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PAPER

Emergent behaviours of a non-abelian quantum synchronisation model over the unitary group

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Abstract
We introduce a new non-abelian quantum synchronisation model over the unitary group, represented as a gradient flow, where state matrices asymptotically converge to a common one up to phase translation. We provide a sufficient framework leading to quantum synchronisation based on Riccati-type differential inequalities. In addition, uniform time-delayed interaction is considered for modelling realistic communication, and we demonstrate that quantum synchronisation is persistent when a small time delay is allowed. Finally, numerical simulation is performed to visualise qualitative behaviours and support theoretical results.

1. Introduction

Collective synchronous behaviour of a many-body quantum system has attracted much attention from many scientific disciplines, particularly from quantum optics and quantum information [3, 4, 8, 15, 17, 20, 23]. In order to provide a mathematically rigorous analysis of such a phenomenon, several mathematical models (even phenomenological ones) have been proposed in literature after Winfree [22], Kuramoto [12] and Vicsek [21]. Among tractable candidates, we are concerned with the following model [14] on the unitary group $\mathbf{U}(d)$ of degree d:

$$\mathrm{i}\dot{U}_j U_j^\dagger = H_j + \frac{\mathrm{i}\kappa}{2N}\sum_{k=1}^{N} a_{jk}(U_k U_j^\dagger - U_j U_k^\dagger), \quad t > 0, \tag{1.1}$$

subject to initial data:

$$U_j(0) = U_j^0 \in \mathbf{U}(d), \quad j \in [N] := \{1, 2, \dots, N\}. \tag{1.2}$$

Here, U_j is a state matrix on j-th node, H_j plays a role of intrinsic frequency on j-th node, $\kappa > 0$ measures a (uniform) coupling strength between nodes, and $a_{ij} > 0$ describes network structures. Note that system (1.1)–(1.2) preserves unitarity and is represented as a gradient flow. It has been shown that when $H_i \equiv H$ for all $i \in [N]$, then every state asymptotically collapses to a common one, and when $H_i \neq H_j$ for some $i \neq j \in [N]$, relative correlation matrix $U_i U_j^\dagger$ converges to a definite one for each i, j. For the latter case, convergence of U_j itself is still unknown. For detailed statements of the convergence results, we refer the reader to [6, 7, 14].

However, it is more natural to expect the situation that state converges to another state up to *phase translation*. This is more reasonable and fits better with quantum mechanics where gauge invariance

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is allowed. In this regard, we suggest a new model on the unitary group satisfying following the two properties:

- (P1): Each state converges to a stationary state.
- (P2): Such stationary states are identical up to phase translation.

For this purpose, we introduce a new model on the unitary group:

$$\dot{U}_j = \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\langle U_j, U_k\rangle U_k - \langle U_k, U_j\rangle U_j U_k^\dagger U_j\Big), \quad t>0, \tag{1.3}$$

subject to initial data:

$$U_j(0) = U_j^0 \in \mathbf{U}(d), \quad j \in [N]. \tag{1.4}$$

Here, the inner product between matrices is defined by the Frobenius inner product, and the corresponding Frobenius norm is defined as follows:

$$\langle A, B\rangle := \mathrm{tr}(AB^\dagger), \quad \|A\| := \sqrt{\langle A, A\rangle}.$$

One can easily verify that unitary group $\mathbf{U}(d)$ is positively invariant under system (1.3) (see Lemma 2.1).

For (P1), we show that (1.3) is represented as a gradient flow with the following analytical potential:

$$\mathcal{V}(\mathcal{U}) := \frac{\kappa}{2N}\sum_{i,j=1}^{N}(d^2 - |\langle U_i, U_j\rangle|^2), \quad \mathcal{U} = (U_1, \ldots, U_N) \in (\mathbf{U}(d))^N. \tag{1.5}$$

Since the unitary group is compact, we deduce from the Łojasiewicz inequality that U_j converges to a definite state, say $U_j^\infty \in \mathbf{U}(d)$, for each $j \in [N]$. See Lemma 2.3 for detailed verification.

On the other hand for (P2), since (1.3) is a gradient flow with potential (1.5), it would be expected that a solution approaches a state that minimises the potential so that $\mathcal{V}(\mathcal{U})$ becomes zero. In this situation, there exists $\alpha_{ij} \in \mathbb{C}$ such that

$$|\langle U_i^\infty, U_j^\infty\rangle| = d^2 \quad \Longleftrightarrow \quad U_i^\infty = \alpha_{ij} U_j^\infty, \quad |\alpha_{ij}| = 1$$

which implies (P2). Hence, in what follows, our analysis is dedicated to rigorously providing this minimising process. To be more specific, we find a sufficient condition leading to

$$\lim_{t\to\infty} |\langle U_i, U_j\rangle(t)|^2 = d^2$$

which is called *quantum synchronisation* (see Definition 2.1). For our argument, we define the matrix-valued synchronisation quantity:

$$F_{ij} := \langle U_i, U_j\rangle I_d - dU_i U_j^\dagger, \tag{1.6}$$

and show that F_{ij} tends to zero (see Theorem 3.1). This plausible scenario is verified for $N = 2$ as a simple motivation in Section 2.4.

Recently, when quantum synchronisation is considered, time-delayed interaction would be introduced for the emission time of some experiments at the laboratory level [1, 19]. One of the simplest implementations of the time delay is to consider a uniform delay time $\tau > 0$ in the interaction. For example, what U_j receives in (1.3) at time t is the information of U_k at time $t-\tau$. Hence, for possible realistic application of the model, we also introduce the following delayed system for (1.3):

$$\dot{U}_j = \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\langle U_j, U_k^\tau\rangle U_k^\tau - \langle U_k^\tau, U_j\rangle U_j U_k^{\tau,\dagger} U_j\Big), \quad t>0, \tag{1.7}$$

subject to initial data

$$U_j(t) = \Phi_j(t) \in \mathbf{U}(d), \quad -\tau \le t \le 0, \quad j \in [N]. \tag{1.8}$$

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Here, $\tau > 0$ is a uniform time delay among all states, and U_j^τ is defined as

$$U_j^\tau(t) := U_j(t-\tau),$$

and initial data $\{\Phi_j(t)\}$ are given to be Lipschitz continuous function. Thus, the system (1.7)–(1.8) admits a local solution from the standard Cauchy-Lipschitz theory, and this local solution directly extended to a unique global one due to the uniform boundedness of $\dot{U}_j$. Then, our second goal is to verify an emergence of quantum synchronisation for (1.7) when τ is sufficiently small by analysing the synchronisation quantity F_{ij} in (1.6). For the analysis, we closely follow [5] where a uniform time delay for (1.1) is considered. See Theorem 4.1 for the result.

Lastly, we perform numerical simulations for both (1.3)–(1.4) and (1.7)–(1.8) to compare theoretical results with numerical ones. We numerically verify that the potential $\mathcal{V}$ (in fact, rescaled one) converges to zero with and without (small) delay for randomly chosen initial data.

The rest of this paper is organised as follows. In Section 2, we study elementary properties and detailed description of our model and review previous relevant results. Moreover, motivation is provided by analysing the case of $N = 2$. In Section 3, we show that quantum synchronisation emerges for (1.3) under some frameworks, and in Section 4, we verify that quantum synchronisation still emerges when the time-delay between nodes is sufficiently small. Finally, we present numerical simulation results in Section 5 to observe that quantum synchronisation of (1.3) and (1.7) indeed appears for the generic initial data. Finally, Section 6 is devoted to conclusion of the paper and further discussion.

2. Preliminaries

In this section, we provide several dynamical properties and detailed description for the system (1.3) and review several relevant previous results.

2.1. Dynamical properties

As a synchronisation model on the unitary group, it should be guaranteed that the unitary group is a positively invariant manifold of the system (1.3). Of course, positive invariance of the unitary group directly follows from the projection operator in (2.4). However, for readers' convenience, we provide an alternative proof in analytic way.

Lemma 2.1. *Let $\{U_j\}$ be a solution to (1.3)–(1.4). Then, the unitary group is positively invariant under the flow (1.3):*

$$U_j(t) \in \mathbf{U}(d), \quad t \geq 0.$$

Proof. We first define a critical time T_* as a maximal time that the Frobenius norms of U_i are smaller than $\sqrt{d}+1$:

$$T_* = \sup \left\{ t \geq 0 \,\middle|\, \sup_{0\leq s\leq t} \max_{1\leq i,\leq N} \|U_i(s)\| < \sqrt{d}+1 \right\}.$$

Since the initial data U_i^0 are on the unitary group, which implies $\|U_i^0\| = \sqrt{d}$, we have $T_* > 0$. We will show that $T_* = +\infty$. Suppose to the contrary that $T_* < +\infty$. Then, there exists an index i_0 such that $\|U_{i_0}(T_*)\| = \sqrt{d}+1$. On the other hand, it is straightforward to observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_d - U_iU_i^\dagger) = -\frac{\kappa}{N}\sum_{k=1}^N \Big((I_d - U_iU_i^\dagger)V_{jk}U_i^\dagger + U_jV_{jk}^\dagger(I_d - U_iU_i^\dagger) \Big) \tag{2.1}$$

where $V_{jk} := \langle U_j, U_k\rangle U_k$. Therefore for $0 \leq t \leq T_*$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|I_d - U_iU_i^\dagger\|^2 \leq C(d)\|I_d - U_iU_i^\dagger\|^2,$$

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where we used a boundedness of $\|U_i(t)\|$ in the time interval $[0, T_*]$. Then, since $\|I_d - U_i^0(U_i^0)^\dagger\| = 0$, Grönwall's inequality implies

$$I_d - U_i(t)U_i(t)^\dagger = 0, \quad 0 \le t \le T_*, \quad i \in [N],$$

and in particular, $U_{i_0}(T_*) \in \mathbf{U}(d)$, and therefore, $\|U_{i_0}(T_*)\| = \sqrt{d}$. This contradicts the choice of index i_0, and therefore, we conclude that $T_* = +\infty$. Then, we return to the estimate (2.1) and use the same argument as above to conclude that $U_i(t) \in \mathbf{U}(d)$ for all $i \in [N]$ and $t \ge 0$. □

Next, we show that system (1.3)–(1.4) is invariant under phase translation.

