+2} preserving a symmetric quadratic form. We take the quadratic form given in the standard basis $e_1, \ldots, e_{2n+2}$ by

$$\langle e_i, e_{2n+3-i}\rangle = (-1)^{i-1}$$

for $1 \leqslant i \leqslant n+1$, and $\langle e_i, e_j\rangle = 0$ otherwise.

REMARK 5.1. Note that the signature of the quadratic form is $(n+1, n+1)$, so that taking real points gives the split real form of SO_{2n+2}. The cluster algebra structure on $\mathrm{Conf}_m\, \mathcal{A}_{Spin_{2n+2}}$ gives another way of defining the positive structure on $\mathcal{A}_{Spin_{2n+2},S}$, which gives a parameterization of the Hitchin component for the group $Spin_{2n+2}$ and the surface S.

The maps

$$Spin_{2n+2} \twoheadrightarrow SO_{2n+2} \hookrightarrow SL_{2n+2}$$

induce maps

$$\mathrm{Conf}_m\, \mathcal{A}_{Spin_{2n+2}} \to \mathrm{Conf}_m\, \mathcal{A}_{SO_{2n+2}} \to \mathrm{Conf}_m\, \mathcal{A}_{SL_{2n+2}}.$$

Let us describe these maps concretely. The variety $\mathcal{A}_{SO_{2n+2}}$ parameterizes chains of isotropic vector spaces

$$V_1 \subset V_2 \subset \cdots \subset V_{n+1} \subset V$$

inside the $2n+2$-dimensional standard representation V, where $\dim V_i = i$, and where each V_i is equipped with a volume form.

Equivalently, a point of $\mathcal{A}_{SO_{2n+2}}$ is given by a sequence of vectors

$$v_1, v_2, \ldots, v_{n+1},$$

where

$$V_i := \langle v_1, \ldots, v_i \rangle$$

is isotropic, and where v_i is only determined up to adding linear combinations of v_j for $j < i$.

The volume form on V_i is then $v_1 \wedge \cdots \wedge v_i$.

From the sequence of vectors $v_1, \ldots, v_{n+1}$, we can complete to a basis $v_1, v_2, \ldots, v_{2n+2}$, where $\langle v_i, v_{2n+3-i} \rangle = (-1)^{i-1}$, and $\langle v_i, v_j \rangle = 0$ otherwise. Equivalently, the quadratic form induces an isomorphism $\langle -, - \rangle : V \to V^*$. At the same time, there are perfect pairings

$$\bigwedge^k V \times \bigwedge^k V^* \to F$$

$$\bigwedge^{2n+2-k} V \times \bigwedge^k V \to F$$

that induce an isomorphism

$$\bigwedge^{2n+2-k} V \simeq \bigwedge^k V^*.$$

Composing this with the inverse of the isomorphism

$$\langle -, - \rangle : \bigwedge^k V \to \bigwedge^k V^*$$

gives an isomorphism

$$\bigwedge^{2n+2-k} V \simeq \bigwedge^k V^* \simeq \bigwedge^k V.$$

Then $v_{n+2}, \ldots, v_{2n+2}$ are chosen so that this isomorphism takes $v_1 \wedge \cdots \wedge v_k$ to $v_1 \wedge \cdots \wedge v_{2n+2-k}$ for $i \leqslant n+1$.

Then $v_1, v_2, \ldots, v_{2n+2}$ determines a point of $\mathcal{A}_{SL_{2n+2}}$, as $\mathcal{A}_{SL_{2n+2}}$ parameterizes chains of vector subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_{2n+2} = V$$

along with volume forms $v_1 \wedge \cdots \wedge v_i$, $1 \leqslant i \leqslant 2n+1$.

From the embedding

$$\mathcal{A}_{SO_{2n+2}} \hookrightarrow \mathcal{A}_{SL_{2n+2}},$$

one naturally gets an embedding $\operatorname{Conf}_m \mathcal{A}_{SO_{2n+2}} \hookrightarrow \operatorname{Conf}_m \mathcal{A}_{SL_{2n+2}}$. We can then pull back functions from $\operatorname{Conf}_m \mathcal{A}_{SL_{2n+2}}$ to get functions on $\operatorname{Conf}_m \mathcal{A}_{SO_{2n+2}}$. However, we are ultimately interested in functions on $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+2}}$.

The functions on $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+2}}$ that we will use to define the cluster structure on $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+2}}$ will be invariants of tensor products of representations of $Spin_{2n+2}$. For $m = 3$, they will lie inside

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^G$$

where λ, μ, ν are elements of the dominant cone inside the weight lattice. In general, not all such functions will come from pulling back functions on $\operatorname{Conf}_m \mathcal{A}_{SL_{2n+2}}$.

However, suppose that

$$f \in [V_\lambda \otimes V_\mu \otimes V_\nu]^G \subset \mathcal{O}(\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+2}}).$$

Then

$$f^2 \in [V_{2\lambda} \otimes V_{2\mu} \otimes V_{2\nu}]^G \subset \mathcal{O}(\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+2}}).$$

However, because $2\lambda, 2\mu, 2\nu$ are dominant weights for SO_{2n+2}, f^2 may be viewed as a function on $\operatorname{Conf}_m \mathcal{A}_{SO_{2n+2}}$. This function is then a pull-back of a function on $\operatorname{Conf}_m \mathcal{A}_{SL_{2n+2}}$. Therefore functions on $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+2}}$ which are tensor invariants are either the pull-backs of functions on $\operatorname{Conf}_m \mathcal{A}_{SL_{2n+2}}$ or square roots of such functions. The square root here corresponds to the fact that $Spin_{2n+2}$ is a double cover of SO_{2n+2}. The choice of the branch of the square root that we take is determined by the positive structure on $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+2}}$: if f is a positive function on $\operatorname{Conf}_m \mathcal{A}_{SL_{2n+2}}$ such that its square root is a function on $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+2}}$, there is a unique choice of $\sqrt{f}$ that is positive on $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+2}}$. That is the square root that we will always take. We discuss this issue further in the section on signs.

It will often be convenient to write down functions on $\operatorname{Conf}_m \mathcal{A}_{Spin_{2n+2}}$ in a slightly different way. Recall that $Spin_{2n+2}$ is associated to the Dynkin diagram D_{n+1}:

This diagram has an order two automorphism that gives an outer automorphism of $Spin_{2n+2}$ of order two. Under this map, the flag given by the sequence of vectors $v_1, v_2, \ldots, v_{2n+2}$ gets sent to the flag given by the sequence of vectors

$$v_1, v_2, \ldots, v_n, v_{n+2}, v_{n+1}, v_{n+3}, \ldots, v_{2n+2}.$$

In other words, the vectors v_{n+1} and v_{n+2} switch places.

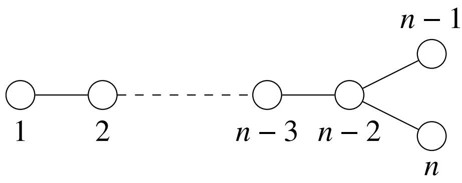

Figure 31. D_n Dynkin diagram.

The existence of this automorphism of $Spin_{2n+2}$ means that there is a second map

$$\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+2}} \to \mathrm{Conf}_m \mathcal{A}_{SL_{2n+2}}.$$

We can also pull back functions from $\mathrm{Conf}_m \mathcal{A}_{SL_{2n+2}}$ under this second map. We do not get any new functions in this way, but we will often get simpler expressions for our functions in this way. Thus the functions we consider will involve the volume forms $v_1 \wedge \cdots \wedge v_i$ as well as the volume form

$$v_1 \wedge \cdots \wedge v_n \wedge v_{n+2}.$$

5.1. Construction of the seed. We are now ready to construct the seed for the cluster structure on $\mathrm{Conf}_m \mathcal{A}$ when $G = Spin_{2n+2}$. Throughout this section, $G = Spin_{2n+2}$ unless otherwise noted.

The nodes of the diagram correspond to $n+1$ roots that all have the same length. To describe the cluster structure on $\mathrm{Conf}_3 \mathcal{A}$, we need to give the following data: the set I parameterizing vertices, the functions on $\mathrm{Conf}_3 \mathcal{A}$ corresponding to each vertex, and the B-matrix for this seed.

The B-matrix is encoded via a quiver which consists of $n^2 + 3n + 2$ vertices, all of which are black. There are $n + 1$ edge functions for each edge of the triangle, and $n^2 - 1$ face functions.

In Figure 32, we see the quiver for $Spin_8$. The generalization for other values of n should be clear.

The diagram is busy because the vertices x_{i3} and y_3 are doubled by the vertices x_{i3^*} and y_{3^*}.

For simplicity, all future diagrams will only contain the vertices x_{in} and y_n, and not the vertices x_{in^*} and y_{n^*}, which merely double them.

Label the vertices of the quiver x_{ij} and y_k, where $0 \leqslant i \leqslant n$, $j = 1, 2, 3, \ldots, n, n^*$, $k = 1, 2, 3, \ldots, n, n^*$. The vertices y_k and x_{ij} for $i = 0$ or n are frozen. We will sometimes write $x_{i,j}$ for x_{ij} for orthographic reasons.

Let us now recall some facts about the representation theory of $Spin_{2n+2}$. The fundamental representations of $Spin_{2n+2}$ are labeled by the fundamental weights

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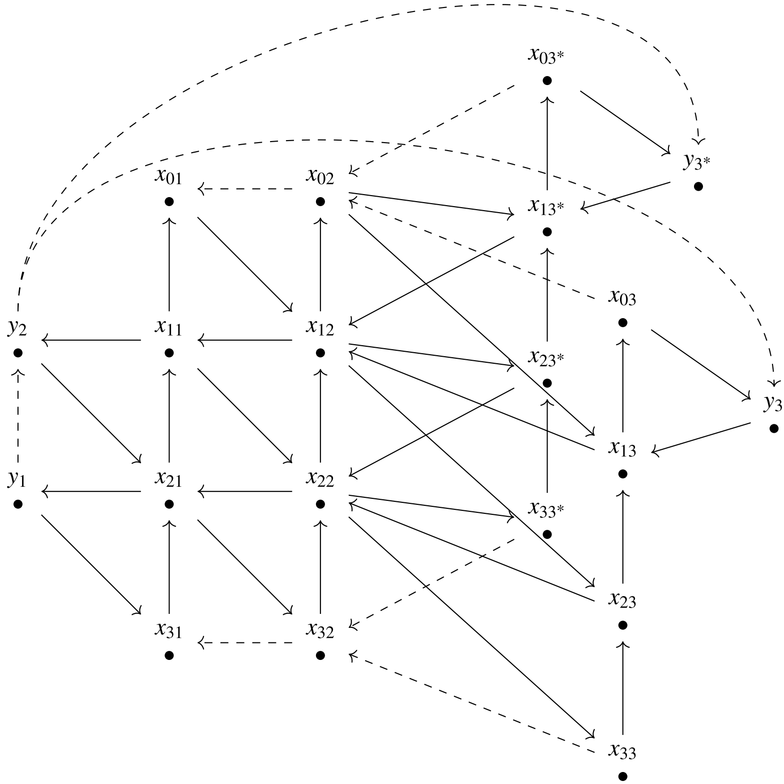

Figure 32. Quiver encoding the cluster structure for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_8}$.

$\omega_1, \ldots, \omega_n, \omega_{n^*}$. $Spin_{2n+2}$ has a standard $2n+1$-dimensional representation V. Let $\langle -, - \rangle$ be the orthogonal pairing. Then for $i < n$ the representation V_{ω_i} corresponding to ω_i is precisely $\bigwedge^i V$. The representations V_{ω_n}, $V_{\omega_{n^*}}$ are the *spin representations* of $Spin_{2n+2}$. When n is even, the spin representations are dual to each other. When n is odd, the spin representations are self-dual. The direct sum of the representations $V_{2\omega_{n^*}}$ and $V_{2\omega_n}$ is isomorphic to $\bigwedge^{n+1} V$, and $V_{\omega_n+\omega_{n^*}}$ is isomorphic to $\bigwedge^n V$.

We now say which functions are attached to the vertices of the quiver. Recall the functions defined via the webs from Figures 1, 2, and 6. It will be convenient to describe the functions attached to x_{ij} for $j \leqslant n-1$ and y_k for $k \leqslant n-1$ first:

(1) For $k \leqslant n-1$, assign the function $\binom{k}{2n+2-k}(-1)^k = \binom{2n+2-k}{k}$ to y_k.

(2) When $i \geqslant j$, assign the function $\binom{\quad n-i \quad}{n+2+i-j \quad j}(-1)^{n-i}$ to x_{ij}.

(3) When $i < j$ and $i \neq 0$, we assign the function $\binom{n-i, 2n+2+i-j \quad\quad j}{n+2}(-1)^{n-i}$ to x_{ij}.

(4) When $i = 0$, we assign the function $\binom{2n+2-j \quad j}{}$ to x_{ij}.

Now, when j or k are equal to n or n^*, the story is somewhat more complicated. As in the case of $Spin_{2n+1}$, the functions involve square roots. But then there are two additional complications: first, we will need to slightly modify some of the functions which we previously defined in order to deal with the fact that there are two spin representations of $Spin_{2n+2}$; and second, the somewhat different behavior of these spin representations for n odd and n even means we will need to treat these cases separately.

Recall that in Figure 3, we defined functions of the form $\binom{a,b \quad d}{c}$. We will now need to define some new functions of the form $\binom{a,b \quad n+1*}{c}$, where $a+b+c = 3n+3$. This is a function on the space of configurations of three principal flags for the group $Spin_{2n+2}$. Suppose these flags are given in terms of the three flags

$$u_1, \ldots, u_N;$$

$$v_1, \ldots, v_N;$$

$$w_1, \ldots, w_N;$$

where $N = 2n + 2$.

Now consider the forms

$$U_a := u_1 \wedge \cdots \wedge u_a,$$

$$U_b := u_1 \wedge \cdots \wedge u_b,$$

$$V_c := v_1 \wedge \cdots \wedge v_c,$$

$$W_{n+1*} := w_1 \wedge \cdots \wedge w_n \wedge w_{n+2}.$$

The function $\binom{a,b \quad n+1^*}{c}$ is defined in the same way as the function $\binom{a,b \quad n+1}{c}$, except everywhere where one had $W_{n+1} := w_1 \wedge \cdots \wedge w_n \wedge w_{n+1}$, one replaces this by $W_{n+1^*} := w_1 \wedge \cdots \wedge w_n \wedge w_{n+2}$.

Recall that there is an outer automorphism of the group $Spin_{2n+2}$. On the level of flags, this automorphism takes the $n + 1$-form W_{n+1} to the $n + 1$-form W_{n+1*}. All the new functions we will need to define the cluster algebra structure on $\mathrm{Conf}_3\, \mathcal{A}_{Spin_N}$ will involve taking a previously defined function and substituting $n+1*$ for $n+1$ for some subset of the arguments. For example, it is straightforward to define the functions $\sqrt{-\begin{pmatrix} n+1^* \\ n+1 \end{pmatrix}}$ and $\sqrt{\begin{pmatrix} n-i,n+2+i & & \\ & & n+1^* \\ & n+1 & \end{pmatrix}}$, which we will use below.

We will distinguish two cases: n even or n odd. First suppose n is even. Then we assign functions as follows:

(1) Assign the function $\sqrt{-\begin{pmatrix} n+1^* \\ n+1 \end{pmatrix}}$ to y_n and $\sqrt{\begin{pmatrix} n+1 \\ n+1^* \end{pmatrix}}$ to y_{n^*}.

(2) Assign the function $\sqrt{-\begin{pmatrix} & n+1 \\ n+1^* & \end{pmatrix}}$ to x_{nn} and $\sqrt{\begin{pmatrix} & n+1^* \\ n+1 & \end{pmatrix}}$ to x_{nn^*}.

(3) When $0 < i < n$, i odd, we assign the function $\sqrt{-\begin{pmatrix} n-i,n+2+i & & \\ & & n+1 \\ & n+1 & \end{pmatrix}}$ to x_{in} and $\sqrt{\begin{pmatrix} n-i,n+2+i & & \\ & & n+1^* \\ & n+1^* & \end{pmatrix}}$ to x_{in^*}.

(4) When $0 < i < n$, i even, we assign the function $\sqrt{-\begin{pmatrix} n-i,n+2+i & & \\ & & n+1 \\ & n+1^* & \end{pmatrix}}$ to x_{in} and $\sqrt{\begin{pmatrix} n-i,n+2+i & & \\ & & n+1^* \\ & n+1 & \end{pmatrix}}$ to x_{in^*}.

(5) Assign the function $\sqrt{-\begin{pmatrix} n+1^* & \\ & n+1 \end{pmatrix}}$ to x_{0n} and $\sqrt{\begin{pmatrix} n+1 & \\ & n+1^* \end{pmatrix}}$ to x_{0n^*}.

When n is odd, we assign functions as follows:

(1) Assign the function $\sqrt{\begin{pmatrix} n+1 \\ n+1 \end{pmatrix}}$ to y_n and $\sqrt{-\begin{pmatrix} n+1^* \\ n+1^* \end{pmatrix}}$ to y_{n^*}.

(2) Assign the function $\sqrt{\begin{pmatrix} & n+1 \\ n+1 & \end{pmatrix}}$ to x_{nn} and $\sqrt{-\begin{pmatrix} & n+1^* \\ n+1^* & \end{pmatrix}}$ to x_{nn^*}.

