2
Differential geometry

The space–time structure discussed in the next chapter, and assumed through the rest of this book, is that of a manifold with a Lorentz metric and associated affine connection.

In this chapter, we introduce in § 2.1 the concept of a manifold and in § 2.2 vectors and tensors, which are the natural geometric objects defined on the manifold. A discussion of maps of manifolds in § 2.3 leads to the definitions of the induced maps of tensors, and of submanifolds. The derivative of the induced maps defined by a vector field gives the Lie derivative defined in § 2.4; another differential operation which depends only on the manifold structure is exterior differentiation, also defined in that section. This operation occurs in the generalized form of Stokes' theorem.

An extra structure, the connection, is introduced in § 2.5; this defines the covariant derivative and the curvature tensor. The connection is related to the metric on the manifold in § 2.6; the curvature tensor is decomposed into the Weyl tensor and Ricci tensor, which are related to each other by the Bianchi identities.

In the rest of the chapter, a number of other topics in differential geometry are discussed. The induced metric and connection on a hypersurface are discussed in § 2.7, and the Gauss–Codacci relations are derived. The volume element defined by the metric is introduced in § 2.8, and used to prove Gauss' theorem. Finally, we give a brief discussion in § 2.9 of fibre bundles, with particular emphasis on the tangent bundle and the bundles of linear and orthonormal frames. These enable many of the concepts introduced earlier to be reformulated in an elegant geometrical way. § 2.7 and § 2.9 are used only at one or two points later, and are not essential to the main body of the book.

2.1 Manifolds

A manifold is essentially a space which is locally similar to Euclidean space in that it can be covered by coordinate patches. This structure permits differentiation to be defined, but does not distinguish intrinsically between different coordinate systems. Thus the only concepts defined by the manifold structure are those which are independent of the choice of a coordinate system. We will give a precise formulation of the concept of a manifold, after some preliminary definitions.

Let R^n denote the *Euclidean space of n dimensions*, that is, the set of all n-tuples $(x^1, x^2, \ldots, x^n)$ $(-\infty < x^i < \infty)$ with the usual topology (open and closed sets are defined in the usual way), and let $\frac{1}{2}R^n$ denote the 'lower half' of R^n, i.e. the region of R^n for which $x^1 \leqslant 0$. A map ϕ of an open set $\mathscr{O} \subset R^n$ (respectively $\frac{1}{2}R^n$) to an open set $\mathscr{O}' \subset R^m$ (respectively $\frac{1}{2}R^m$) is said to be of class C^r if the coordinates $(x'^1, x'^2, \ldots, x'^m)$ of the image point $\phi(p)$ in $\mathscr{O}'$ are r-times continuously differentiable functions (the rth derivatives exist and are continuous) of the coordinates $(x^1, x^2, \ldots, x^n)$ of p in $\mathscr{O}$. If a map is C^r for all $r \geqslant 0$, then it is said to be C^∞. By a C^0 map, we mean a continuous map.

A function f on an open set $\mathscr{O}$ of R^n is said to be locally Lipschitz if for each open set $\mathscr{U} \subset \mathscr{O}$ with compact closure, there is some constant K such that for each pair of points $p, q \in \mathscr{U}$, $|f(p) - f(q)| \leqslant K|p-q|$, where by $|p|$ we mean

$$\{(x^1(p))^2 + (x^2(p))^2 + \ldots + (x^n(p))^2\}^{\frac{1}{2}}.$$

A map ϕ will be said to be locally Lipschitz, denoted by C^{1-}, if the coordinates of $\phi(p)$ are locally Lipschitz functions of the coordinates of p. Similarly, we shall say that a map ϕ is C^{r-} if it is C^{r-1} and if the $(r-1)$th derivatives of the coordinates of $\phi(p)$ are locally Lipschitz functions of the coordinates of p. In the following we shall usually only mention C^r, but similar definitions and results hold for C^{r-}.

If $\mathscr{P}$ is an arbitrary set in R^n (respectively $\frac{1}{2}R^n$), a map ϕ from $\mathscr{P}$ to a set $\mathscr{P}' \subset R^m$ (respectively $\frac{1}{2}R^m$) is said to be a C^r map if ϕ is the restriction to $\mathscr{P}$ and $\mathscr{P}'$ of a C^r map from an open set $\mathscr{O}$ containing $\mathscr{P}$ to an open set $\mathscr{O}'$ containing $\mathscr{P}'$.