Lemma 2.2. *Let $\{U_j\}$ be a solution to (1.3)–(1.4). Then, the system is invariant under the phase translation, that is, for a set of constants $\{\theta_1, \theta_2, \ldots, \theta_N\}$, if $\{U_1, U_2, \ldots, U_N\}$ is a solution to (1.3), then $\{e^{\mathrm{i}\theta_1}U_1, e^{\mathrm{i}\theta_2}U_2, \ldots, e^{\mathrm{i}\theta_N}U_N\}$ also becomes a solution to (1.3).*

Proof. One can easily notice that if $\{U_j\}$ is a solution to (1.3), then

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\mathrm{i}\theta_i}U_i) = \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\langle e^{\mathrm{i}\theta_i}U_i, e^{\mathrm{i}\theta_k}U_k\rangle e^{\mathrm{i}\theta_k}U_k - \langle e^{\mathrm{i}\theta_k}U_k, e^{\mathrm{i}\theta_i}U_i\rangle(e^{\mathrm{i}\theta_i}U_i)(e^{\mathrm{i}\theta_k}U_k)^\dagger(e^{\mathrm{i}\theta_i}U_i)\Big). \tag{2.2}$$

Thus, if we define $W_i := e^{\mathrm{i}\theta_i}U_i$, then (2.2) can be written as

$$\dot{W}_i = \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\langle W_i, W_k\rangle W_k - \langle W_k, W_i\rangle W_i W_k^\dagger W_i\Big).$$

Therefore, W_i is also a solution to (1.3) subject to the initial data $W_i(0) = e^{\mathrm{i}\theta_i}U_i(0)$. □

In most of the synchronisation models on the unitary group, such as (1.1), the system is said to be synchronised if the difference between two oscillators U_i and U_j tends to zero:

$$\lim_{t\to\infty}\|U_i(t) - U_j(t)\| = 0.$$

However, a generalised definition of synchronisation, called *quantum synchronisation* was introduced in [9], based on the idea that if the one state is the phase factor multiplication of the other state, then the two states are indistinguishable.

Definition 2.1. *[9] For a solution $\{U_i\}$ to system (1.3)–(1.4), we say that the system exhibits quantum synchronisation if for each $i, j \in [N]$, there exists a constant $\alpha_{ij} \in \mathbb{R}$ such that*

$$\lim_{t\to\infty}\|U_i(t) - e^{\mathrm{i}\alpha_{ij}}U_j(t)\| = 0, \quad \textit{or equivalently} \quad \lim_{t\to\infty}\|U_i(t)U_j(t)^\dagger - e^{\mathrm{i}\alpha_{ij}}I_d\| = 0.$$

In particular, if $\alpha_{ij} \equiv 0$ for all $i, j \in [N]$, we say that the system exhibits complete quantum synchronisation.

Next, we show that the system (1.3) can be represented as a gradient flow on the unitary group.

Lemma 2.3. *The system (1.3) can be represented as a gradient flow of the following analytical potential:*

$$\mathcal{V}(\mathcal{U}) := \frac{\kappa}{2N}\sum_{k,\ell=1}^{N}(d^2 - |\langle U_k, U_\ell\rangle|^2), \quad \mathcal{U} = (U_1, \ldots, U_N) \in (\mathbf{U}(d))^N. \tag{2.3}$$

Moreover, for each $i \in [N]$, there exists a constant unitary matrix $U_i^\infty \in \mathbf{U}(d)$ such that

$$\lim_{t\to\infty}U_i(t) = U_i^\infty.$$

Proof. In order to show that system (1.3) is a gradient flow, we first calculate $\frac{\partial|\langle U_i, U_j\rangle|^2}{\partial U_i}$ and then take the orthogonal projection of the resulting relation by using the projection formula.

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Note that projection of any $d \times d$ matrix X onto $T_U\mathbf{U}(d)$, the tangent space of $\mathbf{U}(d)$ at $U \in \mathbf{U}(d)$, is given by

$$\mathbb{P}_U(X) = \frac{1}{2}(X - UX^\dagger U) \in T_U\mathbf{U}(d). \tag{2.4}$$

We now compute the gradient of $|\langle U_i, U_j\rangle|^2$ for fixed indices i and j. Since $U_i \in \mathcal{M}_{d,d}(\mathbb{C}) = \mathbb{R}^{2d^2}$, where $\mathcal{M}_{d,d}(\mathbb{C})$ denotes the set of all $d \times d$ complex-valued matrices, partial derivatives of U_i can be achieved by the partial derivatives of real and imaginary components of U_i in $\mathbb{R}^{2d^2}$. Similarly, taking partial derivatives of a scalar-valued function of $\mathcal{U}$ can be obtained by taking partial derivatives of the function with respect to real and imaginary components. Let us denote the (k, l)-component of U_i as $[U_i]_{kl} = a^i_{kl} + \mathrm{i}b^i_{kl}$. We first consider the case when $i \neq j$. Then, since

$$\begin{aligned}
|\langle U_i, U_j\rangle|^2 &= \left|\sum_{k,l} [U_i]_{kl}\overline{[U_j]_{kl}}\right|^2 \\
&= \left|\sum_{k,l} (a^i_{kl}a^j_{kl} + b^i_{kl}b^j_{kl}) + \mathrm{i}(a^j_{kl}b^i_{kl} - a^i_{kl}b^j_{kl})\right|^2 \\
&= \left(\sum_{k,l} (a^i_{kl}a^j_{kl} + b^i_{kl}b^j_{kl})\right)^2 + \left(\sum_{k,l} (a^j_{kl}b^i_{kl} - a^i_{kl}b^j_{kl})\right)^2,
\end{aligned}$$

we have

$$\begin{aligned}
\frac{\partial|\langle U_i, U_j\rangle|^2}{\partial a^i_{kl}} &= 2\left(\sum_{k,l} (a^i_{kl}a^j_{kl} + b^i_{kl}b^j_{kl})\right) a^j_{kl} - 2\left(\sum_{k,l} (a^j_{kl}b^i_{kl} - a^i_{kl}b^j_{kl})\right) b^j_{kl} \\
&= 2\mathrm{Re}\left(\langle U_i, U_j\rangle[U_j]_{kl}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial|\langle U_i, U_j\rangle|^2}{\partial b^i_{kl}} &= 2\left(\sum_{k,l} (a^i_{kl}a^j_{kl} + b^i_{kl}b^j_{kl})\right) b^j_{kl} + 2\left(\sum_{k,l} (a^j_{kl}b^i_{kl} - a^i_{kl}b^j_{kl})\right) a^j_{kl} \\
&= 2\mathrm{Im}\left(\langle U_i, U_j\rangle[U_j]_{kl}\right).
\end{aligned}$$

Similarly, when $i = j$, then we have

$$\frac{\partial|\langle U_i, U_i\rangle|^2}{\partial a^i_{kl}} = 4\mathrm{Re}(\langle U_i, U_i\rangle[U_i]_{kl}), \qquad \frac{\partial|\langle U_i, U_i\rangle|^2}{\partial b^i_{kl}} = 4\mathrm{Im}(\langle U_i, U_i\rangle[U_i]_{kl}).$$

Thus, the derivative of $|\langle U_i, U_j\rangle|^2$ by U_i is

$$\frac{\partial|\langle U_i, U_j\rangle|^2}{\partial U_i} = (2 + 2\delta_{ij})\langle U_i, U_j\rangle U_j,$$

where δ_{ij} denotes the Kronecker delta. This yields the following formula of the gradient of $|\langle U_i, U_j\rangle|^2$ at U_i:

$$\nabla_{U_i}|\langle U_i, U_j\rangle|^2 = \mathbb{P}_{U_i}\left((2 + 2\delta_{ij})\langle U_i, U_j\rangle U_j\right) = \langle U_i, U_j\rangle U_j - U_i(\langle U_i, U_j\rangle U_j)^\dagger U_i,$$

since $\mathbb{P}_{U_i}(\langle U_i, U_i\rangle U_i) = 0$. Hence, the gradient of potential is given by

$$\begin{aligned}
\nabla_{U_i}\mathcal{V} &= \frac{\kappa}{2N}\sum_{k,\ell=1}^{N}\left(-\nabla_{U_i}|\langle U_k, U_\ell\rangle|^2\right) = -\frac{\kappa}{N}\sum_{j=1}^{N}\nabla_{U_i}|\langle U_i, U_j\rangle|^2 \\
&= -\frac{\kappa}{N}\sum_{j=1}^{N}\left(\langle U_i, U_j\rangle U_j - \langle U_j, U_i\rangle U_i U_j^\dagger U_i\right).
\end{aligned}$$

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Therefore, the system (1.3) is a gradient flow with the potential $\mathcal{V}$ in (2.3), that is

$$\dot{U}_j = -\nabla_{U_j}\mathcal{V}.$$

On the other hand, it is known that the gradient flow on a compact manifold must converge [6, Theorem 5.1], by using Łojasiewicz inequality. Since $(\mathbf{U}(d))^N$ is a compact manifold, we conclude that the system (1.3) must converge, that is, there exists $U_i^\infty \in \mathbf{U}(d)$ such that $\lim_{t\to\infty} U_i(t) = U_i^\infty$. □

2.2. *Model description*

Since (1.3) is a gradient flow equipped with potential (2.3), a solution is expected to converge to a minimiser of (2.3), that is, a solution converges to some definite states tending to minimise the potential $\mathcal{V}$. When a solution reaches global minimum of $\mathcal{V}(\mathcal{U})$, then we have for all $i, j \in [N]$:

$$|\langle U_i, U_j\rangle|^2 = d^2 \quad \Longleftrightarrow \quad U_i = \alpha_{ij} U_j, \quad |\alpha_{ij}| = 1.$$

This condition exactly coincides with the emergence of quantum synchronisation in Definition 2.1. Hence, it is natural to expect that the system (1.3)–(1.4) exhibits the quantum synchronisation, instead of the complete quantum synchronisation.

Another motivation for quantum synchronisation is as follows. We note that the only difference between classical synchronisation system (1.1) with $(H_i, a_{ij}) = (O_d, 1)$ and the system (1.3) is that U_k in (1.1) is replaced by $\frac{\langle U_i, U_k\rangle}{d} U_k$, by rescaling $\kappa \mapsto d\kappa$. Since a solution to (1.1) with $(H_i, a_{ij}) = (O_d, 1)$ tends to a common state, that is, $\|U_i(t) - U_k(t)\| \to 0$ as $t \to \infty$ [7], one can expect that our model would exhibit the following convergence:

$$\left\| U_i(t) - \frac{\langle U_i(t), U_k(t)\rangle}{d} U_k(t) \right\| \to 0, \quad \text{as} \quad t \to \infty,$$

which also coincides with the definition of quantum synchronisation in Definition 2.1.