(3) When $0 < i < n$, i odd, we assign the function $\sqrt{\begin{pmatrix} n-i,n+2+i & & \\ & & n+1 \\ & n+1^* & \end{pmatrix}}$ to x_{in} and $\sqrt{-\begin{pmatrix} n-i,n+2+i & & \\ & & n+1^* \\ & n+1 & \end{pmatrix}}$ to x_{in^*}.

(4) When $0 < i < n$, i even, we assign the function $\sqrt{\begin{pmatrix} n-i,n+2+i & & \\ & & n+1 \\ & n+1 & \end{pmatrix}}$ to x_{in} and $\sqrt{-\begin{pmatrix} n-i,n+2+i & & \\ & & n+1^* \\ & n+1^* & \end{pmatrix}}$ to x_{in^*}.

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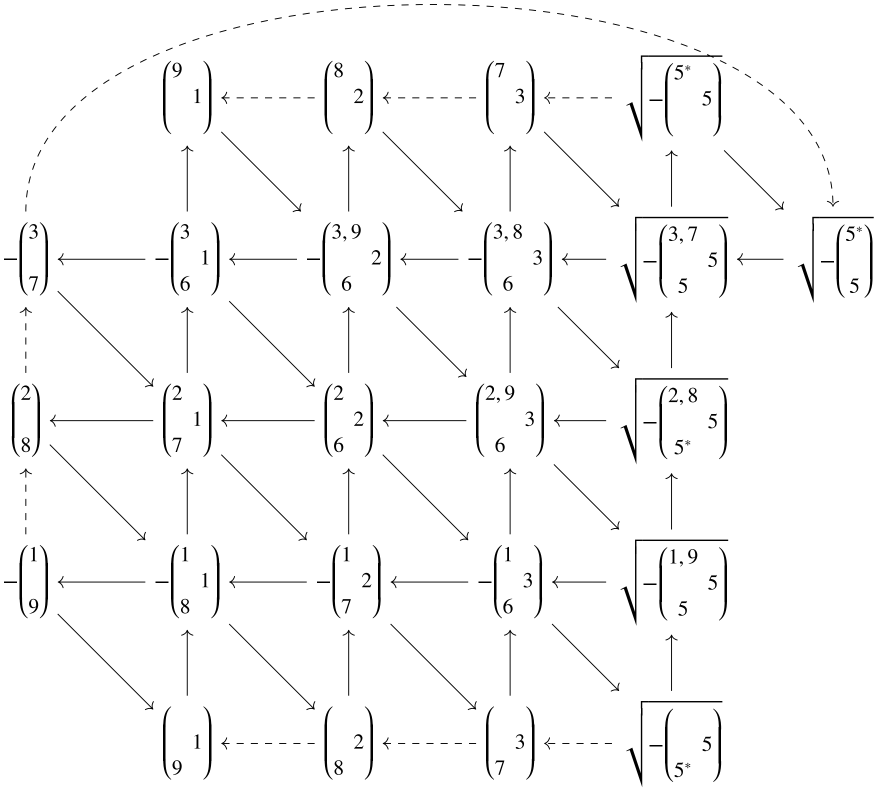

Figure 33. A cluster structure for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{10}}$.

(5) Assign the function $\sqrt{\begin{pmatrix} n+1 & \\ & n+1 \end{pmatrix}}$ to x_{0n} and $\sqrt{-\begin{pmatrix} n+1^* & \\ & n+1^* \end{pmatrix}}$ to x_{0n^*}.

This completely describes the cluster structure on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+2}}$. The fact that we can take the square roots of the functions assigned to x_{in} and y_n and get functions that are well-defined on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+2}}$ follows from the computations with reduced words that we perform later. Note that the cluster structure is not symmetric with respect to the three flags. Performing various S_3 symmetries, we obtain six different possible cluster structures on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+2}}$. These six structures are related by sequences of mutations that we describe in the next section. Below, in Figure 33, we depict the standard cluster described above.

Unlike the cases of $G = Sp_{2n}$ and $G = Spin_{2n+1}$, the functions in a cluster coming from an S_3 symmetry do not come from permuting the arguments in our notation for the function. Permuting the arguments only gives the correct functions up to a sign. This can be seen in the examples above. This is important enough that we will discuss this separately in the next section.

5.2. Signs and spin representations. In this section, we discuss in detail the signs involved in defining the cluster variables for $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+2}}$. This is a somewhat delicate issue, and can certainly be ignored on a first reading. For the remainder of the paper, we would like to ignore sign issues, on the one hand, for the sake of simplicity, and on the other hand, to avoid having to treat the different cases that, as our calculations become more complex, ultimately will depend on the value of $n \bmod 4$. Instead of keeping track of all these signs, we will give a framework for computing them.

In defining the various functions we have used, like $\begin{pmatrix} a,b & \\ & d \\ c & \end{pmatrix}$, we have been careful to define them so that they would be positive functions on $\mathrm{Conf}_m \mathcal{A}_{SL_n}$, $\mathrm{Conf}_m \mathcal{A}_{Sp_{2n}}$, and $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+1}}$. Unfortunately, these functions are sometimes positive and sometimes negative on $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+2}}$. Moreover, as we saw above, the sign of these functions also depends on the parity n. As we will see, if we additionally look at the rotations of these functions, the signs will depend on $n \bmod 4$. The divergence of all these cases is reflection of Bott periodicity. We will attempt to clarify the situation by isolating the various difficulties and dealing with them separately.

There are two main difficulties. The first source of complication is that the functions we have defined are well adapted to computations in SL_N. In types B and C, the maps $Sp_{2n} \hookrightarrow SL_{2n}$ and $Spin_{2n+1} \to SL_{2n+1}$ come from folding of the Dynkin diagrams, and therefore preserve positive structures. However, in type D, this is not the case. The map $Spin_{2n+2} \to SL_{2n+2}$ does not preserve positive structures, that is, the pull-back of a positive function on $\mathrm{Conf}_m \mathcal{A}_{SL_{2n+2}}$ is not necessarily positive on $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+2}}$.

The second issue is the twisted cyclic shift map, which behaves differently for SL_{2n+2} and $Spin_{2n+2}$. In the following, we will usually let $N = 2n + 2$, but we will also allow N to be odd for sake of comparison, and to also emphasize the relationship with Bott periodicity.

Suppose we have the functions for a given cluster on $\mathrm{Conf}_3 \mathcal{A}_{Spin_N}$. If we want to give the functions in the cluster coming from a rotation of the original cluster, we must pull back by the twisted cyclic shift map for $Spin_N$. However, the elements s_G for $G = Spin_N$ and $G = SL_N$ are not necessarily the same. In fact, $s_{Spin_N} \in Spin_N$ is the nontrivial lift of the identity in SO_N when

$$N \equiv 3, 4, 5, 6 \mod 8 \tag{5.1}$$

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and s_{Spin_N} is the identity when

$$N \equiv 0, 1, 2, 7 \mod 8. \tag{5.2}$$

Here is one way of determining when s_{Spin_N} is the identity. Let ρ be the half-sum of the positive roots, as usual. In the case that $N = 2n+2$, ω_n and ω_{n^*} are the highest weights of the spin representations, and s_{Spin_N} is the identity when $\langle \omega_n, 2\rho \rangle = \langle \omega_{n^*}, 2\rho \rangle$ is even and the nontrivial lift of the identity when $\langle \omega_n, 2\rho \rangle$ is odd. In the case that $N = 2n + 1$, ω_n is the highest weight of the spin representation, and s_{Spin_N} is the identity when $\langle \omega_n, 2\rho \rangle$ is even and the nontrivial lift of the identity when $\langle \omega_n, 2\rho \langle$ is odd.

On the other hand, $s_{SL_N} \in SL_N$ is negative of the identity element when N is even and the identity element when N is odd.

(Under the embedding $Sp_{2n} \hookrightarrow SL_{2n}$, $s_{Sp_{2n}}$ is sent to $s_{SL_{2n}}$. Under the map $Spin_{2n+1} \to SL_{2n+1}$, the element $s_{Spin_{2n+1}}$ is sent to $s_{SL_{2n+1}}$. Note that in the latter case, $s_{Spin_{2n+1}}$ is some lift of the identity in SO_{2n+1}, while $s_{SL_{2n+1}}$ is the identity in SL_{2n+1}.)

Let us compare what the elements s_{Spin_N} and s_{SL_N} do on the level of flags. Let $N = 2n + 2$ be even. Suppose we have a principal affine flag $A \in \mathcal{A}_{Spin_N}$. Then this flag is given an SL_N flag given by a chain of vector spaces

$$V_1 \subset V_2 \subset \cdots \subset V_N$$

with volume forms on each of these subspaces. Note that the spaces $V_1, \ldots, V_{n+1}$ along with their volume forms determine the rest of the flag. This flag can be represented by the sequence of vectors $v_1, v_2, \ldots, v_N$, where V_i has volume form $v_1 \wedge v_2 \wedge \cdots \wedge v_i$. Additionally, to make this a $Spin_N$ flag, we choose a 'square root' of the form $v_1 \wedge v_2 \wedge \cdots \wedge v_{n+1}$, which is given by choosing a highest weight vector in the spin representation V_{ω_n}. This then forces a choice of a 'square root' of the form $v_1 \wedge v_2 \wedge \cdots \wedge v_n \wedge v_{n+2}$, given by a highest weight vector in the spin representation $V_{\omega_{n^*}}$.

Now s_{SL_N} acts by -1 on V, so

$$s_{SL_N}(v_1 \wedge v_2 \wedge \cdots \wedge v_i) = (-1)^i v_1 \wedge v_2 \wedge \cdots \wedge v_i.$$

On the other hand, s_{Spin_N} acts by 1 on V, so

$$s_{Spin_N}(v_1 \wedge v_2 \wedge \cdots \wedge v_i) = v_1 \wedge v_2 \wedge \cdots \wedge v_i.$$

However, in the cases where s_{Spin_N} is a nontrivial element ($N \equiv 4, 6 \bmod 8$), it acts by -1 on the spin representations.

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More generally, the functions on $\mathcal{A}_{Spin_N}$ are naturally isomorphic to the direct sum of its irreducible representations:

$$\bigoplus_{\lambda\in\Lambda_+} V_\lambda.$$

The representations come in two types: those that factor through SO_N, and those that do not. s_{Spin_N} acts by 1 on the former and ±1 on the latter, depending on N mod 8.

For example, if T is the twisted cyclic shift on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_N}$, then rotation of the function $\begin{pmatrix} n-i,2n+2+i-j & \\ & j \\ n+2 & \end{pmatrix}$ is given by

$$(T^{-1})^*\begin{pmatrix} n-i,2n+2+i-j & \\ & j \\ n+2 & \end{pmatrix} = \begin{pmatrix} j & \\ & n+2 \\ n-i,2n+2+i-j & \end{pmatrix}(-1)^j.$$

As another example, let us apply the twisted cyclic shift to the function

$$\sqrt{-\begin{pmatrix} n-i,n+2+i & \\ & n+1 \\ n+1 & \end{pmatrix}}.$$

Suppose that $N\equiv 4,6 \mod 8$. Note that

$$(T^{-1})^*\begin{pmatrix} n-i,n+2+i & \\ & n+1 \\ n+1 & \end{pmatrix} = (-1)^{n+1}\begin{pmatrix} n+1 & \\ & n+1 \\ n-i,n+2+i & \end{pmatrix}.$$

In other words, the function $(T_{-1})^*\begin{pmatrix} n-i,n+2+i & \\ & n+1 \\ n+1 & \end{pmatrix}$ is just a strict rotation of the function $\begin{pmatrix} n-i,n+2+i & \\ & n+1 \\ n+1 & \end{pmatrix}$. However,

$$(T^{-1})^*\sqrt{-\begin{pmatrix} n-i,n+2+i & \\ & n+1 \\ n+1 & \end{pmatrix}} = -\sqrt{(-1)^n\begin{pmatrix} n+1 & \\ & n+1 \\ n-i,n+2+i & \end{pmatrix}}.$$

What this means is that to obtain the function $(T^{-1})^*\sqrt{-\begin{pmatrix} n-i,n+2+i & \\ & n+1 \\ n+1 & \end{pmatrix}}$, we rotate the function $\sqrt{-\begin{pmatrix} n-i,n+2+i & \\ & n+1 \\ n+1 & \end{pmatrix}}$ to get $\sqrt{(-1)^n\begin{pmatrix} n+1 & \\ & n+1 \\ n-i,n+2+i & \end{pmatrix}}$ and then multiply by -1.

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The reason for the multiplication by -1 is as follows. Note that the function

$$\sqrt{-\begin{pmatrix} n-i, n+2+i & \\ & n+1 \\ n+1 & \end{pmatrix}}$$

lies in the space

$$[V_{\omega_{n-i}} \otimes V_{\omega_n} \otimes V_{\omega_n}]^{Spin_N}.$$

Applying the twisted cyclic shift will give a function in the space

$$[V_{\omega_n} \otimes V_{\omega_{n-i}} \otimes V_{\omega_n}]^{Spin_N}.$$

In the twisted cyclic shift, one factor of V_{ω_n}, which is a spin representation, is moved from the third slot to the first slot. The twisted cyclic shift will then act by -1 on this factor when $N \equiv 4, 6 \mod 8$.

Similarly, when we want to find the cluster structure corresponding to a transposition, then transposing the arguments only gives the correct function up to a sign. We showed the correct signs for the functions for one of the transpositions of the cluster structure in Figure 33. The negative signs, instead of occurring in every other row, occur here on every other lower diagonal. This pattern persists in general. The signs for other transpositions come from applying the twisted cyclic shift map to this cluster.

Finally, we would like to say something about square roots. Some of our functions on $\mathrm{Conf}_m \mathcal{A}_{Spin_N}$ were defined in terms of square roots of other functions. This happens when a function lies in an invariant space where two or more of the representations involved is a spin representation. We would like to pick out the correct square root. In order to do this, we will need a way to describe vectors in these invariant spaces. In the following, we will treat both the cases when $N = 2n + 1$ is even and when $N = 2n$ is odd.

We start by recalling some facts about spin representations. Let V be an N-dimensional vector space with a nondegenerate quadratic form $Q(-, -)$. From such data, we can form the Clifford algebra $C(V)$. It is the quotient of the free tensor algebra on V by the relation

$$v \otimes w + w \otimes v = 2Q(v, w).$$

Let $W \subset V$ be as maximal isotropic subspace. It has dimensions n. Let $\bigwedge^\bullet W$ be the exterior algebra of W. $C(V)$ and $\bigwedge^\bullet W$ have a natural $\mathbb{Z}/2\mathbb{Z}$-grading that comes from considering elements of V to be odd.

Then when $N = 2n$ is even,

$$C(V) \simeq \mathrm{End} \bigwedge\nolimits^\bullet W.$$

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Moreover,

$$C^{\text{even}}(V) \simeq \text{End}\left(\bigwedge^{\text{even}} W\right) \oplus \text{End}\left(\bigwedge^{\text{odd}} W\right).$$

Let us describe the action of $C(V)$ on $\bigwedge^{\bullet} W$. Write $V \simeq W \oplus W'$, where W' is an isotropic subspace complementary to W. Then $w \in W$ acts as

$$w \wedge - : \bigwedge^{i} W \to \bigwedge^{i+1} W.$$

$w' \in W'$ maps $\bigwedge^{i} W$ to $\bigwedge^{i-1} W$. It maps $w \in \bigwedge^{1} W$ to $Q(w', w)$, and the action extends to $\bigwedge^{i} W$ by using Leibniz and the sign rule (recall that $C(V)$ and $\bigwedge^{\bullet} W$ have a natural $\mathbb{Z}/2\mathbb{Z}$-grading.)

When $N = 2n + 1$ is odd, we can $C(V)$ is isomorphic to two copies of $\text{End} \bigwedge^{\bullet} W$. Let us describe two actions of $C(V)$ on $\bigwedge^{\bullet} W$. Write $V \simeq W \oplus U \oplus W'$, where W' is another n-dimensional isotropic subspace, and U is a one-dimensional space spanned by u where $Q(u, u) = 1$. Then as before, we allow $w \in W$ to act by $w\wedge$, and $w' \in W'$ to act by sending $w \in \bigwedge^{1} W$ to $Q(w', w)$, and extending by a signed Leibniz rule. u can act in one of two ways: it can act by 1 on $\bigwedge^{\text{even}} W$ and -1 on $\bigwedge^{\text{odd}} W$; or it can act by -1 on $\bigwedge^{\text{even}} W$ and 1 on $\bigwedge^{\text{odd}} W$. These two maps of $C(V)$ to $\text{End} \bigwedge^{\bullet} W$ realize the isomorphism

$$C(V) \simeq \text{End}\left(\bigwedge^{\bullet} W\right) \oplus \text{End}\left(\bigwedge^{\bullet} W\right).$$

Moreover, we have that

$$C^{\text{even}}(V) \simeq \text{End}\left(\bigwedge^{\bullet} W\right).$$

Recall that the lie algebra $\mathfrak{so}_N \simeq \bigwedge^2 V$ of $Spin_N$ can be embedded in $C(V)$ as the span of elements of the form $v_1 \cdot v_2 - v_2 \cdot v_1$ in $C(V)$. Then when $N = 2n$ is even, we get two representations of $\mathfrak{so}_N \subset C^{\text{even}}(V)$: $\bigwedge^{\text{even}} W$ and $\bigwedge^{\text{odd}} W$. These are precisely the spin representations of $Spin_N$. When n is even, $\bigwedge^{\text{even}} W$ has highest weight ω_n and $\bigwedge^{\text{odd}} W$ has highest weight ω_{n^*}. If n is odd, $\bigwedge^{\text{even}} W$ has highest weight ω_{n^*} and $\bigwedge^{\text{odd}} W$ has highest weight ω_n.

