ON A THEOREM OF BEURLING AND LIVINGSTON

FELIX E. BROWDER

Introduction. In their paper **(1)**, Beurling and Livingston established a generalization of the Riesz–Fischer theorem for Fourier series in L^p using a theorem on duality mappings of a Banach space B into its conjugate space B^*. It is our purpose in the present paper to give another proof of this theorem by deriving it from a more general result concerning monotone mappings related to recent results on non-linear functional equations in Banach spaces obtained by the writer **(2, 3, 4, 5)** and G. J. Minty **(6)**.

In Section 1, we establish our basic results on monotone operators. In Section 2, we derive the Beurling–Livingston theorem for strictly convex reflexive spaces from the results of Section 1.

1. Let B be a real Banach space, B^* its conjugate space, (w, u) the pairing between an element w of B^* and u of B.

DEFINITION 1.1. *If T is a mapping (not necessarily linear) of B into B^*, T is said to be monotone if*

$$(Tu - Tv, u - v) \geqslant 0$$

for all u and v in B.

DEFINITION 1.2. *The mapping T from B to B^* is said to be hemi-continuous if it is continuous from every segment in B to the weak* topology in B^*.*

THEOREM 1. *Let T be a hemi-continuous monotone mapping of the reflexive Banach space B into B^*. Suppose that there exists a function $c(r)$ on R^1 with $c(r) \to +\infty$ as $r \to +\infty$ such that*

$$(Tu, u) \geqslant c(||u||)\{||u|| + ||Tu||\}$$

for all u in B.

Then if C is any closed subspace of B, $C^\perp$ its annihilator in B^*, h an element of B, k an element of B^*, the set

$$T(C + h) \cap (C^\perp + k)$$

is non-empty.

THEOREM 2. *If in addition to the hypothesis of Theorem* 1,

$$(Tu - Tv, u - v) > 0$$

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for all pairs u, v in B with $u \neq v$, then there exists exactly one element v in $T(C + h) \cap (C^{\perp} + k)$.

We shall derive the results stated in Theorems 1 and 2 from the following:

THEOREM 3. *Let T be a hemi-continuous monotone mapping of the reflexive Banach space B into B^* such that there exists a function $c(r)$ on R^1 with $c(r) \to +\infty$ as $r \to +\infty$ such that for all u in B*

$$(Tu, u) \geqslant c(||u||)||u||.$$

Then T maps B onto B^*.

Theorem 3 has been established by the writer in **(2** and **5)** (see **(3** and **4)** for more general results on densely defined operators) and by G. J. Minty in **(6)**.

Proof of Theorem 1. If C is a closed subspace of B, C is a Banach space with respect to the norm of B. Let j be the injection map of C into B. Then j^* maps B^* onto C^*.

We define a mapping W of C into C^* as follows:

$$Wu = j^*T(u + h), \qquad u \in C.$$

We assert that

$$T(C + h) \cap (C^{\perp} + k) \neq \emptyset$$

if and only if j^*k lies in the range of W. Indeed suppose that there exists an element w in $T(C + h) \cap (C^{\perp} + k)$, i.e.

$$\begin{cases} w - k = w_1 \in C^{\perp}, \\ w = T(u + h), \qquad u \in C. \end{cases}$$

Then for all v in C

$$(T(u + h) - k, v) = (w_1, v) = 0,$$

i.e. $j^*\{T(u + h) - k\} = 0$. Hence $j^*T(u + h) = Wu = j^*k$. For the converse, we see similarly that if $Wu = j^*k$, then for all v in C

$$(T(u + h) - k, v) = 0,$$

i.e.

$$T(u + h) \in (C^{\perp} + k).$$

To establish the conclusion of Theorem 1, we need therefore only to show that the range of W is all of C^*. We establish this fact by applying Theorem 3 to the reflexive Banach space C and the mapping W. We remark that:

(i) W is hemi-continuous since T is hemi-continuous and j^* is weak* continuous.

(ii) For u and v in C

$$\begin{aligned}(Wu - Wv, u - v) &= (j^*T(u + h) - j^*T(v + h), u - v)\\ &= (T(u + h) - T(v + h), u - v)\\ &= (T(u + h) - T(v + h), (u + h) - (v + h)) \geqslant 0,\end{aligned}$$

i.e. W is monotone.