Remark 2.1. (1) We also mention that the model (1.3) is essentially high-order model. If we consider the case when $d = 1$ and parameterise $U_i = e^{\mathrm{i}\theta_i}$, then the system (1.3) reduces to

$$\dot{\theta}_j = 0, \quad \text{or,} \quad \theta_j(t) = \theta_j(0), \quad t > 0,$$

which is a trivial dynamics. However, with the same parameterisation, the system (1.1) with $(H_i, a_{ij}) = (O_d, 1)$ reduces to the well-known Kuramoto model [12]:

$$\dot{\theta}_j = \frac{\kappa}{N} \sum_{k=1}^N \sin(\theta_k - \theta_j).$$

The reason why the system (1.3) reduces to the trivial dynamics is that the two phases are considered to be identical if they are only different in phase factor. In this sense, all the one-dimensional phases are identical, and therefore, the dynamics becomes trivial. Therefore, the new model (1.3) is essentially high-dimensional, and it is qualitatively different from the usual synchronisation model.
(2) One might consider the non-identical extension of the system (1.3) as in (1.1):

$$\dot{U}_j = -\mathrm{i}H_j U_j + \frac{\kappa}{N} \sum_{k=1}^N \left(\langle U_j, U_k\rangle U_k - \langle U_k, U_j\rangle U_j U_k^\dagger U_j \right). \tag{2.5}$$

If we consider the diagonal Hamiltonian, that is,

$$H_j = a_j I_d, \quad a_j \in \mathbb{R},$$

then (2.5) becomes

$$\dot{U}_j = -\mathrm{i}a_j U_j + \frac{\kappa}{N} \sum_{k=1}^N \left(\langle U_j, U_k\rangle U_k - \langle U_k, U_j\rangle U_j U_k^\dagger U_j \right).$$

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However, if we introduce $V_j := e^{\mathrm{i}a_jt}U_j$, then by using similar argument as in Lemma 2.2, V_j exactly satisfies (1.3):

$$\dot{V}_j = \frac{\kappa}{N}\sum_{k=1}^{N}\left(\langle V_j, V_k\rangle V_k - \langle V_k, V_j\rangle V_j V_k^\dagger V_j\right).$$

Therefore, the effect of a_j disappears up to the phase factor. See also Remark 3.2.

2.3. Previous results

In this part, we review previous results on the quantum synchronisation models that are closely related to (1.3). First of all, the first non-abelian quantum synchronisation model (1.1) on the unitary group was introduced in [14]. Here, we briefly recall the previous results on (1.1). For detailed statements and proofs, we refer the reader to [7, 11].

Theorem 2.1. [7, 11]

1. [7] *Suppose that $H_i = O_d$ for all $i \in [N]$ and initial data $\{U_j^0\}$ satisfy*

$$\max_{1\le i,j\le N}\|U_i^0 - U_j^0\| < \sqrt{2}$$

and let $\{U_j\}$ be a solution to (1.1) with initial data $\{U_j^0\}$. Then, we have

$$\lim_{t\to\infty}\max_{1\le i,j\le N}\|U_i(t) - U_j(t)\| = 0.$$

2. [7] *Suppose that system parameters satisfy*

$$\kappa > \frac{54}{17}\max_{1\le i,j\le N}\|H_i - H_j\| > 0, \quad \max_{1\le i,j\le N}\|U_i^0 - U_j^0\| \ll 1,$$

and let $\{U_j\}$ be a solution to (1.1) with initial data $\{U_j^0\}$. Then, the limit $\lim_{t\to\infty}(U_iU_j^\dagger)(t)$ exists.

3. [11] *Suppose that system parameters satisfy*

$$H_i = -a_iI_d, \quad \sum_{i=1}^{N} a_i = 0, \quad \kappa \gg \max_{1\le i,j\le N}|a_i - a_j|, \quad \max_{1\le i,j\le N}\|U_i^0 - U_j^0\| \ll 1$$

and let $\{U_j\}$ be a solution to (1.1) with initial data $\{U_j^0\}$. Then, there exists a constant unitary matrix $V \in \mathbf{U}(d)$ and a real number θ_j^∞ such that

$$\lim_{t\to\infty}U_j(t) = e^{\mathrm{i}\theta_j^\infty}V.$$

We remark that the results in Theorem 2.1(3) exactly coincide with quantum synchronisation in Definition 2.1, whereas Theorem 2.1(2) is, in fact, not quantum synchronisation.

On the other hand, the system (1.3) is also closely related to a swarming model on the complex sphere in the Hilbert space [9] suggested by the present authors, which reads as:

$$\dot{\psi}_i = \frac{\kappa}{2N}\sum_{k=1}^{N}(\psi_k - \langle\psi_k, \psi_i\rangle\psi_i), \quad t > 0, \tag{2.6}$$

subject to initial data

$$\psi_i(0) = \psi_i^0 \in \mathcal{H}, \quad \|\psi_i^0\| = 1, \quad i \in [N]. \tag{2.7}$$

Here, $\mathcal{H}$ is a complex Hilbert space and $\psi_i = \psi_i(t)$ is a state vector on the i-th node. In addition, the inner product in $\mathcal{H}$ is linear in the first argument and conjugate linear in the second argument just as the Frobenius inner product: for $c \in \mathbb{C}$ and $x, y \in \mathcal{H}$,

$$\langle cx, y\rangle = c\langle x, y\rangle, \quad \langle x, cy\rangle = \bar{c}\langle x, y\rangle.$$

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Then, the authors showed the following statement.

Theorem 2.2 (9, Theorem 4). *Let $\{\psi_i\}$ be a solution to (2.6)–(2.7). Suppose that the norm of the average of initial data is strictly positive, that is, $\|\frac{1}{N}\sum_{i=1}^N \psi_i^0\| > 0$. Then, there exist complex-valued functions $\alpha_{ij}(t)$ such that*

$$\lim_{t\to\infty} \|\psi_j(t) - \alpha_{ij}(t)\psi_i(t)\| = 0, \quad \lim_{t\to\infty} |\alpha_{ij}(t)| = 1.$$

If we adopt our definition for quantum synchronisation in Definition 2.1 to the swarming model (2.6), then one can say that (2.6) exhibits quantum synchronisation (see also Definition 1 in [9]).

Finally, it should be mentioned that (2.6) is also related to the Schrödinger–Lohe model [13] where $\mathcal{H} = L^2(\mathbb{R}^d)$ which reads as:

$$\dot{\psi}_i = \frac{\kappa}{2N}\sum_{k=1}^N (\psi_k - \langle \psi_i, \psi_k\rangle \psi_i), \quad t > 0. \tag{2.8}$$

The only difference between (2.6) and (2.8) is the order of the inner product. Precisely, $\langle \psi_k, \psi_i\rangle$ is given in (2.6) and $\langle \psi_i, \psi_k\rangle$ is given in (2.8). However, with this change, their asymptotic behaviours are completely different.

2.4. Two-state system

As a starting point of the analysis, we first consider a two-state system of (1.3):

$$\begin{aligned} \dot{U}_1 &= \frac{\kappa}{2}(\langle U_1, U_2\rangle U_2 - \langle U_2, U_1\rangle U_1 U_2^\dagger U_1), \\ \dot{U}_2 &= \frac{\kappa}{2}(\langle U_2, U_1\rangle U_1 - \langle U_1, U_2\rangle U_2 U_1^\dagger U_2), \end{aligned} \tag{2.9}$$

subject to initial data:

$$(U_1, U_2)(0) = (U_1^0, U_2^0) \in \mathbf{U}(d) \times \mathbf{U}(d). \tag{2.10}$$

We define a correlation matrix G for (2.9):

$$G(t) := U_1(t)U_2^\dagger(t), \quad t > 0.$$

Below, we study the temporal evolution of the correlation matrix G.

Lemma 2.4. *Let $\{U_1, U_2\}$ be a solution to (2.9)–(2.10). Then, G satisfies*

$$\dot{G} = \kappa(\mathrm{tr}(G)I_d - \mathrm{tr}(G^\dagger)G^2), \quad t > 0. \tag{2.11}$$

Proof. It directly follows from the governing equations (2.9). □

We note that the matrix G is normal, and therefore, it is always diagonalisable. By following the idea in [10], we show that it suffices to consider the diagonalisation of G.

Lemma 2.5. *Let $G = G(t)$ be a solution to (2.11) with initial datum G_0 whose diagonalisation is given as:*

$$G_0 =: V_0 D_0 V_0^\dagger, \tag{2.12}$$

where V_0 is a $d \times d$ unitary matrix and D_0 is a $d \times d$ complex diagonal matrix. Then, $G(t)$ is determined by:

$$G(t) = V_0 D(t) V_0^\dagger$$

where $D(t)$ is a solution to the following Cauchy problem:

$$\begin{cases} \dot{D} = \kappa(\mathrm{tr}(D)I_d - \mathrm{tr}(D^\dagger)D^2), & t > 0, \\ D(0) = D_0. \end{cases} \tag{2.13}$$

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Proof. Let D be a solution to (2.13). We multiply V_0 and $V_0^\dagger$ in the left- and right-hand sides of (2.13). Then, one has

$$V_0\dot{D}V_0^\dagger = \kappa(\mathrm{tr}(D)I_d - \mathrm{tr}(D^\dagger)V_0D^2V_0^\dagger) = \kappa(\mathrm{tr}(V_0DV_0^\dagger)I_d - \mathrm{tr}(V_0D^\dagger V_0^\dagger)(V_0DV_0^\dagger)(V_0DV_0^\dagger)).$$

If we define

$$X(t) := V_0D(t)V_0^\dagger, \quad X(0) = V_0D_0V_0^\dagger,$$

then, $X = X(t)$ satisfies

$$\begin{aligned} &\dot{X} = \kappa(\mathrm{tr}(X)I_d - \mathrm{tr}(X^\dagger)X^2), \\ &X(0) = V_0D_0V_0^\dagger, \end{aligned}$$

which is the same governing equation (2.11) for $G(t)$. Since the initial datum G_0 of $G(t)$ is also decomposed as (2.12), the desired assertion follows from the uniqueness of the solution to (2.11). □

We use Lemma 2.5 and a well-known result for the Kuramoto synchronisation model to show that the system (2.9)–(2.10) exhibits the quantum synchronisation for generic initial data.

Theorem 2.3. *Let $\{U_1, U_2\}$ be a solution to (2.9)–(2.10). Then, for almost all initial data, there exists a constant $\theta^\infty \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} (U_1U_2^\dagger)(t) = e^{\mathrm{i}\theta^\infty} I_d.$$

Proof. It follows from Lemma 2.4 that $G = U_1U_2^\dagger$ satisfies

$$\dot{G} = \kappa(\mathrm{tr}(G)I_d - \mathrm{tr}(G^\dagger)G^2)$$

and we assume that initial datum G_0 is decomposed into $G_0 = V_0D_0V_0^\dagger$ where D_0 is diagonal and V_0 is unitary. Then, $G(t)$ is completely determined by the relation $G(t) = V_0D(t)V_0^\dagger$ where D satisfies (2.13). Since D is a $d \times d$ diagonal and unitary matrix, we can parameterise D as:

$$D = \mathrm{diag}(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_d}). \tag{2.14}$$

By substituting the representation (2.14) of D into (2.13), we obtain

$$\mathrm{i}\dot{\theta}_i e^{\mathrm{i}\theta_i} = \kappa\left(\sum_{k=1}^d e^{\mathrm{i}\theta_k} - \sum_{k=1}^d e^{-\mathrm{i}\theta_k}e^{2\mathrm{i}\theta_i}\right), \quad i \in [d]$$

or, equivalently,

$$\dot{\theta}_i = \kappa \sum_{k=1}^d \sin(\theta_k - \theta_i),$$

which is nothing but a classical Kuramoto synchronisation model. On the other hand, it is already known in [2] that for almost all initial data, there exists $\theta^\infty \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \theta_i(t) = \theta^\infty, \quad i \in [d].$$

This yields

$$\lim_{t\to\infty} D(t) = e^{\mathrm{i}\theta^\infty} I_d.$$

Finally, the relation $U_1(t)U_2(t)^\dagger = G(t) = V_0D(t)V_0^\dagger$ yields the desired convergence. □

Theorem 2.3 implies that quantum synchronisation for (1.3) with $N = 2$ occurs for generic initial data. Thus, it is naturally expected that quantum synchronisation also emerges for $N > 2$ at least under some appropriate condition on the parameters and initial data. We will show the general results for $N > 2$ in Section 3.