When N is odd, we get one spin representation, given by $\bigwedge^{\bullet} W$.

Now we are ready to define invariants in tensor products of representations. First we will define invariants of a tensor product of two spin representations.

Let $N = 2n$ be even. Let us write down bases for the spin representations. First choose a basis $e_1, \ldots, e_{2n}$ where $Q(e_i, e_{2n+1-i}) = (-1)^{i-1}$ for $i \leqslant n$. Then there is a maximal torus consisting of diagonal elements that preserve Q. For each i, $1 \leqslant i \leqslant n$, there is a cocharacter given by sending $\lambda \in \mathbb{C}^*$ to the map which takes

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e_i to λe_i and e_{2n+1-i} to $\lambda^{-1}e_{2n+1-i}$ and leaves all other basis elements fixed. These n cocharacters form a basis for the Cartan $\mathfrak{h}$. There is a dual basis $L_1, \ldots, L_n$ of $\mathfrak{h}^*$. We may let W be the span of $e_1, \ldots, e_n$. Then if $I = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$ where $i_1 < i_2 < \cdots < i_k$, then let

$$e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

Then e_I has weight

$$\omega_I := \frac{1}{2}\Big(\sum_{i\in I} L_i - \sum_{j\notin I} L_j\Big).$$

First suppose n is even. We will then define a pairing $\phi : V_{\omega_n} \times V_{\omega_n} \to \mathbb{C}$. Then let $e_I, e_J \in \bigwedge^{\text{even}} W$. If I and J are subsets of $\{1, 2, \ldots, n\}$, then we will declare $\phi(e_I, e_J) = 0$ unless I and J are complementary subsets of $\{1, 2, \ldots, n\}$, in which case

$$\phi(e_I, e_J) = (-1)^{\langle \omega_n - \omega_I, 2\rho\rangle}.$$

Similarly, we have a pairing $\phi : V_{\omega_{n^*}} \times V_{\omega_{n^*}} \to \mathbb{C}$. Then let $e_I, e_J \in \bigwedge^{\text{odd}} W$. If I and J are subsets of $\{1, 2, \ldots, n\}$, then we will declare $\phi(e_I, e_J) = 0$ unless I and J are complementary subsets of $\{1, 2, \ldots, n\}$, in which case

$$\phi(e_I, e_J) = (-1)^{\langle \omega_{n^*} - \omega_I, 2\rho\rangle}.$$

Now suppose n is odd. We can then define a pairing $\phi : V_{\omega_n} \times V_{\omega_{n^*}} \to \mathbb{C}$. Then let $e_I \bigwedge^{\text{even}} W$ and $e_J \in \bigwedge^{\text{odd}} W$. If I and J are subsets of $\{1, 2, \ldots, n\}$, then we will declare $\phi(e_I, e_J) = 0$ unless I and J are complementary subsets of $\{1, 2, \ldots, n\}$, in which case

$$\phi(e_I, e_J) = (-1)^{\langle \omega_n - \omega_I, 2\rho\rangle}.$$

Similarly, we have a pairing $\phi : V_{\omega_{n^*}} \times V_{\omega_n} \to \mathbb{C}$. Then let $e_I \in \bigwedge^{\text{odd}} W$ and $e_J \in \bigwedge^{\text{even}} W$. If I and J are subsets of $\{1, 2, \ldots, n\}$, then we will declare $\phi(e_I, e_J) = 0$ unless I and J are complementary subsets of $\{1, 2, \ldots, n\}$, in which case

$$\phi(e_I, e_J) = (-1)^{\langle \omega_{n^*} - \omega_I, 2\rho\rangle}.$$

Then ϕ in all the above cases gives an invariant of the tensor product of two spin representations. This gives correct square roots for functions of the form $\sqrt{(n; n)}$, $\sqrt{(n^*; n^*)}$, $\sqrt{(n; n^*)}$ and $\sqrt{(n^*; n)}$, respectively.

Now let $N = 2n + 1$ be odd. Here, we may choose a basis $e_1, \ldots, e_{2n+1}$ where $Q(e_i, e_{2n+2-i}) = (-1)^{i-1}$ for $i \leqslant n$. Then there is a maximal torus consisting of diagonal elements that preserve Q. For each i, $1 \leqslant i \leqslant n$, there is a cocharacter given by sending $\lambda \in \mathbb{C}^*$ to the map which takes e_i to λe_i and e_{2n+2-i} to $\lambda^{-1}e_{2n+1-i}$

and leaves all other basis elements fixed. These n cocharacters form a basis for the Cartan $\mathfrak{h}$. There is a dual basis $L_1, \ldots, L_n$ of $\mathfrak{h}^*$. We may let W be the span of $e_1, \ldots, e_n$. Then if $I = \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$ where $i_1 < i_2 < \cdots < i_k$, then let

$$e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

Then e_I has weight

$$\omega_I := \frac{1}{2}\bigg(\sum_{i \in I} L_i - \sum_{j \notin I} L_j\bigg).$$

We can then define a pairing $\phi : V_{\omega_n} \times V_{\omega_n} \to \mathbb{C}$. If I and J are subsets of $\{1, 2, \ldots, n\}$, then we will declare $\phi(e_I, e_J) = 0$ unless I and J are complementary subsets of $\{1, 2, \ldots, n\}$, in which case

$$\phi(e_I, e_J) = (-1)^{\langle \omega_n - \omega_I, 2\rho \rangle}.$$

This defines the correct square root for the function $\sqrt{(n; n+1)}$.

Finally, let S_1 and S_2 be two spin representations of $Spin_N$, where $N = 2n$ or $2n+1$. Let $k \leqslant n-1$. We would like to define an invariant, when it exists, in the space

$$[V_{\omega_k} \otimes S_1 \otimes S_2]^{Spin_N}.$$

We can do this for simultaneously for N even or odd. We will do this by constructing a $Spin_N$-invariant map

$$V_{\omega_k} \otimes S_1 \otimes S_2 \to \mathbb{C}.$$

Note that there is a natural map

$$\bigwedge^k V \to C(V)$$

given by

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k \to \frac{1}{k!} \sum_{\sigma \in S_n} v_{\sigma(1)} \cdot v_{\sigma(2)} \cdot \cdots \cdot v_{\sigma(k)}.$$

When k is even or odd, $\bigwedge^k V$ maps to $C^{\text{even}}(V)$ or $C^{\text{odd}}(V)$, respectively. Thus in the case when N is even, an element of $\bigwedge^k V$, viewed inside $C(V)$, gives a map from each spin representation to itself k is even, and gives a map from one spin representation to the other one when k is odd. (When N is odd an element of $\bigwedge^k V$ always gives a map from the unique spin representation to itself.) Then if we have $v \otimes s_1 \otimes s_2 \in V_{\omega_k} \otimes S_1 \otimes S_2$, we can map this to $\phi s_1, v \cdot s_2$ in the instances where $v \cdot s_2$ lies in the spin representation dual to S_1 (which depends on k, and so on).

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The above invariant in $[V_{\omega_k} \otimes S_1 \otimes S_2]^{Spin_N}$ gives the correct square root for the functions $\sqrt{\binom{k,N-k}{n}\ {}_{n+1}}$ when N is odd, and $\sqrt{\pm(k, N-k; n^{(*)}; n^{(*)})}$ when n is even.

(The presence of the superscripts $*$ will depend on the parities of k and n, while the sign under the radical will depend on the presence of the superscripts $*$.)

From now on, we will suppress all signs for the sake of simplicity in all our future computations. The analysis above allows the interested reader to supply signs for all the functions that arise in the mutations that follow.

5.3. Folding. Let us now discuss how the quiver for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+2}}$ comes from an unfolding for the quiver for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$. From the description above, it is clear that there is an automorphism σ of the quiver for the cluster algebra structure on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+2}}$. Namely, we can define

$$\sigma(x_{in}) = x_{in*}$$

$$\sigma(x_{in*}) = x_{in}$$

$$\sigma(y_n) = x_{n*}$$

$$\sigma(y_{n*}) = x_n$$

and σ fixes all other vertices. Then folding the quiver under the automorphism σ gives the quiver for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$.

Let us say this in another way. Let σ be the Dynkin diagram automorphism of D_{n+1} having quotient B_n. Then σ induces a map on the root system for $Spin_{2n+2}$, and hence on the fundamental weights and the dominant weights. It also induces an outer automorphism of $Spin_{2n+2}$ having fixed locus $Spin_{2n+1}$, and an involution on the spaces $\mathrm{Conf}_m\,\mathcal{A}_{Spin_{2n+1}}$. Let π be the map from the vertices of D_{n+1} to the vertices of B_n. This induces a map π sending fundamental weights to corresponding fundamental weights, and therefore projects the weight space for $Spin_{2n+2}$ to the weight space for $Spin_{2n+1}$.

It turns out that the cluster algebra structure on $\mathrm{Conf}_3\,\mathcal{A}_{SL_{2n}}$ is preserved by this involution, and that, moreover, the initial seed that we constructed above is preserved by this involution. Folding this seed gives the cluster algebra structure on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+1}}$.

OBSERVATION 5.2. Let f be a function on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+2}}$ that lies in the invariant space

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^{Spin_{2n+2}}.$$

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Then $\sigma^*(f)$ lies in the invariant space

$$[V_{\sigma(\lambda)} \otimes V_{\sigma(\mu)} \otimes V_{\sigma(\nu)}]^{Spin_{2n+2}}.$$

OBSERVATION 5.3. Let f be a function on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$ that lies in the invariant space

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^{SL_{2n}}.$$

Then as a function on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$, f lies in the invariant space

$$[V_{\pi(\lambda)} \otimes V_{\pi(\mu)} \otimes V_{\pi(\nu)}]^{Spin_{2n+1}}.$$

OBSERVATION 5.4. Consider our initial cluster for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$. Suppose v is a vertex in the quiver for this cluster. Let f_v be the function attached to v. Then

$$f_v \in [V_\lambda \otimes V_\mu \otimes V_\nu]^{Spin_{2n+2}},$$

$$f_{\sigma(v)} \in [V_{\sigma(\lambda)} \otimes V_{\sigma(\mu)} \otimes V_{\sigma(\nu)}]^{Spin_{2n+2}}.$$

However, on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$, $f_v = f_{\sigma(v)}$. This means that we must have $\pi(\lambda) = \pi(\sigma(\lambda))$, $\pi(\mu) = \pi(\sigma(\mu))$, and $\pi(\nu) = \pi(\sigma(\nu))$.

As we mutate the cluster for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$, we continue to get clusters that unfold to give clusters for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$. We will later give sequences of mutations that realize various S_3 symmetries for the cluster algebra on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$, and also the flip on $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+2}}$. All these sequences of mutations will just be unfoldings of the analogous sequence of mutations for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$ or $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+1}}$. This gives us the following principle which will underlie the computations of the S_3 symmetries on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$ and the flip on $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+2}}$:

OBSERVATION 5.5. We can compute the formulas for the cluster variables on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$ and $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+2}}$ that appear at various stages of mutation in the following way: start with the formula for the corresponding cluster variable on $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+1}}$. Replace every instance of 'a' where $1 \leqslant a \leqslant n-1$ by 'a,' and replace every instance of '$2n+1-a$' where $1 \leqslant a \leqslant n-1$ by '$2n+2-a$.' Every instance of 'n' should be replaced by either 'n,' '$n+1$,' or '$n+1^*$,' depending on the context. One then obtains the formula for the cluster variable on $\mathrm{Conf}_m \mathcal{A}_{Spin_{2n+2}}$.

All the formulas we derive for $Spin_{2n+2}$ will follow this principle.

Finally, because any two vertices that are identified under the folding of cluster variables (for example x_{in} and x_{in^*}) are exchanged under the involution σ, we

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have that the formulas for computing the functions attached to these vertices obey another principle:

OBSERVATION 5.6. Suppose we have a cluster that is fixed under the involution σ. (This is the case for our initial cluster and any cluster obtained from the initial one in which whenever we mutate a vertex v we also mutate $\sigma(v)$.) Then if v is a vertex in this cluster, the formula for $f_{\sigma(v)}$ is obtained from the formula for f_v by switching all occurrences of $n+1$ and $n+1^*$.

Finally, let us briefly say something about folding on the level of reduced words in the Weyl group. We will see in the next section that the cluster algebra structure on $\operatorname{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$ comes from the longest word in the Weyl group for $Spin_{2n+2}$. Let the generators for this Weyl group be $s_1, s_2, \ldots, s_{n-1}, s_n, s_{n*}$. Then the longest element of the Weyl group is

$$(s_n' s_{n-1}' \ldots s_1')^n \to (s_n s_{n*} s_{n-1} s_{n-2} \ldots s_1)^n.$$

Let $s_1', s_2', \ldots, s_n'$ be the generators for the Weyl group of $Spin_{2n+1}$. There is an injection from the Weyl group of $Spin_{2n+1}$ to the Weyl group of $Spin_{2n+2}$ that takes

$$s_n' \to s_n s_{n*},$$
$$s_i' \to s_i.$$

This map carries the longest element of the Weyl group of $Spin_{2n+1}$ to the longest element of the Weyl group of $Spin_{2n+2}$:

$$(s_n' s_{n-1}' \ldots s_1')^n \to (s_n s_{n*} s_{n-1} s_{n-2} \ldots s_1)^n.$$

Therefore the reduced word for the longest element of the Weyl group of $Spin_{2n+2}$ folds to give the reduced word for the longest element of the Weyl group of $Spin_{2n+1}$, and the folding that gives the cluster structure on $\operatorname{Conf}_3 \mathcal{A}_{Sp_{2n}}$ from the cluster structure on $\operatorname{Conf}_3 \mathcal{A}_{SL_{2n}}$ really takes place on the level of Weyl groups.

5.4. Reduced words. We now relate the cluster structure on $\operatorname{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$ given in the previous section to Berenstein, Fomin and Zelevinsky's cluster structure on B, the Borel in the group G [BFZ].

Consider the map

$$i : b \in B^- \to (U^-, \overline{w_0} U^-, b \cdot \overline{w_0} U^-) \in \operatorname{Conf}_3 \mathcal{A}_{Spin_{2n+2}}.$$

PROPOSITION 5.7. *The cluster algebra constructed above on* $\operatorname{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$, *when restricted to the image of* i, *coincides with the cluster algebra structure given in* [BFZ] *on*$G^{w_0,e} \subset B^-$.

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Proof. We choose a reduced word for w_0. In the numbering of the nodes of the Dynkin diagram given above for $Spin_{2n+2}$, we choose the reduced word expression

$$w_0 = (s_n s_{n^*} s_{n-1} \cdots s_2 s_1)^n.$$

Here our convention is that the above word corresponds to the string $i_1, i_2, \ldots, i_{n-1}, i_{n^*}, i_n$ repeated n times.

In our situation, we are interested in such minors when $u, v = e$, or when $v = e$ and $u = u_{ij} = (s_n s_{n^*} s_{n-1} \cdots s_2 s_1)^{i-1} s_n s_{n^*} s_{n-1} \cdots s_j$ for $1 \leqslant i \leqslant n$ and $n \geqslant j \geqslant 1$ or $j = n^*$.

Then the cluster functions on B^- given in [BFZ] are $\Delta_{\omega_i,\omega_i}$ for $1 \leqslant i \leqslant n$ (these are the functions associated to $u, v = e$), and

$$\Delta_{u_{ij}\omega_j,\omega_j},$$

which are the functions associated to $v = e$ and $u = u_{ij} = (s_n s_{n-1} \cdots s_2 s_1)^i s_n s_{n-1} \cdots s_j$. Note that u_{ij} is the subword of u that stops on the ith iteration of s_j.

We have the following claims:

(1) When $i = 0$, the function we assigned to x_{ij} $\left(\binom{2n+2-j}{\ \ \ \ \ j}\right.$ for $j < n$, $\sqrt{\pm(n+1^{(*)};0;n+1)}$ for $j = n$, and $\left.\sqrt{\pm(n+1^{(*)};0;n+1^*)} \text{ for } j = n^*\right)$ is precisely $\Delta_{\omega_j,\omega_j}$.

(2) In the cases $i \geqslant j \neq n$; $i = n$ and $j = n$; $i = n$ $j = n^*$; the functions we assign to x_{ij} are, respectively

$$\begin{pmatrix} n-i & \\ & j \\ n+1+i-j & \end{pmatrix}(-1)^{n-i} = \Delta_{u_{ij}\omega_j,\omega_j}$$

$$\sqrt{\pm(0;n+1^{(*)};n+1)} = \Delta_{u_{nn}\omega_n,\omega_n}$$

$$\sqrt{\pm(0;n+1^{(*)};n+1^*)} = \Delta_{u_{nn^*}\omega_{n^*},\omega_{n^*}}.$$

(3) When $i < j < n$, the function we assigned to x_{ij}, $\begin{pmatrix} n-i,2n+2+i-j & \\ & j \\ n+2 & \end{pmatrix}(-1)^{n-i}$, is precisely $\Delta_{u_{ij}\omega_j,\omega_j}$.