(iii) For u in C,

$$\begin{aligned}(Wu, u) &= (j^*T(u + h), u)\\ &= (T(u + h), u) = (T(u + h), u + h) - (T(u + h), h)\\ &\geqslant c(||u + h||)||u + h|| + \{c(||u + h||) - c_0\}||T(u + h)||\}.\end{aligned}$$

For $||u|| \geqslant M$, $c(||u + h||) - c_0 \geqslant 0$. Hence for such u

$$\begin{aligned}(Wu, u) &\geqslant c(||u + h||)||u + h||\\ &\geqslant c(||u + h||)\{||u|| - ||h||\}\\ &\geqslant \tfrac{1}{2}c(||u + h||)||u||\end{aligned}$$

for $||u|| \geqslant 2||h||$. If we set

$$c_1(r) = \tfrac{1}{2}\inf_{||u||=r} c(||u + h||),$$

it follows that $c_1(r) \to +\infty$ as $r \to \infty$ and

$$(Wu, u) \geqslant c_1(||u||)||u||$$

for all u in C.

Thus we see that all the hypotheses of Theorem 3 have been verified, W maps C onto C^*, and the conclusion of Theorem 1 follows by our previous remarks.

Proof of Theorem 2. For each w in $T(C + h) \cap (C^\perp + k)$, there exists u in C such that

$$\begin{cases} w = T(u + h),\\ w - k \in C.\end{cases}$$

If T satisfies the hypothesis of Theorem 2,

$$(Tu - Tv, u - v) > 0$$

for u and v in B, $u \neq v$. To show the uniqueness of w, it suffices to prove the uniqueness of u.

Suppose then that for u and v in C,

$$\begin{aligned}T(u + h) - k \in C^\perp,\\ T(v + h) - k \in C^\perp.\end{aligned}$$

Then $T(u + h) - T(v + h) \in C^\perp$, and since $u - v$ lies in C, we have

$$\begin{aligned}0 &= (T(u + h) - T(v + h), u - v)\\ &= (T(u + h) - T(v + h), (u + h) - (v + h)).\end{aligned}$$

Hence $u + h = v + h$, i.e. $u = v$.

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2.

DEFINITION (2.1). *The mapping T of B into B^* is said to be a duality mapping if the following conditions are satisfied:*

(1) *For each u in B,*

$$(Tu, u) = ||Tu|| \cdot ||u||.$$

(2) *For all u in B, $||Tu|| = \phi(||u||)$, where $\phi(r)$ is a given continuous monotonically increasing function on R^1 such that $\phi(0) = 0$, $\phi(r) \to +\infty$ as $r \to +\infty$.*

LEMMA 2.1. *If T is a duality map from B to B^*, T is monotone.*

Proof. Suppose that for some pair u and v of B,

$$(Tu - Tv, u - v) < 0.$$

Then if we expand the left side, we have

$$(Tu, u) + (Tv, v) - (Tv, u) - (Tu, v) < 0.$$

Since

$$|(Tv, u)| \leqslant ||Tv|| \cdot ||u||,$$
$$|(Tu, v)| \leqslant ||Tu|| \cdot ||v||,$$

it follows from (1) of Definition (2.1) that

$$0 > ||Tu|| \cdot ||u|| + ||Tv|| \cdot ||v|| - ||Tu|| \cdot ||v|| - ||Tv|| \cdot ||u||$$
$$= (||Tu|| - ||Tv||)(||u|| - ||v||).$$

By condition (2) of Definition (2.1)

$$(||Tu|| - ||Tv||)(||u|| - ||v||) \geqslant 0, \tag{1}$$

yielding a contradiction.