Finally, we close this section with the Riccati-type differential inequality.

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Lemma 2.6. *Let* $y : [0, \infty) \to \mathbb{R}_+$ *be a positive, almost everywhere differentiable function satisfying*

$$\frac{\mathrm{d}y}{\mathrm{d}t} \le -ay + by^2 + cy^3, \quad t > 0, \quad y(0) = y_0$$

where $a, b, c > 0$ *are positive constants. If*

$$y_0 < \alpha_+ := \frac{-b + \sqrt{b^2 + 4ac}}{2c},$$

then, there exists a constant $\lambda > 0$ *such that*

$$y(t) \le y_0 e^{-\lambda t}, \quad t > 0.$$

Proof. Let us consider a quadratic function $f(x) := cx^2 + bx - a$. Then, the differential inequality that y satisfies becomes

$$\frac{\mathrm{d}y}{\mathrm{d}t} \le yf(y).$$

On the other hand, note that the equation $f(x) = 0$ has two real roots $\alpha_\pm$ satisfying

$$\alpha_- := \frac{-b - \sqrt{b^2 + 4ac}}{2c} < 0 < \frac{-b + \sqrt{b^2 + 4ac}}{2c} =: \alpha_+.$$

Hence, if $0 < y_0 < \alpha_+$, then

$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=0} = y_0 f(y_0) < 0,$$

which implies $y(t)$ decreases from time $t = 0$. However, for $0 < y(t) < y_0 < \alpha_+$,

$$\frac{\mathrm{d}y}{\mathrm{d}t} \le yf(y) \le yf(y_0), \quad \text{i.e.,} \quad y(t) \le y_0 e^{f(y_0)t}.$$

Therefore, we choose $\lambda = -f(y_0) > 0$ to obtain the desired exponential decay. □

3. Emergence of quantum synchronisation of (1.3)

We already observed in the previous section that the model (1.3) with $N = 2$ exhibits quantum synchronisation for generic initial data. In this section, we present the quantum synchronisation estimate of the model (1.3)–(1.4) when $N > 2$. Precisely, we will provide a sufficient condition on the initial data and the model parameters for the quantum synchronisation of (1.3).

Motivated by the heuristic idea that the following quantity

$$\left\| U_i - \frac{1}{d} \langle U_i(t), U_j(t) \rangle U_j \right\|$$

decays to 0 as $t \to \infty$, we define the matrix F_{ij} as:

$$F_{ij} := \langle U_i, U_j \rangle I_d - dU_i U_j^\dagger,$$

which measures a degree of quantum synchronisation between U_i and U_j. Moreover, for simplicity, we also define

$$G_{ij} := U_i U_j^\dagger, \quad h_{ij} := \mathrm{tr}(G_{ij}) = \langle U_i, U_j \rangle.$$

Then, one can observe that the following relations hold

$$F_{ij} = h_{ij} I_d - dG_{ij}, \quad \mathrm{tr}(F_{ij}) = 0.$$

We start with deriving a temporal evolution of F_{ij}.

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Lemma 3.1. *Let $\{U_j\}$ be a solution to (1.3)–(1.4). Then, F_{ij} satisfies*

$$\begin{aligned}\dot{F}_{ij} &= -2\kappa d F_{ij} - \frac{\kappa}{d^2N}\sum_{k=1}^{N}\left(h_{ki}\mathrm{tr}(F_{ik}F_{ij}) + h_{jk}\mathrm{tr}(F_{ij}F_{kj})\right) I_d \\ &\quad + \frac{\kappa}{dN}\sum_{k=1}^{N}\left(\frac{1}{d}\|F_{ki}\|^2F_{ij} + \frac{1}{d}\|F_{kj}\|^2F_{ij} - h_{ki}F_{ik}F_{ij} - h_{jk}F_{ij}F_{kj}\right) \\ &\quad + \frac{\kappa}{N}\sum_{k=1}^{N}(h_{ik}F_{kj} + h_{kj}F_{ik}) - \frac{\kappa}{dN}\sum_{k=1}^{N}(h_{ki}F_{ik} + h_{jk}F_{kj})h_{ij}.\end{aligned} \tag{3.1}$$

Proof. Since the proof is too lengthy, we present it in Appendix A. □

In order to observe the quantum synchronisation, we define the radius of F_{ij}:

$$\mathcal{D}_F(t) := \max_{1\le i,j\le N}\|F_{ij}(t)\|, \quad t>0.$$

Then, it is straightforward to observe that the quantum synchronisation appears if and only if $\mathcal{D}_F$ converges to 0. In the following lemma, we estimate $\mathcal{D}_F$.

Lemma 3.2. *Let $\{U_i\}_{i=1}^N$ be a solution to (1.3)–(1.4). Then, we have*

$$\frac{\mathrm{d}\mathcal{D}_F}{\mathrm{d}t} \le -2\kappa d\mathcal{D}_F + \left(\frac{2\kappa}{\sqrt{d}} + 2\kappa\right)\mathcal{D}_F^2 + \frac{2\kappa}{d^2}\mathcal{D}_F^3, \quad t>0. \tag{3.2}$$

Proof. We use the estimate of F_{ij} in (3.1) and the fact that $\mathrm{tr}(F_{ij}) = 0$ to derive the time derivative of $\|F_{ij}\|^2$ as:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|F_{ij}\|^2 &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\Big[\mathrm{tr}(F_{ij}^\dagger F_{ij})\Big] = \mathrm{Re}\Big[\mathrm{tr}(F_{ij}^\dagger \dot{F}_{ij})\Big] \\ &= \mathrm{Re}\left[\mathrm{tr}\left(-2\kappa d F_{ij}^\dagger F_{ij} + \frac{\kappa}{N}\sum_{k=1}^{N} h_{ik}F_{ij}^\dagger F_{kj} + h_{kj}F_{ij}^\dagger F_{ik}\right)\right] \\ &\quad + \frac{\kappa}{dN}\sum_{k=1}^{N}\mathrm{Re}\left[\mathrm{tr}\left(\frac{1}{d}\|F_{ki}\|^2F_{ij}^\dagger F_{ij} + \frac{1}{d}\|F_{kj}\|^2F_{ij}^\dagger F_{ij} - h_{ki}F_{ij}^\dagger F_{ik}F_{ij} - h_{jk}F_{ij}^\dagger F_{ij}F_{kj}\right)\right] \\ &\quad - \frac{\kappa}{dN}\sum_{k=1}^{N}\mathrm{Re}\left[\mathrm{tr}\left(h_{ki}h_{ij}F_{ij}^\dagger F_{ik} + h_{jk}h_{ij}F_{ij}^\dagger F_{kj}\right)\right] \\ &= -2\kappa d\|F_{ij}\|^2 + \frac{\kappa}{N}\sum_{k=1}^{N}\mathrm{Re}\left[\mathrm{tr}\left(\left(h_{ik} - \frac{1}{d}h_{jk}h_{ij}\right)F_{ij}^\dagger F_{kj} + \left(h_{kj} - \frac{1}{d}h_{ki}h_{ij}\right)F_{ij}^\dagger F_{ik}\right)\right] \\ &\quad + \frac{\kappa}{d^2N}\sum_{k=1}^{N}\left(\|F_{ki}\|^2 + \|F_{kj}\|^2\right)\|F_{ij}\|^2 - \frac{\kappa}{dN}\sum_{k=1}^{N}\mathrm{Re}\left[\mathrm{tr}\left(h_{ki}F_{ij}^\dagger F_{ik}F_{ij} + h_{jk}F_{ij}^\dagger F_{ij}F_{kj}\right)\right].\end{aligned}$$

On the other hand, we observe that the following relation holds:

$$h_{ik} - \frac{1}{d}h_{ij}h_{jk} = \mathrm{tr}\left(\left(U_i - \frac{1}{d}h_{ij}U_j\right)U_k^\dagger\right),$$

which implies

$$\left|h_{ik} - \frac{1}{d}h_{ij}h_{jk}\right| \le \left\|U_i - \frac{1}{d}h_{ij}U_j\right\|\,\|U_k\| = \frac{1}{\sqrt{d}}\|dU_i - h_{ij}U_j\| = \frac{1}{\sqrt{d}}\|dU_iU_j^\dagger - h_{ij}I_d\| = \frac{1}{\sqrt{d}}\|F_{ij}\|,$$

where we used $\|U_k\| = \sqrt{d}$. Similarly, we also obtain

$$\left|h_{kj} - \frac{1}{d}h_{ki}h_{ij}\right| \le \frac{1}{\sqrt{d}}\|F_{ij}\|.$$

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Using $|h_{ij}| = |\mathrm{tr}(U_iU_j^\dagger)| \le d$, we finally obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|F_{ij}\|^2 \le -2\kappa d\|F_{ij}\|^2 + \frac{\kappa}{N}\sum_{k=1}^{N}\left(\frac{1}{\sqrt{d}}\|F_{ij}\|^2\|F_{kj}\| + \frac{1}{\sqrt{d}}\|F_{ik}\|\|F_{ij}\|^2\right)$$
$$+\frac{\kappa}{d^2N}\sum_{k=1}^{N}(\|F_{ki}\|^2 + \|F_{kj}\|^2)\|F_{ij}\|^2 + \frac{\kappa}{N}\sum_{k=1}^{N}\left(\|F_{ij}\|^2\|F_{ik}\| + \|F_{ij}\|^2\|F_{kj}\|\right).$$

Therefore, once we choose the indices i and j so that $\|F_{ij}\| = \mathcal{D}_F$, then we derive the desired differential inequality for $\mathcal{D}_F$:

$$\frac{\mathrm{d}\mathcal{D}_F}{\mathrm{d}t} \le -2\kappa d\mathcal{D}_F + \left(\frac{2\kappa}{\sqrt{d}} + 2\kappa\right)\mathcal{D}_F^2 + \frac{2\kappa}{d^2}\mathcal{D}_F^3. \qquad \square$$

Now we are ready to introduce our first main theorem that concerns the emergence of quantum synchronisation for (1.3). It is nothing but a corollary of Lemmas 2.6 and 3.2

Theorem 3.1. *Suppose that the initial data satisfy*

$$\mathcal{D}_F(0) = \max_{1\le i,j\le N}\|\langle U_i^0, U_j^0\rangle I_d - dU_i^0(U_j^0)^\dagger\| < \frac{2d\sqrt{d}}{\sqrt{d+2\sqrt{d}+5}+\sqrt{d}+1} \tag{3.3}$$

and let $\{U_j\}$ *be a solution to* (1.3)–(1.4). *Then, system* (1.3) *exhibits quantum synchronisation:*

$$\lim_{t\to\infty}\max_{1\le i,j\le N}\left\|U_i(t) - \frac{\langle U_i, U_j\rangle}{d}U_j(t)\right\| = 0,$$

and the decay rate is exponential.