When $i < n$ and $j = n$ or n^*, the functions we assign to x_{ij} are, respectively,

$$\sqrt{\pm(n-i,n+2+i;n+1^{(*)};n+1)} = \Delta_{u_{in}\omega_n,\omega_n},$$

$$\sqrt{\pm(n-i,n+2+i;n+1^{(*)};n+1^*)} = \Delta_{u_{in^*}\omega_{n^*},\omega_{n^*}}.$$

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Thus, in all cases the function assigned to x_{ij} is precisely $\Delta_{u_{ij}\omega_j,\omega_j}$.
The proof of these claims is a straightforward calculation. □

Finally, note that we have the following equalities of functions:

$$
\begin{gathered}
\begin{pmatrix} k \\ \\ 2n+2-k \end{pmatrix} = \begin{pmatrix} 2n+2-k \\ \\ k \end{pmatrix} \\
\begin{pmatrix} n-i & \\ & j \\ n+2+i-j & \end{pmatrix} = \begin{pmatrix} n+2+i & \\ & 2n+2-j \\ n-i+j & \end{pmatrix} \\
\begin{pmatrix} n-i, 2n+2+i-j & \\ & j \\ n+2 & \end{pmatrix} = \begin{pmatrix} n+2+i, j-i & \\ & 2n+2-j \\ n & \end{pmatrix}.
\end{gathered}
\tag{5.3}
$$

These equalities are valid up to sign (see the previous section). They arise because the quadratic form induces a duality isomorphism between $\bigwedge^i V$ and $\bigwedge^{2n+2-i} V$.

5.5. The sequences of mutations realizing S_3 symmetries.

5.5.1. The first transposition. Let $(A, B, C) \in \mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$ be a triple of flags. The sequence of mutations that realizes that S_3 symmetry $(A, B, C) \to (A, C, B)$ is the same as (3.3) (which gave the S_3 symmetry for $G = Sp_{2n}$ and $Spin_{2n+1}$):

$$
\begin{gathered}
x_{11}, x_{21}, x_{22}, x_{12}, x_{13}, x_{23}, x_{33}, x_{32}, x_{31}, \ldots, x_{1,n-1}, \ldots, x_{n-1,n-1}, \ldots, x_{n-1,1}, \\
x_{11}, x_{21}, x_{22}, x_{12}, \ldots x_{1,n-2}, \ldots, x_{n-2,n-2}, \ldots, x_{n-2,1}, \\
\ldots, \\
x_{11}, x_{21}, x_{22}, x_{12}, \\
x_{11}.
\end{gathered}
\tag{5.4}
$$

The sequence can be thought of as follows: At any step of the process, we mutate all x_{ij} such that $\max(i, j)$ is constant. We first mutate the x_{ij} such that $\max(i, j) = 1$, then the x_{ij} such that $\max(i, j) = 2$, then the x_{ij} such that $\max(i, j) = 3$, and so on. The sequence of maximums that we use is

$$1, 2, 3, \ldots, n-1, 1, 2, \ldots, n-2, \ldots, 1, 2, 3, 1, 2, 1.$$

The evolution of the quiver for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$ is just as in the case for $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$, as pictured in Figures 11 and 12, with the only difference being that each white vertex is replaced by two black vertices.

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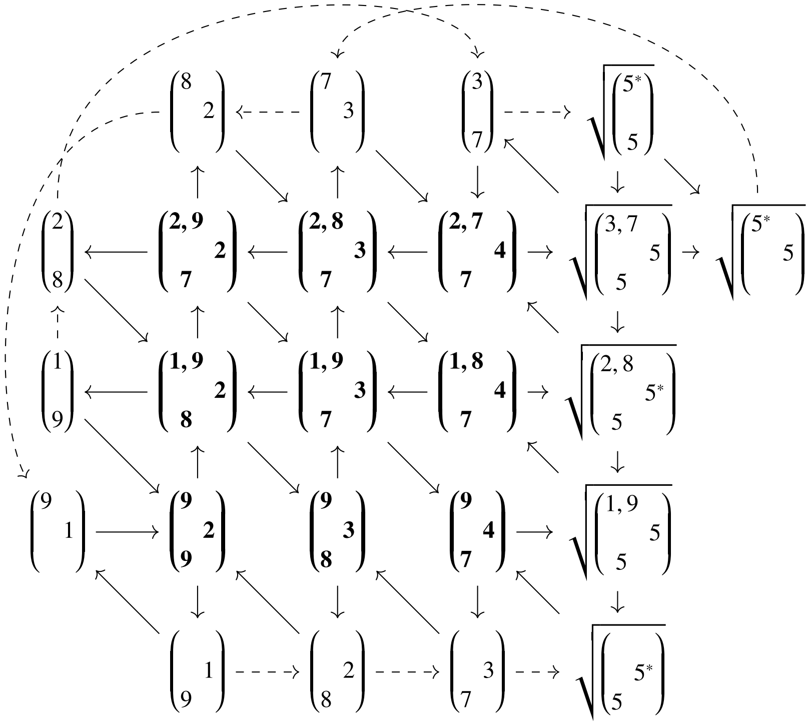

Figure 34. The quiver and the functions for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{10}}$ after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3.

In Figure 34, we depict how the quiver for $\mathrm{Conf}_3 \mathcal{A}_{Spin_10}$ changes after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3.

Note that in Figure 34, we have changed which of the doubled vertices shown. This is because mutation of vertices adjacent to the doubled vertices changes how they are connected to each other. It is not difficult to see that we end up with the functions and quiver as depicted.

In Figure 35, we depict the state of the quiver after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3; 1, 2, 3, 1, 2; and 1, 2, 3, 1, 2, 1.

From these diagrams the various quivers in the general case of $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$ should be clear.

THEOREM 5.8. *The sequence of mutations* (5.4) *realizes the* S_3 *symmetry* $(A, B, C) \to (A, C, B)$ *on* $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$.

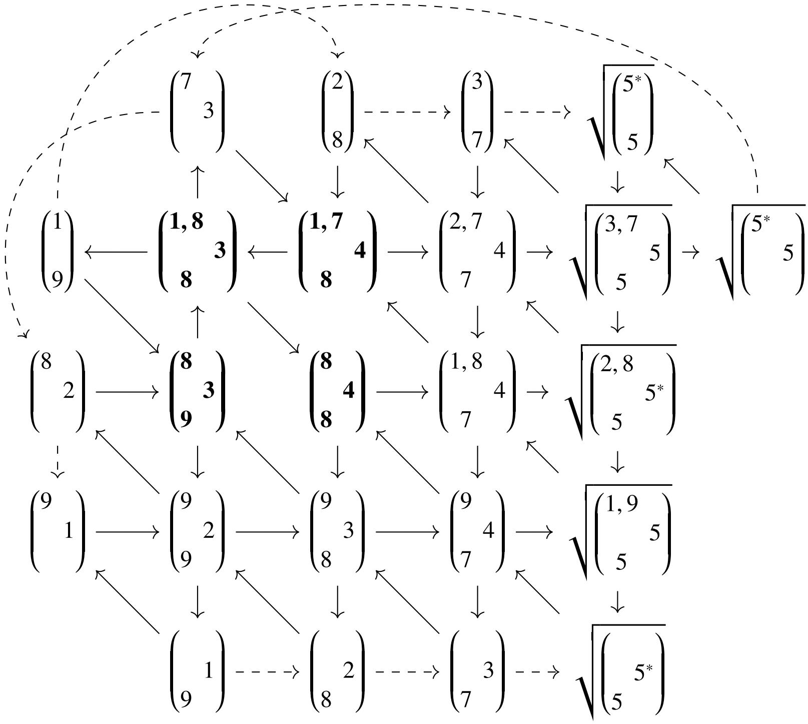

Figure 35a. The quiver and the functions for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{10}}$ after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3, 1, 2.

Proof. If $\max(i, j) = k$, then x_{ij} is mutated a total of $n - k$ times.

Recall that when $i \geqslant j$, we assign the function $\begin{pmatrix} n-i & \\ & j \\ n+2+i-j & \end{pmatrix}$ to x_{ij}. Thus the function attached to x_{ij} transforms as follows:

$$\begin{pmatrix} n-i & \\ & j \\ n+2+i-j & \end{pmatrix} \to \begin{pmatrix} 2n+1, n-i-1 & \\ & j+1 \\ n+i-j+3 & \end{pmatrix} \to \begin{pmatrix} 2n, n-i-2 & \\ & j+2 \\ n+i-j+4 & \end{pmatrix} \to \cdots$$

$$\to \begin{pmatrix} n+i+3, 1 & \\ & n-i+j-1 \\ 2n+1-j & \end{pmatrix} \to \begin{pmatrix} n+i+2 & \\ & n-i+j \\ 2n+2-j & \end{pmatrix} = \begin{pmatrix} n-i & \\ & n+2+i-j \\ j & \end{pmatrix}$$

When $i < j$ and $i \neq 0$, we assign the function $\begin{pmatrix} n-i, 2n+2+i-j & \\ & j \\ n+2 & \end{pmatrix}$ to x_{ij}. Thus the

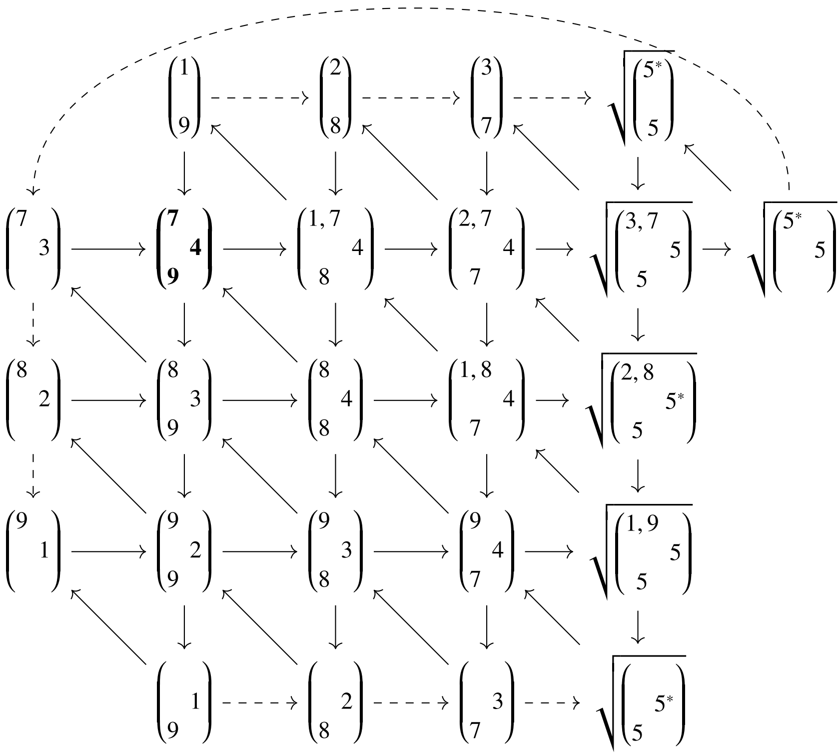

Figure 35b. The quiver and the functions for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{10}}$ after performing the sequence of mutations of x_{ij} having maximums 1, 2, 3, 1, 2, 1.

function attached to x_{ij} transforms as follows:

$$\begin{aligned}
&\begin{pmatrix} n-i, 2n+2+i-j & \\ & j \\ n+2 & \end{pmatrix} \to \begin{pmatrix} n-i-1, 2n+1+i-j & \\ & j+1 \\ n+3 & \end{pmatrix} \\
&\quad \to \begin{pmatrix} n-i-2, 2n+i-j & \\ & j+2 \\ n+4 & \end{pmatrix} \to \cdots \to \begin{pmatrix} j-i+1, n+i+3 & \\ & n-1 \\ 2n+1-j & \end{pmatrix} \\
&\quad \to \begin{pmatrix} j-i, n+i+2 & \\ & n \\ 2n+2-j & \end{pmatrix} = \begin{pmatrix} n-i, 2n+2+i-j & \\ & n+2 \\ j & \end{pmatrix}.
\end{aligned}$$

The proof is identical to the case where $G = Sp_{2n}$. Note that we do not mutate any of the vertices that have square roots. However, the functions with

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square roots are involved in the mutations, as we mutate vertices adjacent to them, namely, the vertices $x_{i,n-1}$.

We then need to use that

$$\sqrt{\begin{pmatrix} n-i, n+2+i & & \\ & & n+1^* \\ & n+1^* & \end{pmatrix}} \cdot \sqrt{\begin{pmatrix} n-i, n+2+i & & \\ & & n+1 \\ & n+1 & \end{pmatrix}} = \begin{pmatrix} n-i, n+2+i & & \\ & & n+2 \\ & n & \end{pmatrix},$$
$$\sqrt{\begin{pmatrix} n-i, n+2+i & & \\ & & n+1^* \\ & n+1 & \end{pmatrix}} \cdot \sqrt{\begin{pmatrix} n-i, n+2+i & & \\ & & n+1 \\ & n+1^* & \end{pmatrix}} = \begin{pmatrix} n-i, n+2+i & & \\ & & n+2 \\ & n & \end{pmatrix}. \tag{5.5}$$

Both these identities are a consequence of the fact that

$$\Delta_{u_{in^*}\omega_{n^*},\omega_{n^*}}(x) \cdot \Delta_{u_{in^*}\omega_n,\omega_n}(x) = \Delta_{u_{in^*}(\omega_n+\omega_{n^*}),\omega_n+\omega_{n^*}}(x).$$

Using these identities, we reduce all the mutation identities to those appearing in the cactus sequence. □

The above sequence of mutations takes us from one seed for the cluster algebra structure on $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$ to another seed where the roles of the second and third principal flags have been reversed. Thus, we have realized the first of the transpositions necessary to construct all the S_3 symmetries of $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$.

5.5.2. The second transposition. Let us now give the sequence of mutations that realizes that S_3 symmetry $(A, B, C) \to (C, B, A)$.

The sequence of mutations is as in the case of Sp_{2n}, (3.4):

$$\begin{gathered} x_{n-1,n}, x_{n-1,n^*}, x_{n-2,n-1}, x_{n-2,n}, x_{n-2,n^*}, x_{n-3,n-2}, x_{n-3,n-1}, x_{n-3,n}, x_{n-3,n^*}, \\ \ldots, x_{1,2}, \ldots, x_{1,n}, x_{1,n^*} \\ x_{n-1,n}, x_{n-1,n^*}, x_{n-2,n-1}, x_{n-2,n}, x_{n-2,n^*}, x_{n-3,n-2}, x_{n-3,n-1}, x_{n-3,n}, x_{n-3,n^*}, \\ \ldots, x_{2,3}, \ldots, x_{2,n}, \ldots, x_{2,n^*}, \\ x_{n-1,n}, x_{n-1,n^*}, x_{n-2,n-1}, x_{n-2,n}, x_{n-2,n^*}, x_{n-3,n-2}, x_{n-3,n-1}, x_{n-3,n}, x_{n-3,n^*}, \\ \ldots, x_{3,4}, \ldots, x_{3,n}, \ldots, x_{3,n^*}, \\ \ldots \\ x_{n-1,n}, x_{n-1,n^*}, x_{n-2,n-1}, x_{n-2,n}, x_{n-2,n^*}, x_{n-3,n-2}, x_{n-3,n-1}, x_{n-3,n}, x_{n-3,n^*}, \\ x_{n-1,n}, x_{n-1,n^*}, x_{n-2,n-1}, x_{n-2,n}, x_{n-2,n^*}, \\ x_{n-1,n}, x_{n-1,n^*}, \end{gathered} \tag{5.6}$$

The sequence can be thought of as follows: we only mutate those x_{ij} with $i < j$. At any step of the process, we mutate all x_{ij} in the kth row (the kth row consists

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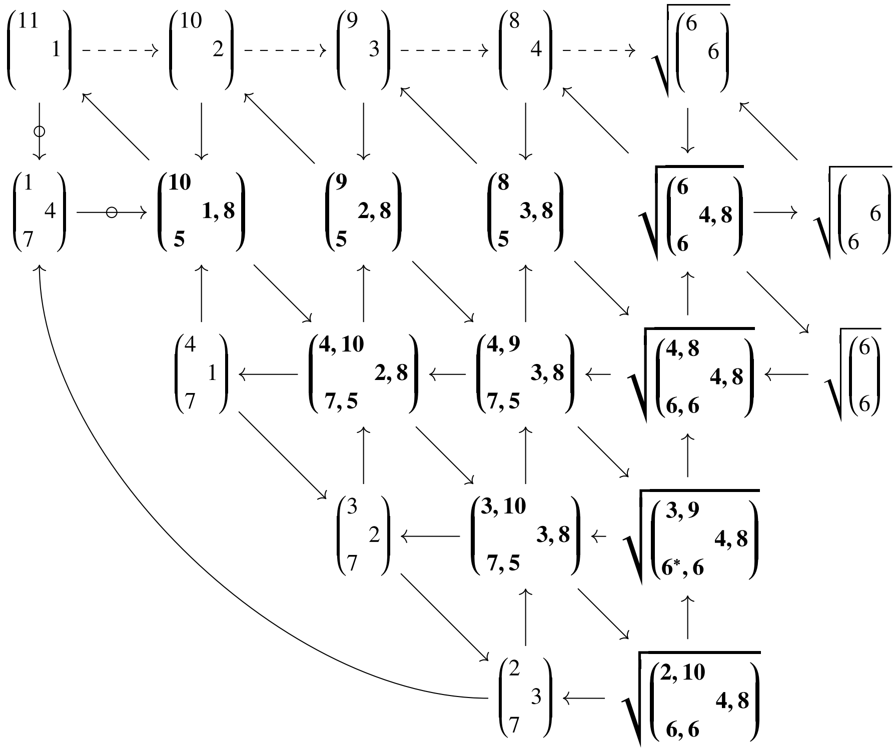

Figure 36. The quiver and functions for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{11}}$ after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1.

of x_{ij} such that $i = k$) such that $i < j$ (we will consider that $n-1 < n$ and $n-1 < n^*$). The sequence of rows that we mutate is

$$n-1, n-2, \ldots, 2, 1, n-1, n-2, \ldots, 2, n-1, \ldots, 3, \ldots, n-1, n-2, n-1.$$

As in the previous transposition, the evolution of the quiver for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$ is just as in the cases for $\mathrm{Conf}_3 \mathcal{A}_{Sp_{2n}}$ and $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+1}}$, as pictured in Figures 15 and 16, with the only difference being that each white vertex is unfolded to give two black vertices.