A C^r *n-dimensional manifold* $\mathscr{M}$ is a set $\mathscr{M}$ together with a C^r *atlas* $\{\mathscr{U}_\alpha, \phi_\alpha\}$, that is to say a collection of charts $(\mathscr{U}_\alpha, \phi_\alpha)$ where the $\mathscr{U}_\alpha$ are subsets of $\mathscr{M}$ and the ϕ_α are one–one maps of the corresponding $\mathscr{U}_\alpha$ to open sets in R^n such that

(1) the $\mathscr{U}_\alpha$ cover $\mathscr{M}$, i.e. $\mathscr{M} = \bigcup_\alpha \mathscr{U}_\alpha$,

(2) if $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$ is non-empty, then the map

$$\phi_\alpha \circ \phi_\beta^{-1}: \phi_\beta(\mathscr{U}_\alpha \cap \mathscr{U}_\beta) \to \phi_\alpha(\mathscr{U}_\alpha \cap \mathscr{U}_\beta)$$

is a C^r map of an open subset of R^n to an open subset of R^n (see figure 4).

Each $\mathscr{U}_\alpha$ is a *local coordinate neighbourhood* with the local coordinates x^a ($a = 1$ to n) defined by the map ϕ_α (i.e. if $p \in \mathscr{U}_\alpha$, then the coordinates of p are the coordinates of $\phi_\alpha(p)$ in R^n). Condition (2) is the requirement that in the overlap of two local coordinate neighbourhoods, the coordinates in one neighbourhood are C^r functions of the coordinates in the other neighbourhood, and vice versa.

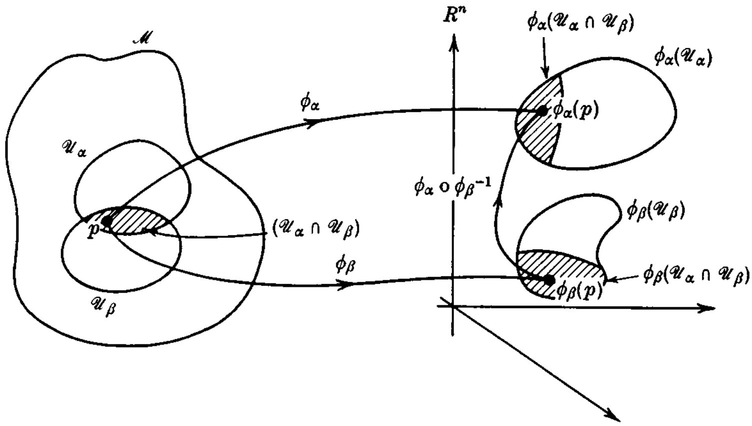

FIGURE 4. In the overlap of coordinate neighbourhoods $\mathscr{U}_\alpha$ and $\mathscr{U}_\beta$, coordinates are related by a C^r map $\phi_\alpha \circ \phi_\beta^{-1}$.

Another atlas is said to be *compatible* with a given C^r atlas if their union is a C^r atlas for all $\mathscr{M}$. The atlas consisting of all atlases compatible with the given atlas is called the *complete atlas* of the manifold; the complete atlas is therefore the set of all possible coordinate systems covering $\mathscr{M}$.

The topology of $\mathscr{M}$ is defined by stating that the open sets of $\mathscr{M}$ consist of unions of sets of the form $\mathscr{U}_\alpha$ belonging to the complete atlas. This topology makes each map ϕ_α into a homeomorphism.

A C^r differentiable manifold with boundary is defined as above, on replacing 'R^n' by '$\frac{1}{2}R^n$'. Then the *boundary of* $\mathscr{M}$, denoted by $\partial\mathscr{M}$, is defined to be the set of all points of $\mathscr{M}$ whose image under a map ϕ_α lies on the boundary of $\frac{1}{2}R^n$ in R^n. $\partial\mathscr{M}$ is an $(n-1)$-dimensional C^r manifold without boundary.

These definitions may seem more complicated than necessary. However simple examples show that one will in general need more than one coordinate neighbourhood to describe a space. The *two-dimensional Euclidean plane* R^2 is clearly a manifold. Rectangular coordinates $(x, y;\ -\infty < x < \infty,\ -\infty < y < \infty)$ cover the whole plane in one coordinate neighbourhood, where ϕ is the identity. Polar coordinates (r, θ) cover the coordinate neighbourhood $(r > 0,\ 0 < \theta < 2\pi)$; one needs at least two such coordinate neighbourhoods to cover R^2. The *two-dimensional cylinder* C^2 is the manifold obtained from R^2 by identifying the points (x, y) and $(x+2\pi, y)$. Then (x, y) are coordinates in a neighbourhood $(0 < x < 2\pi,\ -\infty < y < \infty)$ and one needs two such coordinate neighbourhoods to cover C^2. The *Möbius strip* is the manifold obtained in a similar way on identifying the points (x, y) and $(x+2\pi, -y)$. The *unit two-sphere* S^2 can be characterized as the surface in R^3 defined by the equation $(x^1)^2+(x^2)^2+(x^3)^2 = 1$. Then

$$(x^2, x^3;\ -1 < x^2 < 1,\ -1 < x^3 < 1)$$

are coordinates in each of the regions $x^1 > 0$, $x^1 < 0$, and one needs six such coordinate neighbourhoods to cover the surface. In fact, it is not possible to cover S^2 by a single coordinate neighbourhood. The *n-sphere* S^n can be similarly defined as the set of points

$$(x^1)^2+(x^2)^2+\ldots+(x^{n+1})^2 = 1$$

in R^{n+1}.