Proof. It follows from (3.2) that the diameter $\mathcal{D}_F$ satisfies the differential inequality in Lemma 2.6 with the coefficients:

$$a = 2\kappa d, \quad b = 2\kappa\left(1+\frac{1}{\sqrt{d}}\right), \quad c = \frac{2\kappa}{d^2}.$$

After the straightforward computation, the constant α_+ in Lemma 2.6 becomes

$$\alpha_+ = \frac{2d\sqrt{d}}{\sqrt{d+2\sqrt{d}+5}+\sqrt{d}+1}.$$

Therefore, if $\mathcal{D}_F(0) < \alpha_+$, we conclude that $\mathcal{D}_F(t)$ exponentially decays to 0, which implies the desired convergence. $\square$

Remark 3.1. Since $d \ge 1$, we have

$$\alpha_+ = \frac{2d\sqrt{d}}{\sqrt{(\sqrt{d}+1)^2+4}+\sqrt{d}+1} \ge \frac{2d\sqrt{d}}{\sqrt{(\sqrt{d}+1)^2+(\sqrt{d}+1)^2}+\sqrt{d}+1}$$
$$= \frac{2d\sqrt{d}}{(\sqrt{2}+1)(\sqrt{d}+1)} \ge \frac{d(\sqrt{d}+1)}{(\sqrt{2}+1)(\sqrt{d}+1)} = (\sqrt{2}-1)d \simeq 0.414d.$$

We already observed in Lemma 2.3 that the convergence toward equilibrium is guaranteed. In the following corollary, we show that the convergence rate is indeed exponential by using Theorem 3.1.

Corollary 3.1. *Suppose that the initial data satisfy* (3.3), *and let* $\{U_j\}$ *be a solution to* (1.3)–(1.4). *Then, for each* $j \in [N]$, *there exists* $U_j^\infty \in \mathbf{U}(d)$ *such that*

$$\lim_{t\to\infty} U_j(t) = U_j^\infty.$$

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Here, the convergence rate is exponential. Moreover, there exist $\alpha_j \in \mathbb{C}$ with $|\alpha_j| = 1$ and $U^\infty \in \mathbf{U}(d)$ such that for each $j \in [N]$,

$$\lim_{t\to\infty} U_j^\infty = \alpha_j U^\infty.$$

Proof. We note that the convergence of $U_j(t)$ as $t \to \infty$ is already guaranteed from the gradient flow structure in Lemma 2.3. To show that the convergence is exponential, recall the governing equation (1.3) of U_j:

$$\begin{aligned}\dot{U}_j &= \frac{\kappa}{N}\sum_{k=1}^{N}(\langle U_j, U_k\rangle U_k - \langle U_k, U_j\rangle U_j U_k^\dagger U_j) \\ &= \frac{\kappa}{N}\sum_{k=1}^{N}(\langle U_j, U_k\rangle U_k - dU_j + dU_k U_k^\dagger U_j - \langle U_k, U_j\rangle U_j U_k^\dagger U_j).\end{aligned}$$

Then, we observe

$$\begin{aligned}\|\dot{U}_j\| &\le \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\|\langle U_j, U_k\rangle U_k - dU_j\| + \|dU_k U_k^\dagger U_j - \langle U_k, U_j\rangle U_j U_k^\dagger U_j\|\Big) \\ &= \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\|\langle U_j, U_k\rangle U_k - dU_j\| + \|dU_k - \langle U_k, U_j\rangle U_j\|\Big) \\ &= \frac{\kappa}{N}\sum_{k=1}^{N}(\|F_{jk}\| + \|F_{kj}\|) \le 2\kappa\mathcal{D}_F \le Ce^{-\lambda t}\end{aligned}$$

for some positive constants C and λ by Theorem 3.1. Therefore,

$$\left\|U_j(t) - U_j^\infty\right\| \le \int_t^\infty \|\dot{U}_j(s)\|\,\mathrm{d}s \le \frac{C}{\lambda}e^{-\lambda t},$$

which verifies the exponential convergence toward equilibrium. For the last assertion, we first note that, again by Theorem 3.1, we have

$$U_i^\infty = \frac{\langle U_i^\infty, U_j^\infty\rangle}{d}U_j^\infty,$$

which implies

$$|\langle U_i^\infty, U_j^\infty\rangle(t)| = d.$$

Therefore, for each $i, j \in [N]$, there exists a constant $\alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| = 1$ such that

$$U_j^\infty = \alpha_{ij}U_i^\infty.$$

Fix $i = 1$ and write

$$U^\infty := U_1^\infty, \quad \alpha_j := \alpha_{1j}.$$

Then, $U_j^\infty = \alpha_{1j}U_1^\infty = \alpha_j U^\infty$ for all $j \in [N]$. This completes the proof. □

Remark 3.2. As mentioned in Section 2, we can consider the non-identical model (2.5) of (1.3):

$$\dot{U}_j = -\mathrm{i}H_jU_j + \frac{\kappa}{N}\sum_{k=1}^{N}\left(\langle U_j, U_k\rangle U_k - \langle U_k, U_j\rangle U_j U_k^\dagger U_j\right). \tag{3.4}$$

In this case, the dynamics of G_{ij} becomes

$$\dot{G}_{ij} = -\mathrm{i}H_iG_{ij} + \mathrm{i}G_{ij}H_j + \frac{\kappa}{N}\sum_{k=1}^{N}\left(h_{ik}G_{kj} - h_{ki}G_{ik}G_{ij} + h_{kj}G_{ik} - h_{jk}G_{ij}G_{kj}\right),$$

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and the dynamics of h_{ij} becomes

$$\dot{h}_{ij} = -\mathrm{itr}(H_iG_{ij}) + \mathrm{itr}(G_{ij}H_j) + \frac{\kappa}{N}\sum_{k=1}^{N}\left(h_{ik}h_{kj} - h_{ki}\mathrm{tr}(G_{ik}G_{ij}) + h_{kj}h_{ik} - h_{jk}\mathrm{tr}(G_{ij}G_{kj})\right).$$

Thus, when we consider the dynamics of $\|F_{ij}\|^2$, the additional term is only

$$\mathrm{Re}\left[\mathrm{tr}\left(F_{ij}^{\dagger}\left(-\mathrm{itr}(H_iG_{ij}) + \mathrm{itr}(G_{ij}H_j)\right)I_d\right)\right] - d\mathrm{Re}\left[\mathrm{tr}\left(F_{ij}^{\dagger}(-\mathrm{i}H_iG_{ij} + \mathrm{i}G_{ij}H_j)\right)\right].$$

Using the fact that $\mathrm{tr}(F_{ij}) = 0$, the first term vanishes. On the other hand, by considering the relation $dG_{ij} = h_{ij}I_d - F_{ij}$, the second term is reformulated as:

$$\begin{aligned}&\mathrm{Re}\left[\mathrm{tr}\left(F_{ij}^{\dagger}(\mathrm{i}H_i\left(h_{ij}I_d - F_{ij}\right) - \mathrm{i}(h_{ij}I_d - F_{ij})H_j)\right)\right]\\ &= -\mathrm{Im}\left[\mathrm{tr}\left(F_{ij}^{\dagger}(H_ih_{ij} - h_{ij}H_j)\right)\right]\end{aligned}$$

which also vanishes when $H_j = a_jI_d$. Thus, newly introduced terms from the natural frequencies disappear. Therefore, the results in Theorem 3.1 and Corollary 3.1 are still valid for the non-identical model (3.4) when $H_j = a_jI_d$ with $a_j \in \mathbb{R}$.

4. Emergence of quantum synchronisation with time-delayed interaction

In this section, we show that how quantum synchronisation is robust under time-delayed interaction. Recall the time-delayed model introduced in (1.7):

$$\begin{cases}\dot{U}_j = \dfrac{\kappa}{N}\displaystyle\sum_{k=1}^{N}\langle U_j, U_k^{\tau}\rangle U_k^{\tau} - \langle U_k^{\tau}, U_j\rangle U_jU_k^{\tau,\dagger}U_j, & t > 0,\\ U_j(t) = \Phi_j(t) \in \mathbf{U}(d), \quad -\tau \le t \le 0, & j \in [N].\end{cases}$$

Although time-delayed interaction is employed, the unitary property of $\{U_j\}$ is still guaranteed.

Lemma 4.1. *Let $\{U_j\}$ be a solution to (1.7)–(1.8). Then, we have*

$$U_i(t)U_i^{\dagger}(t) = I_d, \quad t > 0, \quad i \in [N].$$

Proof. Once we notice that the right-hand side of (1.7) can be represented in terms of the projection formula (2.4):

$$\frac{\kappa}{N}\sum_{k=1}^{N}\langle U_j, U_k^{\tau}\rangle U_k^{\tau} - \langle U_k^{\tau}, U_j\rangle U_jU_k^{\tau,\dagger}U_j = \mathbb{P}_{U_j}(X), \quad X := \frac{2\kappa}{N}\sum_{k=1}^{N}\langle U_j, U_k^{\tau}\rangle U_k^{\tau}$$

then the positive invariance directly follows. □

We first study some elementary lemmas which will be crucially used for the quantum synchronisation for the time-delayed model (1.7). We define a delayed fluctuation for U_i:

$$\Delta_i^{\tau}(t) := \|U_i(t) - U_i^{\tau}(t)\|, \quad i \in [N]. \tag{4.1}$$

Then, we can show that the fluctuation is uniformly bounded by $\mathcal{O}(1)\tau$.