In Figure 36, we depict how the quiver for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{12}}$ changes after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1.

In Figure 37a, we depict the state of the quiver after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1, 4, 3, 2; in rows 4, 3, 2, 1, 4, 3, 2, 4, 3; and in rows 4, 3, 2, 1, 4, 3, 2, 4, 3, 4.

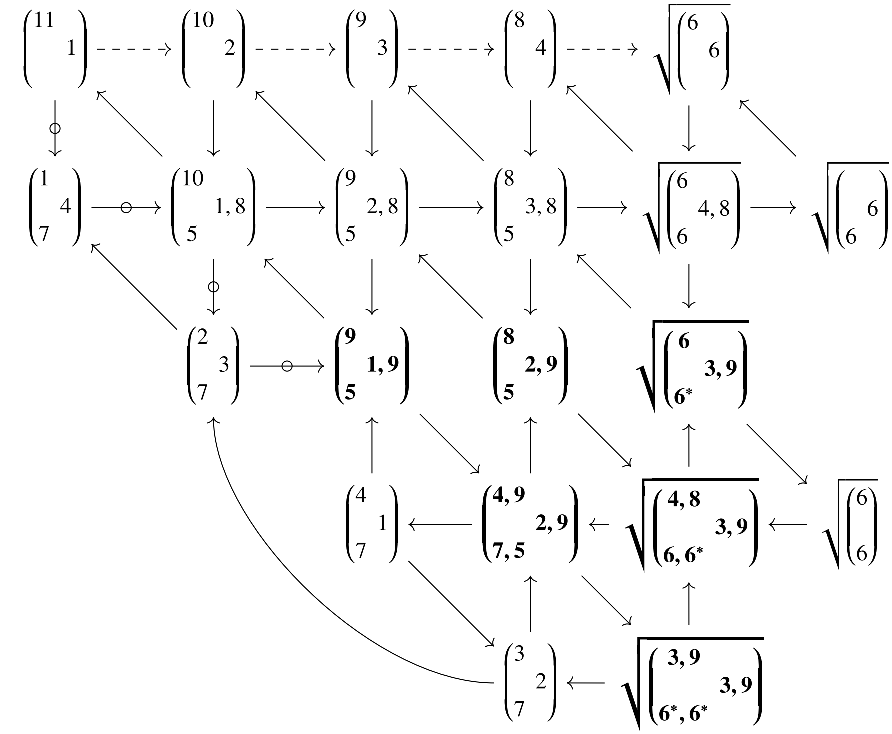

Figure 37a. The quiver and functions for $\mathrm{Conf}_3 \mathcal{A}_{Spin_{11}}$ after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1, 4, 3, 2.

From these diagrams the various quivers in the general case of $\mathrm{Conf}_3 \mathcal{A}_{Spin_{2n+2}}$ should be clear.

Recall the functions defined via Figure 15. They will appear when we perform the sequence of mutations above. In particular, we make use of functions of the form

$$\begin{pmatrix} n-i, n+2+i & & \\ & & n-j, n+2+j \\ n+1^{(*)}, n+1^{(*)} & & \end{pmatrix}.$$

These functions are invariants (unique up to scale) of the tensor products

$$[V_{2\omega_{n-i}} \otimes V_{4\omega_n} \otimes V_{2\omega_{n-j}}]^{Spin_{2n+2}},$$

$$[V_{2\omega_{n-i}} \otimes V_{4\omega_{n^*}} \otimes V_{2\omega_{n-j}}]^{Spin_{2n+2}},$$

or

$$[V_{2\omega_{n-i}} \otimes V_{2\omega_n+2\omega_{n^*}} \otimes V_{2\omega_{n-j}}]^{Spin_{2n+2}}.$$

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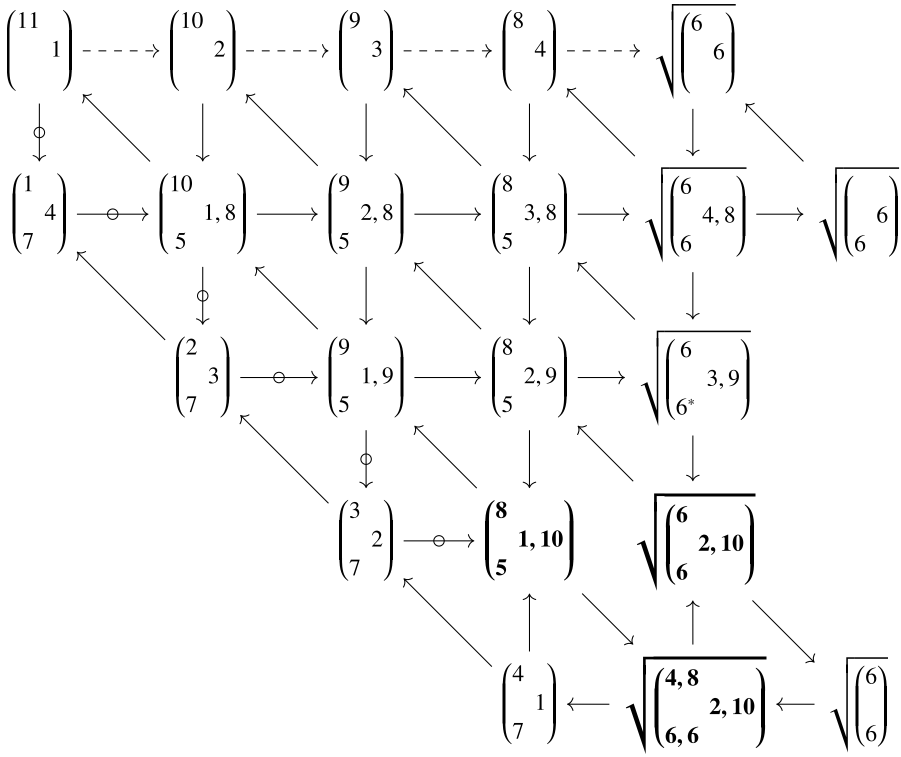

Figure 37b. The quiver and functions for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{11}}$ after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1, 4, 3, 2, 4, 3.

These functions will have square roots which are invariants (again, unique up to scale) of the tensor product

$$[V_{\omega_{n-i}} \otimes V_{2\omega_n} \otimes V_{\omega_{n-j}}]^{Spin_{2n+2}},$$

$$[V_{\omega_{n-i}} \otimes V_{2\omega_{n^*}} \otimes V_{\omega_{n-j}}]^{Spin_{2n+2}},$$

$$[V_{\omega_{n-i}} \otimes V_{\omega_n+\omega_{n^*}} \otimes V_{\omega_{n-j}}]^{Spin_{2n+2}}.$$

Thus

$$\sqrt{\begin{pmatrix} n-i, n+2+i & & \\ & & n-j, n+2+j \\ n+1^{(*)} n+1^{(*)} & & \end{pmatrix}}$$

is a well-defined function on $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+2}}$.

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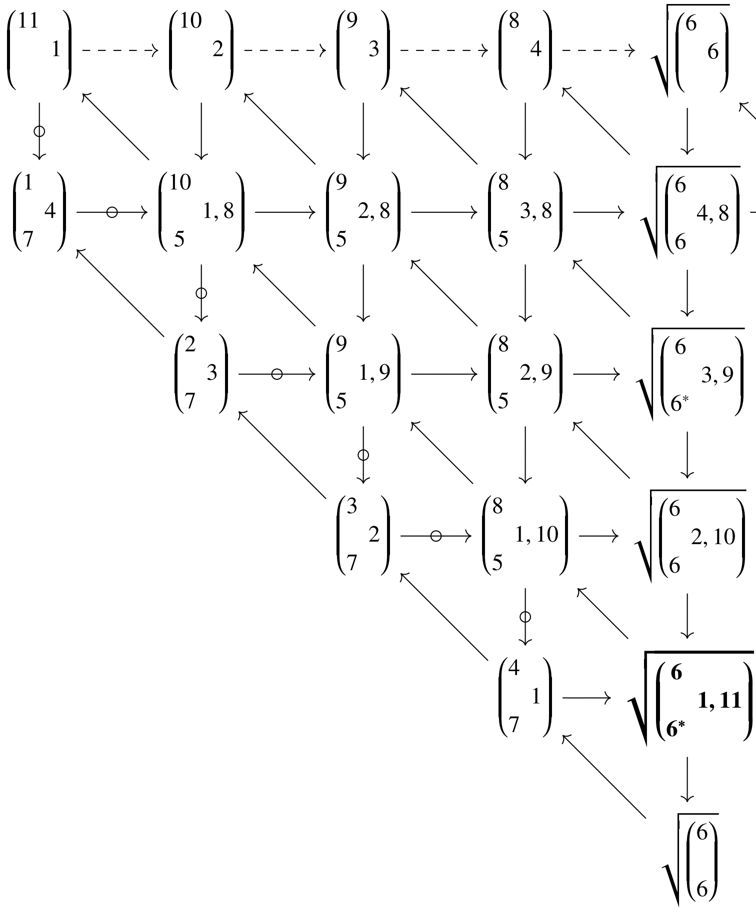

Figure 37c. The quiver and functions for $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{11}}$ after performing the sequence of mutations of x_{ij} in rows 4, 3, 2, 1, 4, 3, 2, 4, 3, 4.

Note that in the above sequence of mutations, x_{ij} is mutated i times if $i < j$. We can now state the main theorem of this section.

THEOREM 5.9. *The sequence of mutations* (5.6) *realizes the* S_3 *symmetry* $(A, B, C) \to (C, B, A)$ *for* $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+2}}$.

Proof. If $i < j$, then x_{ij} is mutated a total of i times. Recall that when $i < j < n$, we assign either the function $\begin{pmatrix} n-i, 2n+1+i-j & \\ & j \\ n & \end{pmatrix}$ to x_{ij}. In these cases, $j < n$, the

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function attached to x_{ij} transforms as follows:

$$\begin{pmatrix} n-i, 2n+2+i-j & \\ & j \\ n+2 & \end{pmatrix} \to \begin{pmatrix} n-i+1, 2n+1+i-j & \\ & j-1, n+3 \\ n, n+2 & \end{pmatrix}$$

$$\to \begin{pmatrix} n-i+2, 2n+i-j & \\ & j-2, n+4 \\ n, n+2 & \end{pmatrix}$$

$$\to \cdots \to \begin{pmatrix} n-1, 2n+3-j & \\ & j-i+1, n+1+i \\ n, n+2 & \end{pmatrix}$$

$$\to \begin{pmatrix} 2n+2-j & \\ & j-i, n+i+2 \\ n & \end{pmatrix} = \begin{pmatrix} j & \\ & n-i, 2n+2+i-j \\ n+2 & \end{pmatrix}.$$

For $j = n$ or n^*, the mutation sequence is similar, but a bit more subtle, and depends on whether $n-i$ is odd or even. First note that the functions attached to x_{in} and x_{in^*} involve square roots. Let us first consider x_{in}. We have the following cases:

- If n is even and i is odd, then the square of the function attached to x_{in} transforms as follows:

$$\begin{pmatrix} n-i, n+i+2 & \\ & n+1 \\ n+1 & \end{pmatrix} \to \begin{pmatrix} n-i+1, n+i+1 & \\ & n-1, n+3 \\ n+1^*, n+1^* & \end{pmatrix}$$

$$\to \begin{pmatrix} n-i+2, n+i & \\ & n-2, n+4 \\ n+1, n+1 & \end{pmatrix}$$

$$\to \cdots \to \begin{pmatrix} n-1, n+3 & \\ & n-i+1, n+i+1 \\ n+1, n+1 & \end{pmatrix}$$

$$\to \begin{pmatrix} n+1^* & \\ & n-i, n+i+2 \\ n+1^* & \end{pmatrix}.$$

- If n is odd and i is even, we have:

$$\begin{pmatrix} n-i, n+i+2 & \\ & n+1 \\ n+1^* & \end{pmatrix} \to \begin{pmatrix} n-i+1, n+i+1 & \\ & n-1, n+3 \\ n+1, n+1 & \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} n-i+2, n+i & & \\ & & n-2, n+4 \\ n+1^*, n+1^* & & \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} n-1, n+3 & & \\ & & n-i+1, n+i+1 \\ n+1, n+1 & & \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} n+1^* & & \\ & & n-i, n+i+2 \\ n+1^* & & \end{pmatrix}.$$

- If n and i are both even, we have:

$$\begin{pmatrix} n-i, n+i+2 & & \\ & & n+1 \\ & n+1^* & \end{pmatrix} \rightarrow \begin{pmatrix} n-i+1, n+i+1 & & \\ & & n-1, n+3 \\ & n+1, n+1^* & \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} n-i+2, n+i & & \\ & & n-2, n+4 \\ n+1^*, n+1 & & \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} n-1, n+3 & & \\ & & n-i+1, n+i+1 \\ n+1, n+1^* & & \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} n+1^* & & \\ & & n-i, n+i+2 \\ n+1 & & \end{pmatrix}.$$

- If n and i are both odd, we have:

$$\begin{pmatrix} n-i, n+i+2 & & \\ & & n+1 \\ & n+1 & \end{pmatrix} \rightarrow \begin{pmatrix} n-i+1, n+i+1 & & \\ & & n-1, n+3 \\ & n+1^*, n+1 & \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} n-i+2, n+i & & \\ & & n-2, n+4 \\ n+1, n+1^* & & \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} n-1, n+3 & & \\ & & n-i+1, n+i+1 \\ n+1, n+1^* & & \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} n+1 & & \\ & & n-i, n+i+2 \\ n+1 & & \end{pmatrix}.$$

The case of x_{in^*} switches all occurrences of $n+1$ and $n+1^*$.

We have already described the quivers at the various stages of mutation. We must then check that the functions above satisfy the identities of the associated cluster transformations.

This is the first sequence of mutations where we have to mutate the vertices x_{in} and x_{in^*}. We will also make use of functions of the form

$$\begin{pmatrix} a, b & & \\ & & c, d \\ x, y & & \end{pmatrix}$$

where $x, y = n+1$ or $n+1^*$, and $a+b+c+d = 2N$. We again use the webs pictured in Figure 15.

The most general identity takes different forms depending on the parity of n, i and j. For example, we need to show:

$$\sqrt{\begin{pmatrix} n-i, n+2+i & \\ & n-j, n+2+j \\ n+1^*, n+1 & \end{pmatrix}} \cdot \sqrt{\begin{pmatrix} n+1-i, n+1+i & \\ & n-1-j, n+3+j \\ n+1, n+1^* & \end{pmatrix}}$$

$$= \sqrt{\begin{pmatrix} n-i, n+2+i & \\ & n-1-j, n+3+j \\ n+1, n+1 & \end{pmatrix}} \cdot \sqrt{\begin{pmatrix} n+1-i, n+1+i & \\ & n-j, n+2+j \\ n+1^*, n+1^* & \end{pmatrix}}$$

$$+ \begin{pmatrix} n+1-i, n+2+i & \\ & n-1-j, n+2+j \\ n+2, n & \end{pmatrix}.$$

Note that by (4.1),

$$\begin{pmatrix} n-i, n+1+i & \\ & n-j, n+3+j \\ n+2, n & \end{pmatrix} = \begin{pmatrix} n+1-i, n+2+i & \\ & n-1-j, n+2+j \\ n+2, n & \end{pmatrix},$$

which explains the seeming asymmetry of the last term.