DEFINITION (2.2). *B is said to be strictly convex if every functional w from B^* attains its norm on at most one point of the unit ball of B.*

LEMMA 2.2. *Let B be a strictly convex Banach space, T a duality map of B into B^*. Then for $u \neq v$ in B,*

$$(Tu - Tv, u - v) > 0.$$

Proof. By Lemma 2.1, we know that

$$(Tu - Tv, u - v) \geqslant 0.$$

Suppose that

$$(Tu - Tv, u - v) = 0.$$

By inequality (1) above, $||u|| = ||v||$. If $u \neq v$, it follows from the fact that Tu assumes its norm on $u/||u||$, that

$$(Tu, v) < ||Tu|| \cdot ||v||$$

and similarly

$$(Tv, u) < ||Tv|| \cdot ||u||.$$

Hence

$$\begin{aligned} 0 &= (Tu - Tv, u - v) \\ &= ||Tu|| \cdot ||u|| + ||Tv|| \cdot ||v|| - (Tu, v) - (Tv, u) \\ &> (||Tu|| - ||Tv||)(||u|| - ||v||) \geqslant 0, \end{aligned}$$

implying a contradiction.

LEMMA 2.3.

(a) *For any Banach space B, there exist duality mappings T of B into B^*.*

(b) *If B is strictly convex, there is exactly one duality mapping T for every function $\phi(r)$ in condition* (2) *of Definition* (2.1).

(c) *If B is reflexive and strictly convex, then the duality map T corresponding to each $\phi(r)$ is a continuous mapping of B into the weak topology of B^*.*

Proof. For the proof of (a), we need to find for each u of B with $||u|| = 1$, an element u^* of B^* with $||u^*|| = 1$ such that $(u, u^*) = 1$. The existence of such u^* follows from the Hahn Banach theorem. We then define

$$T(\lambda u) = \phi(\lambda)u^*,$$

where we assume without loss of generality that $\phi(-\lambda) = -\phi(\lambda)$.

The conclusion of (b) is an immediate consequence of the definition of strict convexity.

For the proof of (c), since $\phi(r)$ is assumed to be continuous, it suffices to show that T is continuous on the unit sphere $S = \{u/||u|| = 1\}$. Suppose then that $u_k \to u_0$ in B, $||u_k|| = 1$. Since $||Tu_k|| = \phi(1)$ while B is reflexive, we may extract a weakly convergent sequence from $\{Tu_k\}$ and assume that $Tu_{k_j} \to w$ weakly in B^*. Then

$$(w, u_0) = \lim_j (Tu_{kj}, u_{k_j}) = \phi(1).$$

Since

$$||w|| \geqslant (\mathrm{w}, u_0) = \phi(1) = \lim_j ||Tu_{k_j}|| \geqslant ||w||,$$

we see that

$$(w, u_0) = ||w|| \cdot ||u_0||, \qquad ||w|| = \phi(||u_0||),$$

i.e. $w = Tu_0$. Since this is true for any weakly convergent subsequence of $\{Tu_k\}$, it follows that $Tu_k \to Tu_0$, i.e. T is continuous from the strong topology of B to the weak topology of B^*.

THEOREM 4 (Beurling–Livingston). *Let B be a reflexive strictly convex Banach space, T a duality map of B into B^*. Let C be a closed subspace of B, h a fixed element of B, k a fixed element of B^*. Then the set*

$$T(C + h) \cap (C^\perp + k)$$

contains exactly one element.

Proof. We apply Theorems 1 and 2 together with Lemmas 2.1, 2.2, and 2.3. We know from the latter that T is a monotone hemi-continuous mapping of B into B^* such that

$$(Tu - Tv, u - v) > 0$$

for $u \neq v$. In addition, for all u in B,

$$\begin{aligned}(Tu, u) &= ||Tu||\cdot||u|| \\ &= \tfrac{1}{2}||u||\cdot||Tu|| + \phi(||u||)||u|| \\ &\geqslant c(||u||)\{||u|| + ||Tu||\},\end{aligned}$$

where

$$c(r) = \min\{\tfrac{1}{2}r, \phi(r)\}.$$

Hence the hypotheses of Theorems 1 and 2 are satisfied and the conclusion of Lemma 4 follows.

Let us remark that while the result of Theorem 4 is stated in (**1**), an explicit proof is only given there for the case in which B is uniformly convex and has a differentiable norm.

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Institute for Advanced Study
and University of Chicago