Lemma 4.2. *Let $\{U_j\}$ be a solution to (1.7)–(1.8). Then, for any $t > 0$, we have*

$$\|\Delta_i^{\tau}(t)\| \le M\tau, \quad M := \max\{\|\Phi_i\|_{\mathrm{Lip}}, 2\kappa d\sqrt{d}\} > 0. \tag{4.2}$$

Proof. We basically follow the proof in [5, Lemma 4.1]. By integrating (1.7) over $[(t-\tau)_+, t]$ for $t > 0$ to find

$$U_i(t) - U_i((t-\tau)_+) = \frac{\kappa}{N}\sum_{k=1}^{N}\int_{(t-\tau)_+}^{t}\left[\langle U_j, U_k^{\tau}\rangle U_k^{\tau} - \langle U_k^{\tau}, U_j\rangle U_jU_k^{\tau,\dagger}U_j\right]\mathrm{d}s.$$

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Here, we used the notation $x_+ := \max\{x, 0\}$. Then, we observe

$$\|\langle U_j, U_k^\tau\rangle U_k^\tau - \langle U_k^\tau, U_j\rangle U_j U_k^{\tau,\dagger} U_j\| \le 2d\sqrt{d}$$

to obtain

$$\begin{aligned}\|U_i(t) - U_i(t-\tau)\| &\le \|U_i(t) - U_i((t-\tau)_+)\| + \|U_i((t-\tau)_+) - U_i(t-\tau)\| \\ &\le \frac{\kappa}{N}\sum_{k=1}^N \int_{(t-\tau)_+}^t \|\langle U_j, U_k^\tau\rangle U_k^\tau - \langle U_k^\tau, U_j\rangle U_j U_k^{\tau,\dagger} U_j\|\, \mathrm{d}s \\ &\quad + ((t-\tau)_+ - (t-\tau))\|\Phi_i\|_{\mathrm{Lip}} \\ &\le 2(t-(t-\tau)_+)\kappa d\sqrt{d} + ((t-\tau)_+ - (t-\tau))\|\Phi_i\|_{\mathrm{Lip}} \le M\tau. \qquad \square\end{aligned}$$

Next, we define quantities that measure a degree of delayed quantum synchronisation, which are analogous to those introduced in Section 3:

$$G_{ij}^\tau(t) := U_i(t)U_j^{\tau,\dagger}(t) = U_i(t)U_j^\dagger(t-\tau), \quad h_{ij}^\tau = \mathrm{tr}(G_{ij}^\tau), \quad F_{ij}^\tau = h_{ij}^\tau I_d - dG_{ij}^\tau. \tag{4.3}$$

For this notation, we observe

$$G_{ij}^{\tau,\dagger}(t) = U_j(t-\tau)U_i^\dagger(t).$$

Similar to Lemma 4.2, we need to estimate the fluctuations for F_{ij}.

Lemma 4.3. *Let $\{U_j\}$ be a solution to (1.7)–(1.8). Then, for any $t > 0$, the following estimates hold*

$$\text{(i)}\ \|F_{ij}(t) - F_{ij}^\tau(t)\| \le 2dM\tau. \qquad \text{(ii)}\ \big|\|F_{ij}(t)\|^2 - \|F_{ij}^\tau(t)\|^2\big| \le 2d^2\sqrt{d}M\tau.$$

Here, M is the constant introduced in (4.2).

Proof. (i) We use Lemma 4.2 to find

$$\begin{aligned}\|F_{ij}(t) - F_{ij}^\tau(t)\| &= \big\|\langle U_i(t), U_j(t)\rangle I_d - dU_i(t)U_j^\dagger(t) - \langle U_i(t), U_j^\tau(t)\rangle I_d + dU_i(t)U_j^{\tau,\dagger}(t)\big\| \\ &\le \|\langle U_i(t), U_j(t) - U_j^\tau(t)\rangle I_d\| + d\|U_i(t)(U_j(t) - U_j^{\tau,\dagger}(t))\| \\ &\le dM\tau + dM\tau = 2dM\tau.\end{aligned}$$

(ii) We first observe

$$\big|\|F_{ij}(t)\|^2 - \|F_{ij}^\tau(t)\|^2\big| = |\mathrm{tr}(F_{ij}F_{ij}^\dagger - F_{ij}^\tau F_{ij}^{\tau,\dagger})|.$$

On the other hand, we have

$$\begin{aligned}\mathrm{tr}(F_{ij}F_{ij}^\dagger) &= \mathrm{tr}\Big[(\langle U_i, U_j\rangle I_d - dU_iU_j^\dagger)(\langle U_j, U_i\rangle I_d - dU_jU_i^\dagger)\Big] \\ &= \mathrm{tr}\Big[|\langle U_i, U_j\rangle|^2 I_d - d\langle U_j, U_i\rangle U_iU_j^\dagger - d\langle U_i, U_j\rangle U_jU_i^\dagger + d^2 I_d\Big] \\ &= d^3 - d|\langle U_i, U_j\rangle|^2,\end{aligned}$$

and by using the same argument, we see

$$\mathrm{tr}(F_{ij}^\tau F_{ij}^{\tau,\dagger}) = d^3 - d|\langle U_i, U_j^\tau\rangle|^2.$$

However, since

$$||\langle U_i, U_j\rangle|^2 - |\langle U_i, U_j^\tau\rangle|^2| = d(|\langle U_i, U_j\rangle| + |\langle U_i, U_j^\tau\rangle|)|\langle U_i, U_j - U_j^\tau\rangle| \le 2d\sqrt{d}M\tau,$$

we obtain the desired control on the difference between $\|F_{ij}\|^2$ and $\|F_{ij}^\tau\|^2$:

$$\big|\|F_{ij}(t)\|^2 - \|F_{ij}^\tau(t)\|^2\big| \le 2d^2\sqrt{d}M\tau. \qquad \square$$

Next, as in Lemma 3.1, we derive a temporal evolution for F_{ij}.

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Lemma 4.4. *Let $\{U_j\}$ be a solution to (1.7)–(1.8). Then, F_{ij} satisfies*

$$\begin{aligned}\dot{F}_{ij} &= -2d\kappa F_{ij} - \frac{\kappa}{d^2N}\sum_{k=1}^{N}\left(\overline{h_{ik}^{\tau}}\mathrm{tr}(F_{ik}^{\tau}F_{ij})I_d + h_{jk}^{\tau}\mathrm{tr}(F_{ij}F_{jk}^{\tau,\dagger})I_d\right) \\ &\quad + \frac{\kappa}{N}\sum_{k=1}^{N}\left[\left(h_{ik}^{\tau} - \frac{1}{d}h_{ij}h_{jk}^{\tau}\right)F_{jk}^{\tau,\dagger} + \left(\overline{h_{jk}^{\tau}} - \frac{1}{d}\overline{h_{ik}^{\tau}}h_{ij}\right)F_{ik}^{\tau}\right] \\ &\quad + \frac{\kappa}{d^2N}\sum_{k=1}^{N}\left(\|F_{ki}^{\tau}\|^2 + \|F_{jk}^{\tau}\|^2\right)F_{ij} + \frac{\kappa}{dN}\sum_{k=1}^{N}\left(\overline{h_{ik}^{\tau}}F_{ik}^{\tau}F_{ij} + h_{jk}^{\tau}F_{ij}F_{jk}^{\tau,\dagger}\right).\end{aligned} \tag{4.4}$$

Proof. The only difference between (1.3) and (1.7) is that $U_k(t)$ in (1.3) is replaced with $U_k^{\tau}(t) = U_k(t-\tau)$. Thus, it suffices to replace the terms U_k involving dummy index k in (3.1) by U_k^{τ}. In particular, we only need to replace the terms whose first index is k such as h_{ki}, F_{ki}, h_{kj} and F_{kj}. For instance, by using (4.3), we observe that $h_{ki}(t) = \mathrm{tr}(G_{ki}(t)) = \mathrm{tr}(U_k(t)U_i^{\dagger}(t))$ becomes $\mathrm{tr}(U_k(t-\tau)U_i^{\dagger}(t)) = \mathrm{tr}(G_{ik}^{\tau,\dagger}(t)) = \overline{h_{ik}^{\tau}}$. Therefore, h_{ki} in (3.1) is now replaced by $\overline{h_{ik}^{\tau}}$. Similarly, F_{ki} becomes $F_{ik}^{\tau,\dagger}$. After the replacement, the governing equation of F_{ij} for the time-delayed model (1.7) becomes (4.4). □

Next, by following Lemma 3.2, we derive the differential inequality for $\mathcal{D}_F$.

Lemma 4.5. *Let $\{U_j\}_{j=1}^N$ be a solution to (1.7)–(1.8). Then, $\mathcal{D}_F$ satisfies*

$$\frac{\mathrm{d}\mathcal{D}_F}{\mathrm{d}t} \le -2\kappa d(1-(2+d\sqrt{d})M\tau)\mathcal{D}_F + 2\kappa\left(1+\frac{1}{\sqrt{d}}\right)\mathcal{D}_F^2 + \frac{2\kappa}{d^2}\mathcal{D}_F^3.$$

Proof. From Lemma 4.4, $\|F_{ij}\|^2$ satisfies

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|F_{ij}\|^2 &= -2d\kappa\|F_{ij}\|^2 + \frac{\kappa}{N}\sum_{k=1}^{N}\mathrm{Re}\Big[(h_{ik}^{\tau} - \frac{1}{d}h_{ij}h_{jk}^{\tau})\mathrm{tr}(F_{jk}^{\tau,\dagger}F_{ji}) + (\overline{h_{jk}^{\tau}} - \frac{1}{d}\overline{h_{ik}^{\tau}}h_{ij})\mathrm{tr}(F_{ik}^{\tau}F_{ji})\Big] \\ &\quad + \frac{\kappa}{d^2N}\sum_{k=1}^{N}(\|F_{ik}^{\tau}\|^2 + \|F_{jk}^{\tau}\|^2)\|F_{ij}\|^2 \\ &\quad + \frac{\kappa}{dN}\sum_{k=1}^{N}\mathrm{Re}\Big[\overline{h_{ik}^{\tau}}\mathrm{tr}(F_{ik}^{\tau}F_{ij}F_{ji}) + h_{jk}^{\tau}\mathrm{tr}(F_{ij}F_{jk}^{\tau,\dagger}F_{ji})\Big].\end{aligned}$$

Here, we observe

$$\begin{aligned}h_{ik}^{\tau} - \frac{1}{d}h_{ij}h_{jk}^{\tau} &= \mathrm{tr}\left(U_iU_k^{\tau,\dagger} - \frac{1}{d}\langle U_i, U_j\rangle U_jU_k^{\tau,\dagger}\right) \\ &= \mathrm{tr}\left(\left(U_iU_j^{\dagger} - \frac{1}{d}\langle U_i, U_j\rangle\right)U_jU_k^{\tau,\dagger}\right) \le \frac{1}{\sqrt{d}}\|F_{ij}\|.\end{aligned}$$

Similarly, we have

$$\overline{h_{jk}^{\tau}} - \frac{1}{d}\overline{h_{ik}^{\tau}}h_{ij} \le \frac{1}{\sqrt{d}}\|F_{ij}\|.$$

Hence, we derive a differential inequality for $\|F_{ij}\|$:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|F_{ij}\|^2 &\le -2d\kappa\|F_{ij}\|^2 + \frac{\kappa}{\sqrt{d}N}\sum_{k=1}^{N}\left(\|F_{ij}\|^2\|F_{kj}^{\tau}\| + \|F_{ij}\|^2\|F_{ik}^{\tau}\|\right) \\ &\quad + \frac{\kappa}{d^2N}\sum_{k=1}^{N}\left((\|F_{ki}^{\tau}\|^2 + \|F_{jk}^{\tau}\|^2)\|F_{ij}\|^2\right) + \frac{\kappa}{N}\sum_{k=1}^{N}\left(\|F_{ik}^{\tau}\|\|F_{ij}\|^2 + \|F_{kj}^{\tau}\|\|F_{ij}\|^2\right).\end{aligned}$$

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We now choose the indices i and j so that $\|F_{ij}\| = \mathcal{D}_F$, and use Lemma 4.3 to conclude that $\mathcal{D}_F$ satisfies

$$\frac{\mathrm{d}\mathcal{D}_F}{\mathrm{d}t} \leq -2\kappa d(1-(2+d\sqrt{d})M\tau)\mathcal{D}_F + 2\kappa\left(1+\frac{1}{\sqrt{d}}\right)\mathcal{D}_F^2 + \frac{2\kappa}{d^2}\mathcal{D}_F^3. \qquad \square$$

We now derive the quantum synchronisation of the time-delayed model (1.7) by combining the differential inequality for $\mathcal{D}_F$ and Lemma 2.6.