The general identity above follows directly from the following identities. Let $i, j < n$. Then

- $$\begin{pmatrix} n-i, n+2+i & \\ & n-j, n+2+j \\ n+2, n & \end{pmatrix} \begin{pmatrix} n+1-i, n+1+i & \\ & n-1-j, n+2+j \\ n+2, n+1^* & \end{pmatrix}$$

  $$= \begin{pmatrix} n+1-i, n+1+i & \\ & n-j, n+2+j \\ n+2, n & \end{pmatrix} \begin{pmatrix} n-i, n+2+i & \\ & n-1-j, n+2+j \\ n+2, n+1^* & \end{pmatrix}$$

  $$+ \begin{pmatrix} n+1-i, n+2+i & \\ & n-1-j, n+2-j \\ n+2, n & \end{pmatrix} \begin{pmatrix} n-i, n+1+i & \\ & n-j, n+2+j \\ n+2, n+1^* & \end{pmatrix}.$$

- $$\begin{pmatrix} n+1-i, n+1+i & \\ & n-1-j, n+2+j \\ n+2, n+1^* & \end{pmatrix}$$

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$$= \sqrt{2\begin{pmatrix} n+1-i, n+1+i & \\ & n-1-j, n+3+j \\ n+1, n+1^* & \end{pmatrix}} \cdot \sqrt{\begin{pmatrix} n+1-i, n+1+i & \\ & n-j, n+2+j \\ n+1^*, n+1^* & \end{pmatrix}},$$

- $$\begin{pmatrix} n-i, n+2+i & \\ & n-1-j, n+2+j \\ n+2, n+1^* & \end{pmatrix}$$

$$= \sqrt{2\begin{pmatrix} n-i, n+2+i & \\ & n-1-j, n+3+j \\ n+1^*, n+1^* & \end{pmatrix}} \cdot \sqrt{\begin{pmatrix} n-i, n+2+i & \\ & n-j, n+2+j \\ n+1, n+1^* & \end{pmatrix}}.$$

- $$\begin{pmatrix} n-i, n+1+i & \\ & n-j, n+2+j \\ n+2, n+1^* & \end{pmatrix}$$

$$= \sqrt{2\begin{pmatrix} n-i, n+2+i & \\ & n-j, n+2+j \\ n+1, n+1^* & \end{pmatrix}} \cdot \sqrt{\begin{pmatrix} n+1-i, n+1+i & \\ & n-j, n+2+j \\ n+1^*, n+1^* & \end{pmatrix}}.$$

These identities are proved as before. Used in combination with (5.5) we get the identity we seek.

The other mutations of black vertices will be degenerate specializations of the above general identity. □

5.6. The sequence of mutations for a flip. In this section, we will give a sequence of mutations that relates two of the clusters coming from different triangulations of the 4-gon.

We will need to label the quiver for $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+2}}$ with vertices x_{ij}, y_k, with $-n \leqslant i \leqslant n$, $j = 1, 2, \ldots, n-1, n, n^*$ and $j = 1, 2, \ldots, n-1, n, n^*, -1, -2, \ldots, -(n-1), -n, -n^*$. The quiver we will start with is the unfolding of the quiver in Figure 29, which would give us the quiver for $Spin_8$. The vertices will be labeled as before, except that the vertices x_{in^*} will double the vertices x_{in}.

Let $N = 2n+2$. We will treat the case of n even. The case where n is odd is similar.

First make an assignment of functions to the edge vertices:

$$
\begin{aligned}
\begin{pmatrix} k & \\ & N-k \\ & \end{pmatrix} &\longleftrightarrow y_k, \quad \text{for } 0 < k \leqslant n-1;\\
\begin{pmatrix} n+1 & \\ & n+1 \\ & \end{pmatrix} &\longleftrightarrow y_n;\\
\begin{pmatrix} n+1^* & \\ & n+1^* \\ & \end{pmatrix} &\longleftrightarrow y_{n^*};\\
\begin{pmatrix} & \\ |k| & \\ & N-|k| \end{pmatrix} &\longleftrightarrow y_k, \quad \text{for } -(n-1) \leqslant k < 0;\\
\begin{pmatrix} & \\ n+1 & \\ & n+1 \end{pmatrix} &\longleftrightarrow y_{-n},\\
\begin{pmatrix} & \\ n+1^* & \\ & n+1^* \end{pmatrix} &\longleftrightarrow y_{-n^*},\\
\begin{pmatrix} & j \\ N-j & \\ & \end{pmatrix} &\longleftrightarrow x_{-n,j} \quad \text{for } 0 < j \leqslant n-1;\\
\begin{pmatrix} & n+1 \\ n+1 & \\ & \end{pmatrix} &\longleftrightarrow x_{-n,n};\\
\begin{pmatrix} & n+1^* \\ n+1^* & \\ & \end{pmatrix} &\longleftrightarrow x_{-n,n^*};\\
\begin{pmatrix} & \\ & j \\ N-j & \end{pmatrix} &\longleftrightarrow x_{nj}; \quad \text{for } 0 < j \leqslant n-1;\\
\begin{pmatrix} & \\ & n+1 \\ n+1 & \end{pmatrix} &\longleftrightarrow x_{nn};\\
\begin{pmatrix} & \\ & n+1^* \\ n+1^* & \end{pmatrix} &\longleftrightarrow x_{nn^*};\\
\begin{pmatrix} j \\ \\ N-j \end{pmatrix} &\longleftrightarrow x_{0j} \quad \text{for } 0 < j \leqslant n-1;
\end{aligned}
$$

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$$\begin{pmatrix} n+1 \\ \\ n+1 \end{pmatrix} \longleftrightarrow x_{0n}$$

$$\begin{pmatrix} n+1^* \\ \\ n+1^* \end{pmatrix} \longleftrightarrow x_{0n^*}.$$

The face functions in the triangle where $i > 0$ are

$$\begin{pmatrix} i+j & \\ & N-i \\ N-j & \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } 0 < i < n, i+j \leqslant n;$$

$$\begin{pmatrix} n & \\ & j+i-n, N-i \\ N-j & \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } 0 < i < n, i+j > n, j < n;$$

$$\begin{pmatrix} n+1^* & \\ & i, N-i \\ n+1 & \end{pmatrix} \longleftrightarrow x_{in}, \quad \text{for } 0 < i < n; i \text{ odd};$$

$$\begin{pmatrix} n+1 & \\ & i, N-i \\ n+1 & \end{pmatrix} \longleftrightarrow x_{in}, \quad \text{for } 0 < i < n; i \text{ even};$$

$$\begin{pmatrix} n+1 & \\ & i, N-i \\ n+1^* & \end{pmatrix} \longleftrightarrow x_{in^*}, \quad \text{for } 0 < i < n; i \text{ odd};$$

$$\begin{pmatrix} n+1^* & \\ & i, N-i \\ n+1^* & \end{pmatrix} \longleftrightarrow x_{in^*}, \quad \text{for } 0 < i < n; i \text{ even};$$

while the face functions in the triangle where $i < 0$ are

$$\begin{pmatrix} & j \\ |i| & \\ & N-|i|-j \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } -n < i < 0, |i|+j \leqslant n;$$

$$\begin{pmatrix} & j \\ |i|, N+n-|i|-j & \\ & n+2 \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } -n < i < 0, |i|+j > n, j < n;$$

$$\begin{pmatrix} & n+1 \\ |i|, N-|i| & \\ & n+1^* \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } -n < i < 0; i \text{ odd};$$

$$\begin{pmatrix} & n+1 \\ |i|, N-|i| & \\ & n+1 \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } -n < i < 0; i \text{ even};$$

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$$\begin{pmatrix} & n+1^* \\ |i|, N-|i| & \\ & n+1 \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } -n < i < 0;\ i \text{ odd};$$

$$\begin{pmatrix} & n+1^* \\ |i|, N-|i| & \\ & n+1^* \end{pmatrix} \longleftrightarrow x_{ij}, \quad \text{for } -n < i < 0;\ i \text{ even}.$$

REMARK 5.10. Note that our labeling of the vertices is somewhat different from before. The vertices labeled x_{ij} here correspond to the vertices labeled $x_{n-|i|,j}$ in the previous sections dealing with $\mathrm{Conf}_3\,\mathcal{A}_{Spin_{2n+2}}$.

REMARK 5.11. Note that to obtain the function attached to x_{in^*} (respectively, y_{n^*}) from the function attached to x_{in} (respectively, y_n), we simply switch every occurrence of $n+1$ and $n+1^*$.

As in the cases of $G = Sp_{2n}$ and $Spin_{2n+1}$, we will have to use some functions which depend on all four flags. Let $N = 2n+2$. Let $0 \leqslant a, b, c, d \leqslant N$ such that $a+b+c+d = 4n+4 = 2N$ and $b+c \leqslant N$. Then we use the web in Figure 17 to define the function

$$\begin{pmatrix} & a & \\ b & \diagup & d \\ & c & \end{pmatrix}.$$

We use the notation

$$T^*\begin{pmatrix} & a & \\ b & \diagup & d \\ & c & \end{pmatrix} =: \begin{pmatrix} & b & \\ c & \diagdown & a \\ & d & \end{pmatrix}.$$

We now define a second type of function on $\mathrm{Conf}_4\,\mathcal{A}_{Spin_{2n+2}}$. If $a+b+c+d = 2N+n$, we define the function

$$\begin{pmatrix} & n+2 & \\ a & \diagup & c,d \\ & b & \end{pmatrix}$$

using a web similar to the one in Figure 18a. Using the twist map T, we can also define the functions $\begin{pmatrix} & a & \\ b & \diagdown & n+2 \\ & c,d & \end{pmatrix}$, $\begin{pmatrix} & b & \\ c,d & \diagup & a \\ & n+2 & \end{pmatrix}$, and $\begin{pmatrix} & c,d & \\ n+2 & \diagdown & b \\ & a & \end{pmatrix}$.

Using duality, there is also a function $\begin{pmatrix} & n & \\ a & \diagup & c,d \\ & b & \end{pmatrix}$ on $\mathrm{Conf}_4\,\mathcal{A}_{Spin_{2n+2}}$ for $0 \leqslant a, b, c, d \leqslant N$, $a+b+c+d = 3n+4 = N+n+2$, and $c \leqslant d$. For this function, we use a web similar to the one in Figure 18b.

Using the twist map T, we can also define the functions $\begin{pmatrix} & a & \\ b & \diagdown & n \\ & c,d & \end{pmatrix}$, $\begin{pmatrix} & b & \\ c,d & \diagup & a \\ & n & \end{pmatrix}$, and $\begin{pmatrix} & c,d & \\ n & \diagdown & b \\ & a & \end{pmatrix}$.

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Finally, we will work with the function $\left(a,b \overset{x}{\underset{y}{\diagup}} c,d\right)$ on $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+2}}$, where $x, y = n, n+1, n+1^*$ or $n+2$. These functions are defined by webs similar to the one in Figure 19.

Now if the function $\left(a,b \overset{x}{\underset{y}{\diagup}} c,d\right)$ on $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+2}}$ has either x or $y = n+1^*$, then we pull the function back from $\mathrm{Conf}_4 \mathcal{A}_{SL_N}$ using the map $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+2}} \to \mathrm{Conf}_4 \mathcal{A}_{SL_N}$ that differs from the standard one by the outer automorphism of $Spin_{2n+2}$ on those flags that have an argument with the symbol $*$.

Note that

$$\left(a,b \overset{x}{\underset{y}{\diagup}} c,d\right) = (T^2)^*\left(c,d \overset{y}{\underset{x}{\diagup}} a,b\right).$$

Finally, note that if $a+b = c+d = N$, $x, y = n+1$ or $n+1^*$, then

$$\sqrt{\left(a,b \overset{x}{\underset{y}{\diagup}} c,d\right)}$$

is a well-defined function on $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+2}}$. This is because the representations V_{ω_a} and V_{ω_b} of SL_N give the same representations of $Spin_N$, and V_{ω_x} and V_{ω_y}, as representations of $Spin_N$, have twice the weight of one of the spin representations. Thus, for example, when n is odd and $a+c$ is odd,

$$\sqrt{\left(a,b \quad \overset{n+1}{\underset{n+1^*}{\diagup}} \quad c,d\right)}$$

is a well-defined function on $\mathrm{Conf}_4 \mathcal{A}_{Spin_{2n+2}}$.

Now we give the sequence of mutations realizing the flip of a triangulation. The sequence of mutations is a lift of the sequence of mutations for the cases of Sp_{2n} and $Spin_{2n+1}$. The sequence of mutations leaves $x_{-n,j}, x_{nj}, y_k$ untouched as they are frozen variables. Hence we only mutate x_{ij} for $-n \leqslant i \leqslant n$. We now describe the sequence of mutations. The sequence of mutations will have $3n-2$ stages. At the rth step, we mutate all vertices such that

$$|i| + j \leqslant r,$$

$$j - |i| + 2n - 2 \geqslant r,$$

$$|i| + j \equiv r \mod 2.$$

Note that the first inequality is empty for $r \geqslant 2n-1$, while the second inequality is empty for $r \leqslant n$.

REMARK 5.12. For the sake of the above inequalities, we take $n+1^*=n+1$, so that whenever we mutate x_{in} we will also mutate x_{in^*}. The vertices x_{in} and x_{in^*} will not have arrows between them in the quivers that are obtained in the stages of our mutation process, so they can be mutated in any order.

For example, for $Spin_8$, the sequence of mutations is

$$\begin{gathered}
x_{01},\\
x_{-1,1}, x_{02}, x_{11},\\
x_{-2,1}, x_{-1,2}, x_{01}, x_{03}, x_{03^*}, x_{12}, x_{21},\\
x_{-2,2}, x_{-1,1}, x_{-1,3}, x_{-1,3^*}, x_{02}, x_{11}, x_{13}, x_{13^*}, x_{22},\\
x_{-2,3}, x_{-2,3^*}, x_{-1,2}, x_{01}, x_{03}, x_{03^*}, x_{12}, x_{23}, x_{23^*},\\
x_{-1,3}, x_{-1,3}, x_{02}, x_{13}, x_{13^*},\\
x_{03}, x_{03^*}.
\end{gathered} \tag{5.7}$$

See Section 3.5.1 for the motivation behind this sequence. In Figure 38, we depict how the quiver for $\mathrm{Conf}_4\,\mathcal{A}_{Spin_8}$ changes after each of the seven stages of mutation.

The analogue for $\mathrm{Conf}_4\,\mathcal{A}_{Spin_{2n+2}}$ should be clear.

We now have the main theorem of this section:

THEOREM 5.13. *The sequence of mutations for a flip for* $\mathrm{Conf}_4\,\mathcal{A}_{Spin_{2n+2}}$ *yields the functions as depicted in Figure 38.*

Proof. The proof comes down to a handful of identities used in conjunction, as in previous proofs of this type.

The main novelty occurs when mutating x_{ij} for $j=n$ or n^*. We will handle the case when $j=n$. Here we will need to derive some new identities. The general mutation identity when $j=n$ has one of the following forms.

If $a+b$ is even, we have

$$\begin{aligned}
&\sqrt{\left(a, N-a \;\overset{n+1}{\underset{n+1}{\diagup}}\; b, N-b\right)}\sqrt{\left(a+1, N-a-1 \;\overset{n+1^*}{\underset{n+1^*}{\diagup}}\; b+1, N-b-1\right)}\\
&= \sqrt{\left(a+1, N-a-1 \;\overset{n+1}{\underset{n+1^*}{\diagup}}\; b, N-b\right)}\sqrt{\left(a, N-a \;\overset{n+1^*}{\underset{n+1}{\diagup}}\; b+1, N-b-1\right)}\\
&\quad + \left(a+1, N-a \;\overset{n}{\underset{n+2}{\diagup}}\; b, N-b-1\right).
\end{aligned}$$

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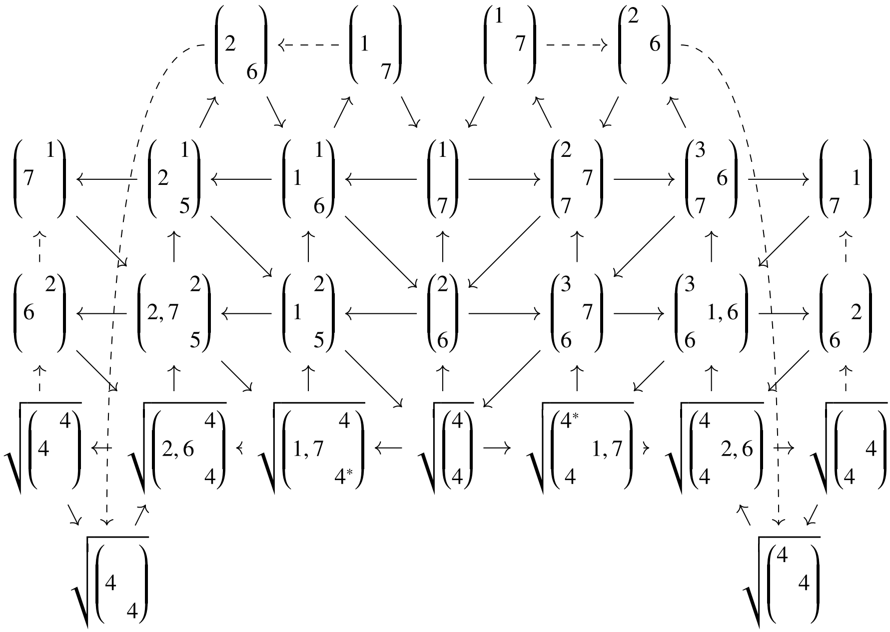

Figure 38a. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_8}$.

If $a+b$ is odd, we have

$$\sqrt{\left(\begin{array}{ccc} & n+1^* & \\ a, N-a & / & b, N-b \\ & n+1 & \end{array}\right)}\sqrt{\left(\begin{array}{ccc} & n+1 & \\ a+1, N-a-1 & / & b+1, N-b-1 \\ & n+1^* & \end{array}\right)}$$

$$= \sqrt{\left(\begin{array}{ccc} & n+1^* & \\ a+1, N-a-1 & / & b, N-b \\ & n+1^* & \end{array}\right)}\sqrt{\left(\begin{array}{ccc} & n+1 & \\ a, N-a & / & b+1, N-b-1 \\ & n+1 & \end{array}\right)}$$

$$+\left(\begin{array}{ccc} & n & \\ a+1, N-a & / & b, N-b-1 \\ & n+2 & \end{array}\right).$$

We will treat the case where $a+b$ is even. The other case is parallel. The above identity in turn follows from the following identities:

- $$\left(\begin{array}{ccc} & n & \\ a, N-a & / & b, N-b \\ & n+2 & \end{array}\right)\left(\begin{array}{ccc} & n & \\ a+1, N-a & / & b+1, N-b-1 \\ & n+1^* & \end{array}\right)$$

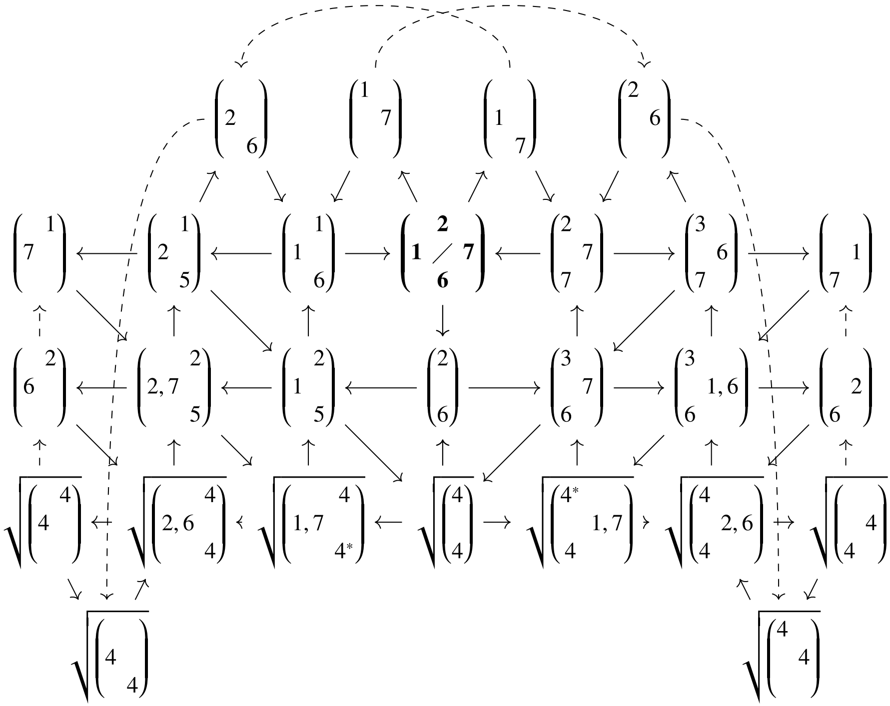

Figure 38b. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_8}$ after the first stage of mutation.

$$= \begin{pmatrix} & n & \\ a+1, N-a & \diagup & b, N-b \\ & n+1^* & \end{pmatrix}\begin{pmatrix} & n & \\ a, N-a & \diagup & b+1, N-b-1 \\ & n+2 & \end{pmatrix}$$

$$+\begin{pmatrix} & n & \\ a+1, N-a & \diagup & b, N-b-1 \\ & n+2 & \end{pmatrix}\begin{pmatrix} & n & \\ a, N-a & \diagup & b+1, N-b \\ & n+1^* & \end{pmatrix}.$$

- $$\begin{pmatrix} & n & \\ a+1, N-a & \diagup & b+1, N-b-1 \\ & n+1^* & \end{pmatrix}$$

$$= \sqrt{2\begin{pmatrix} & n+1 & \\ a, N-a & \diagup & b+1, N-b-1 \\ & n+1^* & \end{pmatrix}\begin{pmatrix} & n+1^* & \\ a+1, N-a-1 & \diagup & b+1, N-b-1 \\ & n+1^* & \end{pmatrix}}.$$

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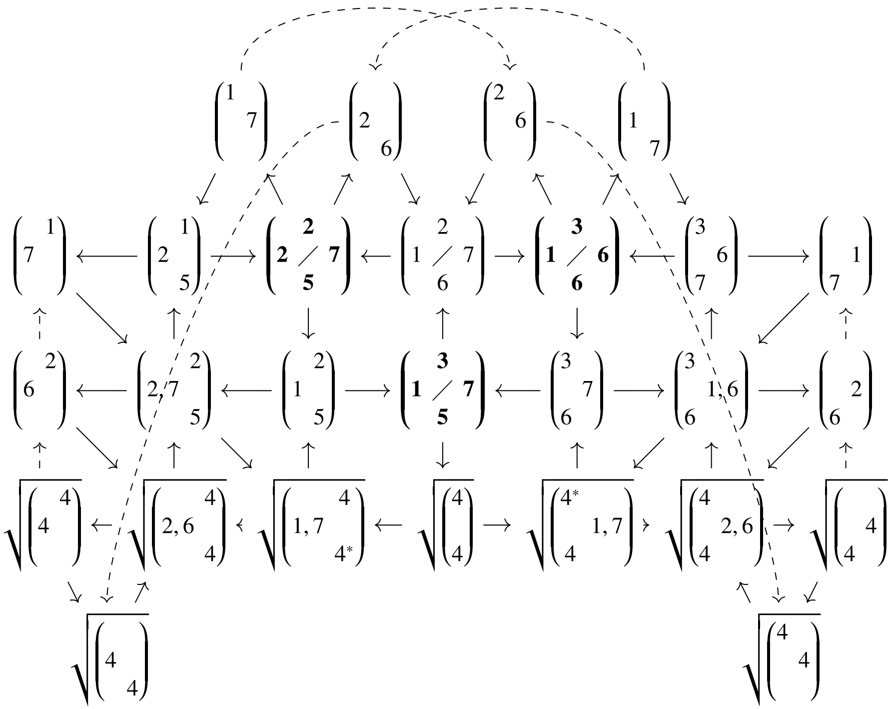

Figure 38c. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_8}$ after the second stage of mutation.

- $$\begin{pmatrix} & n & \\ a+1, N-a & \diagup & b, N-b \\ & n+1^* & \end{pmatrix}$$
$$= \sqrt{2\begin{pmatrix} & n+1^* & \\ a, N-a & \diagup & b, N-b \\ & n+1^* & \end{pmatrix}\begin{pmatrix} & n+1 & \\ a+1, N-a-1 & \diagup & b, N-b \\ & n+1^* & \end{pmatrix}}.$$

- $$\begin{pmatrix} & n & \\ a, N-a & \diagup & b+1, N-b \\ & n+1^* & \end{pmatrix}$$
$$= \sqrt{2\begin{pmatrix} & n+1^* & \\ a, N-a & \diagup & b+1, N-b-1 \\ & n+1^* & \end{pmatrix}\begin{pmatrix} & n+1 & \\ a, N-a & \diagup & b, N-b \\ & n+1^* & \end{pmatrix}}.$$

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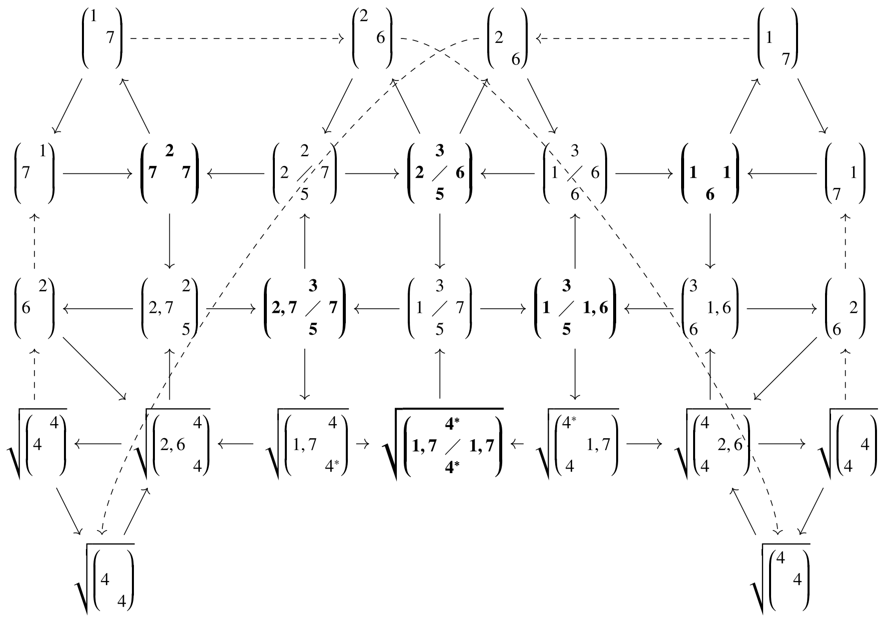

Figure 38d. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4\,\mathcal{A}_{Spin_8}$ after the third stage of mutation.

Simply substitute each term on the left-hand side of the last three identities with the corresponding term on the right-hand side into the first identity. Canceling will give the general mutation identity. All other mutation identities for $j = n$ come from this one using degeneracies. □

6. The space $\mathcal{X}_{G,S}$

In this section, we explain how to derive the structure of a cluster $\mathcal{X}$-variety on $\mathrm{Conf}_m\,\mathcal{B}_G$. In [FG2], the authors explain how to construct from a cluster $\mathcal{A}$-variety the corresponding $\mathcal{X}$-variety. We have shown above that $\mathrm{Conf}_m\,\mathcal{A}_G$ is birational to a cluster $\mathcal{A}$-variety when G is a classical group. We would like to show the following:

THEOREM 6.1. *$\mathrm{Conf}_m\,\mathcal{B}_G$ has the structure of a cluster $\mathcal{X}$-variety. This is the $\mathcal{X}$-variety which is attached, via the constructions of [FG2], to cluster structure that we have constructed on $\mathrm{Conf}_m\,\mathcal{A}_G$.*

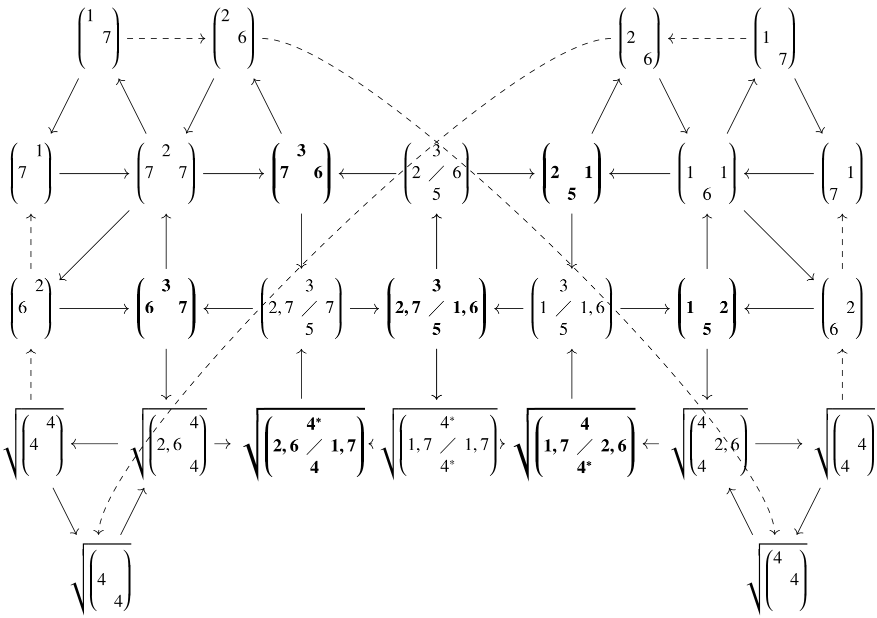

Figure 38e. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_8}$ after the fourth stage of mutation.

Let us recall the constructions of [FG2] discussed in Section 2.6. Suppose we have a cluster $\mathcal{A}$-variety with seed $\Sigma = (I, I_0, B, d)$. Then for every nonfrozen index $i \in I$, there is a cluster variable X_i. There is a map from $p : \mathcal{A}_\Sigma \to \mathcal{X}_\Sigma$ given by

$$p^*(X_i) = \prod_{j \in I} A_j^{B_{ij}}.$$

Let $\mathcal{A}$ be the cluster $\mathcal{A}$-variety associated to $\mathrm{Conf}_3 \mathcal{A}_G$, and let Σ be the initial seed we constructed. Then let us first compute the functions $p^*(X_i)$ and see that they descend to the space $\mathrm{Conf}_3 \mathcal{B}_G$. Recall that all the cluster functions A_j that we constructed on $\mathrm{Conf}_3 \mathcal{A}_G$ were invariants of tensor products:

$$A_j \in [V_\lambda \otimes V_\mu \otimes V_\nu]^G.$$

Now recall that G/U has a left action of H, the Cartan subgroup. The functions on G/U decompose as

$$\bigoplus_{\lambda \in \Lambda_+} V_\lambda.$$

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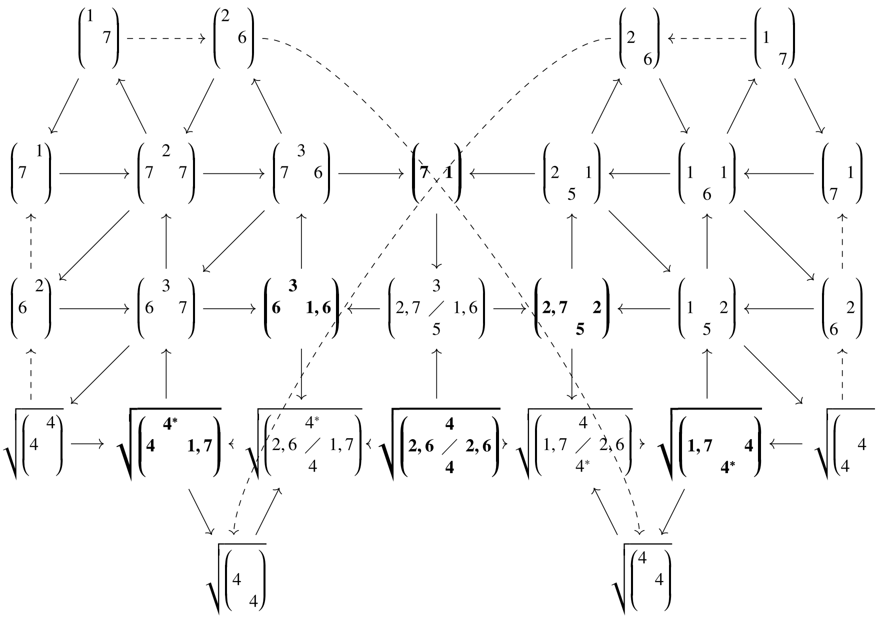

Figure 38f. The functions and quiver for the cluster algebra on $\operatorname{Conf}_4 \mathcal{A}_{Spin_8}$ after the fifth stage of mutation.

Moreover, $h \in H$ acts on the summand V_λ by $\lambda(h)$.

Correspondingly, on $\operatorname{Conf}_3 \mathcal{A}_G$ there is an action of H^3, and (h_1, h_2, h_3) acts on the summand

$$[V_\lambda \otimes V_\mu \otimes V_\nu]^G$$

by

$$\lambda(h_1)\mu(h_2)\nu(h_3).$$

From this action, it is easy to check case by case that the action of H^3 on $p^*(X_i)$ is trivial. In other words, the function $p^*(X_i)$ descends to the quotient of $\operatorname{Conf}_3 \mathcal{A}_G$ by H^3, which is precisely $\operatorname{Conf}_3 \mathcal{B}_G$.

Now we must check that the torus $\mathcal{X}_\Sigma$ is birational to $\operatorname{Conf}_3 \mathcal{B}_G$. From the above, we clearly have a map $p' : \operatorname{Conf}_3 \mathcal{B}_G \to \mathcal{X}_\Sigma$, so that all the functions X_i can be viewed as functions on $\operatorname{Conf}_3 \mathcal{B}_G$. We will show that they parameterize an open set in $\operatorname{Conf}_3 \mathcal{B}_G$.