Theorem 4.1. *Suppose that system parameters and initial data satisfy*

$$0 \leq \tau < \frac{1}{(2+d\sqrt{d})M}, \quad M = \max\{\|\Phi_i\|_{\mathrm{Lip}}, 2\kappa d\sqrt{d}\},$$
$$\mathcal{D}_F(0) < \frac{2d\sqrt{d}(1-(2+d\sqrt{d})M\tau)}{\sqrt{d+2\sqrt{d}+1+4(1-(2+d\sqrt{d})M\tau)}+1+\sqrt{d}},$$

and let $\{U_j\}$ *be a solution to* (1.7)–(1.8). *Then, system* (1.7) *exhibits quantum synchronisation:*

$$\lim_{t\to\infty}\max_{1\leq i,j\leq N}\left\|U_i(t) - \frac{\langle U_i, U_j\rangle(t)}{d}U_j(t)\right\| = 0.$$

Proof. The proof is a direct corollary of Lemma 2.6 and 4.5 with the coefficients

$$a = 2\kappa d(1-(2+d\sqrt{d})M\tau), \quad b = 2\kappa\left(1+\frac{1}{\sqrt{d}}\right), \quad c = \frac{2\kappa}{d^2}. \qquad \square$$

Remark 4.1. When $\tau = 0$, the result of Theorem 4.1 reduces to the result of Theorem 3.1.

5. Numerical simulation

In this section, we numerically simulate (1.3) and (1.7) to observe whether the quantum synchronisation appears or not. We randomly generate the initial data U_j^0 or $\Phi_j(t) \equiv U_j^0$ for $-\tau \leq t \leq 0$ over the unitary group $\mathbf{U}(d)$ and solve the ordinary/delayed differential equations by using the fourth-order Runge–Kutta method. To verify the emergence of quantum synchronisation, we observe the two quantities. First, we observe the potential $\mathcal{V}$ defined in (2.3). Instead of directly tracking the dynamics of $\mathcal{V}$, we rescale it as:

$$\widetilde{\mathcal{V}}(\mathcal{U}) := \frac{1}{d^2N^2}\sum_{k,\ell=1}^{N}(d^2 - |\langle U_k, U_\ell\rangle|^2),$$

so that the value of the potential function lies in [0, 1], regardless of the dimension or the number of particles. We choose $N = 20$ and $\kappa = 1$ and then observe $\widetilde{\mathcal{V}}$ for different values of dimension d in Figure 1.

We observe that regardless of the dimension, the rescaled potential decays to 0 exponentially fast, implying that the system converges to the equilibrium $\{U_j^\infty\}$, where the relation $U_i = \alpha_{ij}U_j$ with $|\alpha_{ij}| = 1$ holds.

Second, we also present the values α_j defined as $U_j^\infty = \alpha_j U_1^\infty$ in Figure 2. As we showed theoretically, α_j are on the unit circle of the complex plane, which also implies that the system reached the minimiser of the potential.

We also conduct numerical simulations for the time-delayed model (1.7) and report the dynamics of rescaled potential $\widetilde{\mathcal{V}}$ and illustrate the results in Figure 3. The numerical simulation results show that although the time delay effect slows down the decay of the potential, and in particular, the larger time delay implies slower convergence, the system eventually reaches equilibrium. Through the simulation, we numerically observe that the conditions in Theorems 3.1 and 4.1 are technical assumptions for proving the quantum synchronisation, and the quantum synchronisation is indeed observed for a generic initial data, as we proved for the case of $N = 2$.

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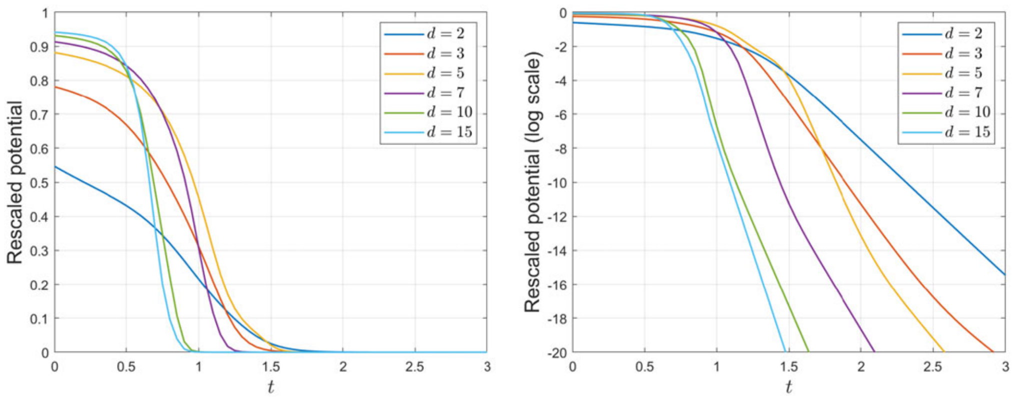

Figure 1. *The evolution of rescaled potential* $\widetilde{\mathcal{V}}$ *with original scale (left) and log scale (right). The potential exponentially decays to 0.*

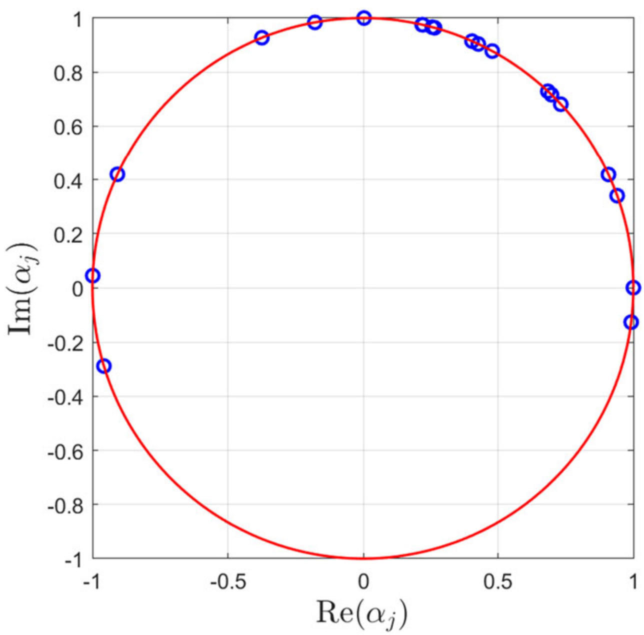

Figure 2. *The values of* α_j*. Each blue dot represents a single value of* α_j*, while the red line denotes the unit circle.*

6. Conclusion

In the present paper, we introduce a modified synchronisation model on the unitary group, which shows a qualitatively different asymptotic behaviour compared to the previous standard synchronisation model on the unitary group. To illustrate the new asymptotic behaviour, we introduce the notion of quantum synchronisation and prove that our model exhibits quantum synchronisation under sufficient conditions on the initial data and model parameters. We also extend the quantum synchronisation analysis to the model with time-delayed interactions and verify the theoretical results by numerical simulations, showing that the quantum synchronisation indeed emerges for generic initial data. Recently, a mean-field limit and kinetic description of the synchronisation models have been widely investigated [16, 18]. Therefore,

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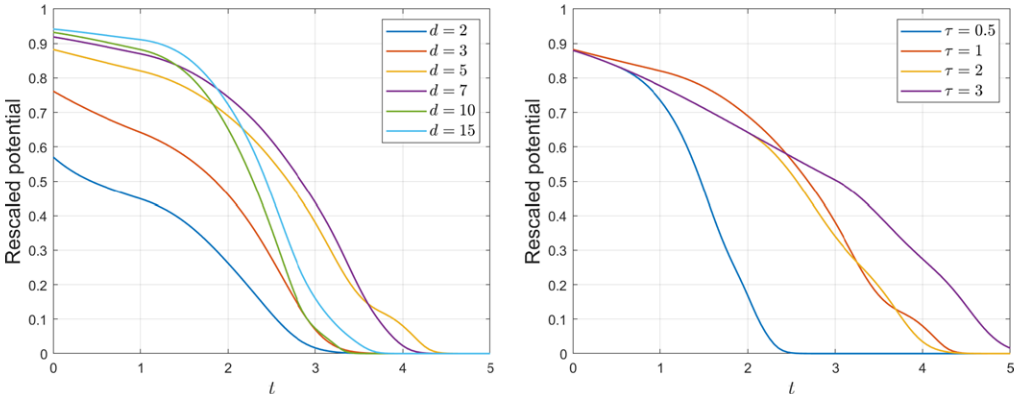

Figure 3. *The evolution of rescaled potential $\widetilde{\mathcal{V}}$ with different dimension with $\tau = 1$ (left) and different time delay with $d = 5$ (right).*

one may extend the quantum synchronisation model to the kinetic level, and this will be one of the possible future perspectives.

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Competing interests. We state that there is no competing interest.

References

[1] Bellomo, B., Giorgi, G. L., Palma, G. M. & Zambrini, R. (2017) Quantum synchronization as a local signature of super- and subradiance. *Phys. Rev. A* **95** (4), 043807.
[2] Dong, J.-G. & Xue, X. (2013) Synchronization analysis of Kuramoto oscillators. *Commun. Math. Sci.* **11** (2), 465–480.
[3] Goychuk, I., Casado-Pascual, J., Morillo, M., Lehmann, J. & Hänggi, P. (2006) Quantum stochastic synchronization. *Phys. Rev. Lett.* **97** (21), 210601.
[4] Giorgi, G. L., Galve, F., Manzano, G., Colet, P. & Zambrini, R. (2012) Quantum correlations and mutual synchronization. *Phys. Rev. A* **85** (5), 052101.
[5] Ha, S.-Y., Kim, D., Kim, D., Park, H. & Shim, W. (2020) Emergent dynamics of the Lohe matrix ensemble on a network under time-delayed interactions. *J. Math. Phys.* **61** (1), 012702.
[6] Ha, S.-Y., Ko, D. & Ryoo, S.-Y. (2018) On the relaxation dynamics of Lohe oscillators on some Riemmanian manifolds. *J. Stat. Phys.* **172** (5), 1427–1428.
[7] Ha, S.-Y. & Ryoo, S.-Y. (2016) On the emergence and orbital stability of phase-locked states for the Lohe model. *J. Stat. Phys.* **163** (2), 411–439.
[8] Kimble, H. J. (2008) The quantum internet. *Nature* **453** (7198), 1023–1030.
[9] Kim, D. & Kim, J. (2022) On the emergent behavior of the swarming models on the complex sphere. *Stud. Appl. Math.* **22** (3), 1303–1338.
[10] Kim, D. & Kim, J. (2024) Low-dimensional reduction of the non–Abelian quantum synchronization models on the unitary group. *Kinet. Relat. Models*. **17** (3), 436–467.
[11] Kim, D. & Park, H. Asymptotic convergence of heterogeneous first-order aggregation models: From the sphere to the unitary group. archived as arXiv: 2206.00984.
[12] Kuramoto, Y. (1975). Self-entrainment of a population of coupled non-linear oscillators. In *International Symposium on Mathematical Problems in Mathematical Physics*, Lecture notes in theoretical physics, Vol. **39**, pp. 420–422. doi:10.1007/BFb0013365.
[13] Lohe, M. A. (2010) Quantum synchronization over quantum networks. *J. Phys. A* **43** (46), 465301.
[14] Lohe, M. A. (2009) Non-Abelian Kuramoto model and synchronization. *J. Phys. A* **42** (39), 395101.
[15] Machida, M., Kano, T., Yamada, S., Okumura, M., Imamura, T. & Koyama, T. (2008) Quantum synchronization effects in intrinsic Josephson junctions. *Phys. C* **468** (7-10), 689–694.
[16] Morales, J. & Poyato, D. (2023) On the trend to global equilibrium for Kuramoto oscillators. *Ann. I. H. Poincaré-AN* **40**, 631–716.
[17] Mari, A., Farace, A., Didier, N., Giovannetti, V. & Fazio, R. (2013) Measures of quantum synchronization in continuous variable systems. *Phys. Rev. Lett.* **111** (10), 103605.

[18] Peszek, J. & Poyato, D. (2023) Heterogeneous gradient flows in the topology of fibered optimal transport. *Calc. Var. Partial Differ. Equ.* **62**, 258.
[19] Palma, G. M., Vaglica, A., Leonardi, C., De Oliveria, F. A. M. & Knight, P. L. (1990) Effects of broad-band squeezing on the quantum onset of superradiance. *Opti. Commun.* **79**, 377–380.
[20] Vinokur, V. M., Baturina, T. I., Fistul, M. V., Mironov, A. Y., Baklanov, M. R. & Strunk, C. (2008) Superinsulator and quantum synchronization. *Nature* **452** (7187), 613–616.
[21] Vicsek, T., Czirók, A., Ben-Jacob, E., Cohen, I. & Shochet, O. (1995) Novel type of phase transition in a system of self-driven particles. *Phys. Rev. Lett.* **75**, 1226–1229.
[22] Winfree, A. T. (1967) Biological rhythms and the behavior of populations of coupled oscillators. *J. Theor. Biol.* **16** (1), 15–42.
[23] Zhirov, O. V. & Shepelyansky, D. L. (2009) Quantum synchronization and entanglement of two qubits coupled to a driven dissipative resonator. *Phys. Rev. B* **80** (1), 014519.

A. Proof of Lemma 3.1

In this appendix, we present the proof of Lemma 3.1. Recall the dynamics of U_j in (1.3):

$$\dot{U}_j = \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\langle U_j, U_k\rangle U_k - \langle U_k, U_j\rangle U_j U_k^\dagger U_j\Big).$$

Then, $G_{ij} = U_i U_j^\dagger$ satisfies

$$\begin{aligned}\dot{G}_{ij} &= \dot{U}_i U_j^\dagger + U_i \dot{U}_j^\dagger \\ &= \frac{\kappa}{N}\sum_{k=1}^{N}\Big(\langle U_i, U_k\rangle U_k U_j^\dagger - \langle U_k, U_i\rangle U_i U_k^\dagger U_i U_j^\dagger \\ &\qquad\qquad + \langle U_k, U_j\rangle U_i U_k^\dagger - \langle U_j, U_k\rangle U_i U_j^\dagger U_k U_j^\dagger\Big) \\ &= \frac{\kappa}{N}\sum_{k=1}^{N}\Big(h_{ik}G_{kj} - h_{ki}G_{ik}G_{ij} + h_{kj}G_{ik} - h_{jk}G_{ij}G_{kj}\Big).\end{aligned} \tag{A.1}$$

Taking the trace to (A.1), we also obtain the dynamics of h_{ij} as:

$$\dot{h}_{ij} = \frac{\kappa}{N}\sum_{k=1}^{N}\big(h_{ik}h_{kj} - h_{ki}\mathrm{tr}(G_{ik}G_{ij}) + h_{kj}h_{ik} - h_{jk}\mathrm{tr}(G_{ij}G_{kj})\big). \tag{A.2}$$

We use the relation $F_{ij} = h_{ij}I_d - dG_{ij}$ and the equations (A.1) and (A.2) to obtain the governing equation for F_{ij}:

$$\begin{aligned}\dot{F}_{ij} &= \dot{h}_{ij}I_d - d\dot{G}_{ij} \\ &= \frac{\kappa}{N}\sum_{k=1}^{N}\big(h_{ik}h_{kj} - h_{ki}\mathrm{tr}(G_{ik}G_{ij}) + h_{kj}h_{ik} - h_{jk}\mathrm{tr}(G_{ij}G_{kj})\big) I_d \\ &\quad - \frac{d\kappa}{N}\sum_{k=1}^{N}\big(h_{ik}G_{kj} - h_{ki}G_{ik}G_{ij} + h_{kj}G_{ik} - h_{jk}G_{ij}G_{kj}\big) \\ &=: \mathcal{I}_1 + \mathcal{I}_2.\end{aligned}$$

Since $dG_{ij} = -F_{ij} + h_{ij}I_d$, we rewrite $\mathcal{I}_2$ as:

$$\begin{aligned}\mathcal{I}_2 &= -\frac{\kappa}{N}\sum_{k=1}^{N}(h_{ik}(-F_{kj} + h_{kj}I_d) + h_{kj}(-F_{ik} + h_{ik}I_d)) \\ &\quad + \frac{\kappa}{dN}\sum_{k=1}^{N}\Big(h_{ki}(-F_{ik} + h_{ik}I_d)\left(-F_{ij} + h_{ij}I_d\right) + h_{jk}\left(-F_{ij} + h_{ij}I_d\right)(-F_{kj} + h_{kj}I_d)\Big).\end{aligned}$$

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Thus, we calculate $\mathcal{I}_1 + \mathcal{I}_2$ as:

$$\begin{aligned}
\mathcal{I}_1 + \mathcal{I}_2 &= -\frac{\kappa}{N}\sum_{k=1}^{N}\left(h_{ki}\mathrm{tr}(G_{ik}G_{ij}) + h_{jk}\mathrm{tr}(G_{ij}G_{kj})\right) I_d + \frac{\kappa}{N}\sum_{k=1}^{N}(h_{ik}F_{kj} + h_{kj}F_{ik}) \\
&\quad + \frac{\kappa}{dN}\sum_{k=1}^{N}(|h_{ki}|^2 I_d - h_{ki}F_{ik})\left(-F_{ij} + h_{ij}I_d\right) + \left(-F_{ij} + h_{ij}I_d\right)(|h_{kj}|^2 I_d - h_{jk}F_{kj}) \\
&= -2\kappa dF_{ij} - \frac{\kappa}{N}\sum_{k=1}^{N}\left(h_{ki}\mathrm{tr}(G_{ik}G_{ij}) + h_{jk}\mathrm{tr}(G_{ij}G_{kj})\right) I_d + \frac{\kappa}{N}\sum_{k=1}^{N}(h_{ik}F_{kj} + h_{kj}F_{ik}) \\
&\quad + \frac{\kappa}{dN}\sum_{k=1}^{N}((d^2 - |h_{ki}|^2)F_{ij} + (d^2 - |h_{kj}|^2)F_{ij} + h_{ki}F_{ik}F_{ij} + h_{jk}F_{ij}F_{kj}) \\
&\quad + \frac{\kappa}{dN}\sum_{k=1}^{N}(|h_{ki}|^2 I_d + |h_{kj}|^2 I_d - h_{ki}F_{ik} - h_{jk}F_{kj})h_{ij}.
\end{aligned}$$

On the other hand, we note that

$$\begin{aligned}
\|F_{ij}\|^2 &= \mathrm{Re}\left[\mathrm{tr}(F_{ij}^\dagger F_{ij})\right] = \mathrm{Re}\left[\mathrm{tr}\left(\left(\overline{h_{ij}}I_d - dG_{ij}^\dagger\right)\left(h_{ij}I_d - dG_{ij}\right)\right)\right] \\
&= \mathrm{Re}\left[\mathrm{tr}\left(|h_{ij}|^2 I_d - dh_{ij}G_{ij}^\dagger - d\overline{h_{ij}}G_{ij} + d^2 I_d\right)\right] \\
&= d^3 - d|h_{ij}|^2,
\end{aligned}$$

and

$$\mathrm{tr}(G_{ik}G_{ij}) = \frac{1}{d^2}\left(\mathrm{tr}(F_{ik}F_{ij}) - h_{ik}\mathrm{tr}(F_{ij}) - h_{ij}\mathrm{tr}(F_{ik}) + dh_{ik}h_{ij}\right) = \frac{1}{d^2}\mathrm{tr}(F_{ik}F_{ij}) + \frac{1}{d}h_{ik}h_{ij},$$

where we used $\mathrm{tr}(F_{ij}) = 0$. Thus, we substitute the above two relations into the estimate of $\mathcal{I}_1 + \mathcal{I}_2$ to obtain the desired equation for F_{ij}:

$$\begin{aligned}
\dot{F}_{ij} = \mathcal{I}_1 + \mathcal{I}_2 &= -2\kappa dF_{ij} - \frac{\kappa}{d^2N}\sum_{k=1}^{N}\left(h_{ki}\mathrm{tr}(F_{ik}F_{ij}) + h_{jk}\mathrm{tr}(F_{ij}F_{kj})\right) I_d \\
&\quad + \frac{\kappa}{N}\sum_{k=1}^{N}\left(h_{ik}F_{kj} + h_{kj}F_{ik}\right) \\
&\quad + \frac{\kappa}{dN}\sum_{k=1}^{N}\left(\frac{1}{d}\|F_{ki}\|^2 F_{ij} + \frac{1}{d}\|F_{kj}\|^2 F_{ij} + h_{ki}F_{ik}F_{ij} + h_{jk}F_{ij}F_{kj}\right) \\
&\quad - \frac{\kappa}{dN}\sum_{k=1}^{N}(h_{ki}F_{ik} + h_{jk}F_{kj})h_{ij}.
\end{aligned}$$

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