To do this, we will adapt the results of [FG4] and [W] on parameterization of double Bruhat cells $G^{u,v}$, applied to the particular Bruhat cell $G^{w_0,e}$. Let us recall the setup. Recall that the functions on $\operatorname{Conf}_3 \mathcal{A}_G$ were associated to a reduced

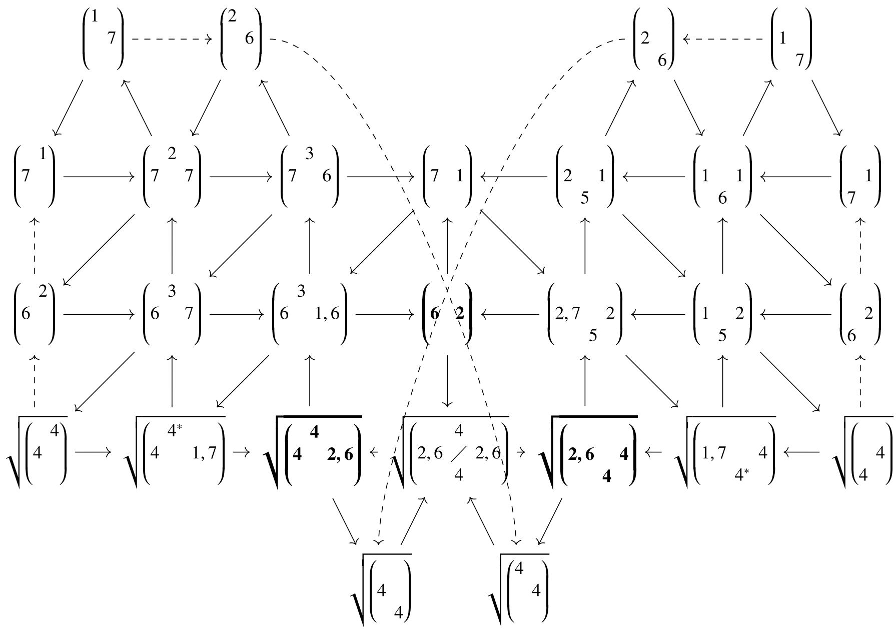

Figure 38g. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_8}$ after the sixth stage of mutation.

word composition for w_0. For the cases where $G = Sp_{2n}, Spin_{2n+1}, Spin_{2n+2}$, they were as follows:

$$w_{0,Sp_{2n}} = w_{0,Spin_{2n+1}} = (s_n s_{n-1} \cdots s_2 s_1)^n.$$

$$w_{0,Spin_{2n+2}} = (s_n s_{n^*} s_{n-1} \cdots s_2 s_1)^n.$$

The functions A_{ij} attached to the vertices x_{ij} for $i > 0$ were associated to the subwords

$$u_{ij} = (s_n s_{n-1} \cdots s_2 s_1)^{i-1} s_n s_{n-1} \cdots s_j,$$

$$u_{ij} = (s_n s_{n^*} s_{n-1} \cdots s_2 s_1)^{i-1} s_n s_{n^*} s_{n-1} \cdots s_j.$$

Let X_{ij} be the $\mathcal{X}$-function attached to the vertex x_{ij} for $0 < i < n$ (the vertices x_{ij} for $i = 0$ or n are frozen, so do not give variables on the $\mathcal{X}$-space). It is known that there is a parameterization of $\mathrm{Conf}_3 \mathcal{B}_G$ given by three flags

$$(B^+, u^- B^+, B^-),$$

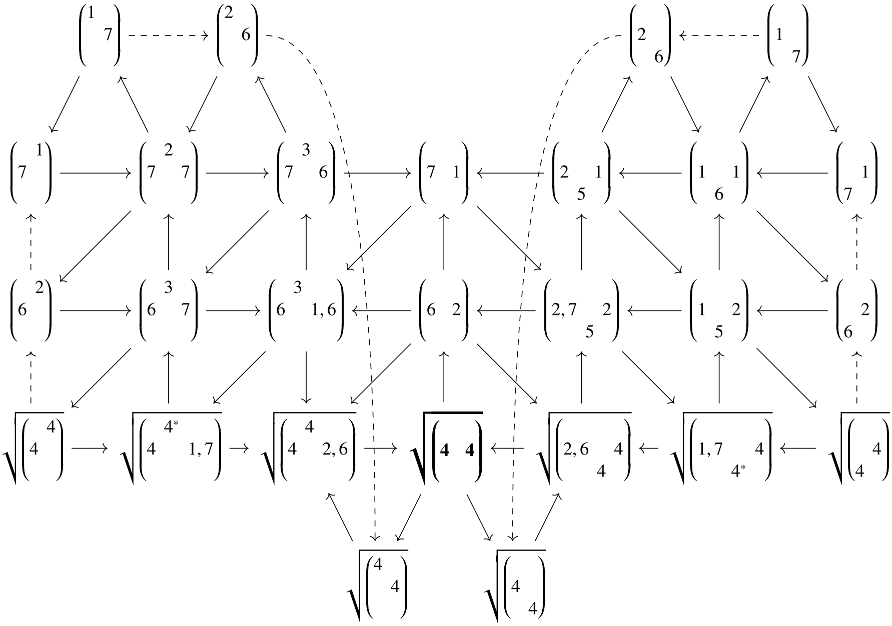

Figure 38h. The functions and quiver for the cluster algebra on $\mathrm{Conf}_4 \mathcal{A}_{Spin_8}$ after the seventh and last stage of mutation.

where u^- is determined up to the adjoint action of H. Let b^- be an element of B^-. Then there is a natural projection

$$\pi : B^- \to H = B/[B, B].$$

The choice of opposite flags B^+ and B^- gives an inclusion

$$i : H \to B^-.$$

Then let

$$\rho(b^-) := i(\pi(b^-))^{-1} b^-.$$

This associates to each element of B^- an element of U^-. We will be interested in $\rho(b^-)$ up to the adjoint action of H.

Then the coordinates X_{ij} give a parameterization of u^- by the following formula:

$$u^- = \rho(b^-),$$

where

$$b^- = F_n H_{\omega_n^\vee}(X_{n-1,n}^{-1}) \dots F_2 H_{\omega_2^\vee}(X_{n-1,2}^{-1}) F_1 H_{\omega_1^\vee}(X_{n-1,1}^{-1}) \dots \\ F_n H_{\omega_n^\vee}(X_{1n}^{-1}) \dots F_2 H_{\omega_2^\vee}(X_{12}^{-1}) F_1 H_{\omega_1^\vee}(X_{11}^{-1}) F_n \dots F_3 F_2 F_1 \tag{6.1}$$

when $G = Sp_{2n}$ or $Spin_{2n+1}$, and

$$b^- = F_n H_{\omega_n^{\vee *}}(X_{n-1,n}^{-1}) F_{n^*} H_{\omega_{n^*}^{\vee *}}(X_{n-1,n^*}^{-1}) \dots F_2 H_{\omega_2^\vee}(X_{n-1,2}^{-1}) F_1 H_{\omega_1^\vee}(X_{n-1,1}^{-1}) \dots$$

$$F_n H_{\omega_n^{\vee *}}(X_{1n}^{-1}) F_{n^*} H_{\omega_{n^*}^{\vee *}}(X_{1n^*}^{-1}) \dots F_2 H_{\omega_2^\vee}(X_{12}^{-1}) F_1 H_{\omega_1^\vee}(X_{11}^{-1}) F_n F_{n^*} \dots F_3 F_2 F_1$$

when $G = Spin_{2n+2}$. Let us explain the notation above. Here, F_i are the usual generators of the u^- associated to the simple roots. $\omega_i^\vee$ is the fundamental weight attached to the i^{th} node of the Dynkin diagram for the simply connected form of $G^\vee$; $\omega_i^{\vee *}$ is the fundamental weight for the representation dual to the one with weight $\omega_i^\vee$. (All the representations of $Spin_{2n+2}$ are self-dual except possibly the spin representations, which are either self-dual or dual to each other depending on n, so we could have more uniformly written $\omega_i^{\vee *}$ everywhere.) The weights for $G^\vee$ are the coweights for the adjoint form of G, and $H_{\omega_i^\vee}$ is the cocharacter attached to this coweight.

Thus the functions X_{ij} give a parameterization of $\mathrm{Conf}_3\,\mathcal{B}_G$.

REMARK 6.2. Replacing the B-matrix by its negation does nothing to change the $\mathcal{A}$ space, but it replaces all the functions X_{ij} on the $\mathcal{X}$-space by their inverses. It is a matter of convention how one chooses the signs of the B-matrix. If one uses the opposite convention to the one we have chosen, one would replace all the expressions $H_{\omega_j^\vee}(X_{ij}^{-1})$ above by $H_{\omega_j^\vee}(X_{ij})$, slightly simplifying the above formulas.

REMARK 6.3. An astute reader will notice that our formulas differ slightly from those found in [**FG1**]. We correct here an error in that paper. The formulas in [**FG1**] give a factorization of an element in B^- in terms of snakes, which are defined using *intersections* of subspaces coming from a configuration of three flags. The resulting parameters X (which correspond to our X_{ij}) in their formulas are then calculated using snakes. However, the X-coordinates in [**FG1**] are defined in terms of *projections* onto various subspaces. Thus they are *dual* to the snake parameters. This explains the discrepancy above.

We then need to check that for any gluing of triangles to get a structure of an $\mathcal{A}$-space on $\mathrm{Conf}_4\,\mathcal{A}_G$, the $\mathcal{X}$-coordinates on the edge gluing the two triangles parameterize gluings of configurations in $\mathrm{Conf}_3\,\mathcal{B}_G$ to get a configuration in

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$\mathrm{Conf}_4\,\mathcal{B}_G$. In other words, there is an equivalence

$$\mathrm{Conf}_4\,\mathcal{B}_G \simeq \mathrm{Conf}_3\,\mathcal{B}_G \times H \times \mathrm{Conf}_3\,\mathcal{B}_G.$$

Let us examine the $\mathcal{X}$-coordinates on the $\mathcal{X}$-space corresponding to the $\mathcal{A}$-space $\mathrm{Conf}_4\,\mathcal{A}_G$. We showed above that the face $\mathcal{X}$-coordinates parameterize the two copies of $\mathrm{Conf}_3\,\mathcal{B}_G$. Then we need to see that the edge $\mathcal{X}$ coordinates parameterize H, the space of gluings.

Explicitly, we have that H acts by shearing the configuration of four flags in the following way:

$$h : (B^+, u^-B^+, B^-, u^+B^-) \to (B^+, u^-B^+, B^-, hu^+B^-).$$

It is then a simple matter to check that the edge $\mathcal{X}$-coordinates a torsor for H. For simplicity, let us consider the cluster structure on $\mathrm{Conf}_4\,\mathcal{A}_G$ for $G = Sp_{2n}$ with the vertices labeled as in Figure 16: we have vertices x_{ij}, y_k, with $-n \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$ and $1 \leqslant |k| \leqslant n$. Then the edge vertices are x_{0j}. An easy calculation shows that

PROPOSITION 6.4. *An element $h \in H$ acts on x_{0j} by $\alpha_j(h)$, where α_j is the jth simple root of G'.*

Note that because G' is adjoint, the simple roots α_j span the weight lattice.

Proof. The proof reduces to a computation. Suppose we have a configuration of flags

$$(B^-, u^+B^-, B^+, u^-B^+).$$

We can lift it to a configuration in $\mathrm{Conf}_4\,\mathcal{A}_G$:

$$(U^-, u^+U^-, U^+, u^-U^+).$$

We can calculate the action of H on $\mathrm{Conf}_4\,\_G$ by lifting it to an action of H on $\mathrm{Conf}_4\,\mathcal{A}_G$:

$$h : (U^-, A_1, U^+, A_2) \to (U^-, A_1, hU^+, hA_2).$$

Then we can easily calculate how H acts on the functions on $\mathrm{Conf}_4\,\mathcal{A}_G$. $\mathrm{Conf}_4\,\mathcal{A}_G$ is glued from two copies of $\mathrm{Conf}_3\,\mathcal{A}_G$, and the action on these two copies is as follows:

$$h : (U^-, A_1, U^+) \to (U^-, A_1, hU^+).$$

$$h : (U^-, U^+, A_2) \to (U^-, hU^+, hA_2) = (h^{-1}U^-, U^+, A_2).$$

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Now the ring of functions on $\mathrm{Conf}_3 \mathcal{A}_G$ has a triple-grading by dominant weights; there is a grading by dominant weights for each flag. Suppose we have a function

$$f \in [V_\lambda \otimes V_\mu \otimes V_\nu]^G.$$

Then the map

$$h : (U^-, A_1, U^+) \to (U^-, A_1, hU^+)$$

acts on this function by $\nu(h)$, that is, $f(h \cdot x) = \nu(h) f(x)$, and the map

$$h : (U^-, U^+, A_2) \to (h^{-1}U^-, U^+, A_2)$$

acts on this function by $\lambda(w_0(h^{-1}))$, that is, $f(h \cdot x) = \lambda(w_0(h^{-1})) f(x)$. Using the formula $p^*(X_i) = \prod_{j \in I} A_j^{B_{ij}}$, we get our result. □

Thus the edge coordinates on the $\mathcal{X}$ space give the usual 'shear' coordinates. The usual cutting and gluing arguments allow us to conclude the following:

THEOREM 6.5. *The spaces $\mathcal{A}_{G,S}$ and $\mathcal{X}_{G',S}$ together have the structure of a cluster ensemble.*

7. Applications

Let $(\mathcal{A}, \mathcal{X})$ form a cluster ensemble, attached to the seed data $\Sigma = (I, I_0, B, d)$. Then the results of [FG2], [FG3] and [FG5] give a deformation quantization $\mathcal{X}_q$ of the space $\mathcal{X}$, as well as representations of this quantum algebra on a Hilbert space. Moreover, on this Hilbert space, we get a natural projective unitary action of the *cluster modular group* of $(\mathcal{A}, \mathcal{X})$.

The results of this paper can be interpreted as saying that, first of all, $(\mathcal{A}_{G,S}, \mathcal{X}_{G',S})$ is a cluster ensemble whenever G is a classical group; and second, that in these cases, the cluster modular group of $(\mathcal{A}_{G,S}, \mathcal{X}_{G',S})$ contains the mapping class group of S. Thus, we get a projective unitary representation of the mapping class group of S coming from the higher Teichmuller spaces $\mathcal{A}_{G,S}$ and $\mathcal{X}_{G',S}$. This is the kind of data one expects to get from a modular functor. It remains an open question whether these projective unitary representations fit together to give a modular functor. This was conjectured by Fock and Goncharov in [FG5]. In [T], Teschner discusses this conjecture and describes progress in the case of SL_2.

In this section, we sketch how to construct the space $\mathcal{X}_q$ and the Hilbert space it acts on.

Consider the seed $\Sigma = (I, I_0, B, d)$. The B-matrix encodes the Poisson structure on $\mathcal{X}$. Let

$$\widehat{\varepsilon_{ij}} = b_{ij} d_j.$$

There is a canonical Poisson structure that is given in the torus chart $\mathcal{X}_\Sigma$ by

$$\{X_i, X_j\} = \widehat{\varepsilon_{ij}}_{ij} X_i X_j.$$

This at first glance seems to depend on which torus chart $\mathcal{X}_\Sigma$ we are using, but it turns out to be invariant under mutations.

In order to quantize the space $\mathcal{X}$, we first construct a q-deformation of each $\mathcal{X}_\Sigma$, which we will call $\mathcal{X}_{\Sigma,q}$. $\mathcal{X}_{\Sigma,q}$ will be a quantum torus. It is given by generators X_i, $i \in I$ and commutation relations

$$q^{-\widehat{\varepsilon}_{ij}} X_i X_j = q^{-\widehat{\varepsilon}_{ji}} X_j X_i. \tag{7.1}$$

Thus each seed gives a quantum torus. We now need to glue together these quantum tori, that is, find algebra maps between skew fields of fractions of the noncommutative algebras $\mathcal{X}_{\Sigma,q}$ that quantize the maps between the algebras $\mathcal{X}_\Sigma$.

Recall that if Σ' is obtained from Σ by mutation at $k \in I \setminus I_0$, we have a birational map $\mu_k : \mathcal{X}_\Sigma \to \mathcal{X}_{\Sigma'}$. It is defined by

$$\mu_k^*(X_i') = \begin{cases} X_i(1 + X_k^{-1})^{-b_{ik}} & \text{if } b_{ik} > 0, i \neq k, \\ X_i(1 + X_k)^{-b_{ik}} & \text{if } b_{ik} < 0, i \neq k, \\ X_k^{-1} & i = k. \end{cases} \tag{7.2}$$

Now set

$$q_k = q^{d_k}.$$

We define the quantum mutation $\mu_k^q : \mathcal{X}_{\Sigma,q} \dashrightarrow \mathcal{X}_{\Sigma',q}$ by

$$\mu_k^{q*}(X_i') = \begin{cases} X_i((1 + q_k^{-1} X_k^{-1})(1 + q_k^{-3} X_k^{-1}) \dots (1 + q_k^{1-2|b_{ik}|} X_k^{-1}))^{-1} & \text{if } b_{ik} > 0, i \neq k, \\ X_i(1 + q_k X_k)(1 + q_k^3 X_k) \dots (1 + q_k^{2|b_{ik}|-1} X_k) & \text{if } b_{ik} < 0, i \neq k, \\ X_k^{-1} & i = k. \end{cases} \tag{7.3}$$

Setting $q = 1$ recovers the space $\mathcal{X}$. The fact that μ^{q*} gives a map of algebras follows from decomposing it as a quantum torus isomorphism and conjugation by the quantum dilogarithm, [**FG2**].

Now consider the special case of the space $\mathcal{X}_{G',S}$. Suppose that S has n boundary components. Then the monodromy around each boundary component is an element of the Borel subgroup $B' \subset G'$. There is a natural projection $B' \to H$. Therefore, we get a map

$$\pi : \mathcal{X}_{G',S} \to H^n.$$

It turns out that the functions on H^n are precisely the center of the algebra of functions on $\mathcal{X}_{G',S}$, that is, they Poisson-commute with everything. The fibers of π are the symplectic leaves of $\mathcal{X}_{G',S}$.

Every choice of a point in $c \in H^n$ prescribes values for the center of the algebra of functions on $\mathcal{X}_{G',S,q}$. The remaining functions form an algebra very close to a Heisenberg algebra, and hence have a unique representation (by unbounded operators) on a Hilbert space $\mathcal{H}_c$. This Hilbert space gives a projective unitary representation of the mapping class group of S.

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