A manifold is said to be *orientable* if there is an atlas $\{\mathscr{U}_\alpha, \phi_\alpha\}$ in the complete atlas such that in every non-empty intersection $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$, the Jacobian $|\partial x^i/\partial x'^j|$ is positive, where $(x^1, \ldots, x^n)$ and $(x'^1, \ldots, x'^n)$ are coordinates in $\mathscr{U}_\alpha$ and $\mathscr{U}_\beta$ respectively. The Möbius strip is an example of a non-orientable manifold.

The definition of a manifold given so far is very general. For most purposes one will impose two further conditions, that $\mathscr{M}$ is Hausdorff and that $\mathscr{M}$ is paracompact, which will ensure reasonable local behaviour.

A topological space $\mathscr{M}$ is said to be a *Hausdorff space* if it satisfies the Hausdorff separation axiom: whenever p, q are two distinct points in $\mathscr{M}$, there exist disjoint open sets $\mathscr{U}, \mathscr{V}$ in $\mathscr{M}$ such that $p \in \mathscr{U}$, $q \in \mathscr{V}$. One might think that a manifold is necessarily Hausdorff, but this is not so. Consider, for example, the situation in figure 5. We identify the points b, b' on the two lines if and only if $x_b = y_{b'} < 0$. Then each point is contained in a (coordinate) neighbourhood homeomorphic to an open subset of R^1. However there are no disjoint open neighbourhoods

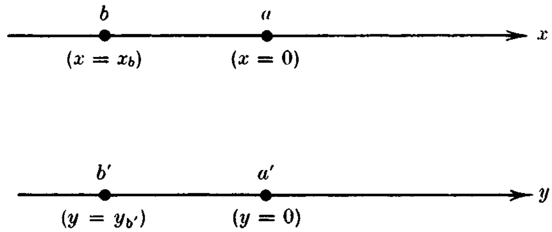

FIGURE 5. An example of a non-Hausdorff manifold. The two lines above are identical for $x = y < 0$. However the two points a $(x = 0)$ and a' $(y = 0)$ are not identified.

$\mathscr{U}, \mathscr{V}$ satisfying the conditions $a \in \mathscr{U}$, $a' \in \mathscr{V}$, where a is the point $x = 0$ and a' is the point $y = 0$.

An atlas $\{\mathscr{U}_\alpha, \phi_\alpha\}$ is said to be *locally finite* if every point $p \in \mathscr{M}$ has an open neighbourhood which intersects only a finite number of the sets $\mathscr{U}_\alpha$. $\mathscr{M}$ is said to be *paracompact* if for every atlas $\{\mathscr{U}_\alpha, \phi_\alpha\}$ there exists a locally finite atlas $\{\mathscr{V}_\beta, \psi_\beta\}$ with each $\mathscr{V}_\beta$ contained in some $\mathscr{U}_\alpha$. A connected Hausdorff manifold is paracompact if and only if it has a countable basis, i.e. there is a countable collection of open sets such that any open set can be expressed as the union of members of this collection (Kobayashi and Nomizu (1963), p. 271).

Unless otherwise stated, *all manifolds considered will be paracompact, connected C^∞ Hausdorff manifolds without boundary.* It will turn out later that when we have imposed some additional structure on $\mathscr{M}$ (the existence of an affine connection, see § 2.4) the requirement of paracompactness will be automatically satisfied because of the other restrictions.

A *function* f on a C^k manifold $\mathscr{M}$ is a map from $\mathscr{M}$ to R^1. It is said to be of class C^r $(r \leqslant k)$ at a point p of $\mathscr{M}$, if the expression $f \circ \phi_\alpha{}^{-1}$ of f on any local coordinate neighbourhood $\mathscr{U}_\alpha$ is a C^r function of the local coordinates at p; and f is said to be a *C^r function* on a set $\mathscr{V}$ of $\mathscr{M}$ if f is a C^r function at each point $p \in \mathscr{V}$.

A property of paracompact manifolds we will use later, is the following: given any locally finite atlas $\{\mathscr{U}_\alpha, \phi_\alpha\}$ on a paracompact C^k manifold, one can always (see e.g. Kobayashi and Nomizu (1963), p. 272) find a set of C^k functions g_α such that

(1) $0 \leqslant g_\alpha \leqslant 1$ on $\mathscr{M}$, for each α;

(2) the support of g_α, i.e. the closure of the set $\{p \in \mathscr{M}: g_\alpha(p) \neq 0\}$, is contained in the corresponding $\mathscr{U}_\alpha$;

(3) $\sum_\alpha g_\alpha(p) = 1$, for all $p \in \mathscr{M}$.

Such a set of functions will be called a *partition of unity*. The result is in particular true for C^∞ functions, but is clearly not true for analytic functions (an analytic function can be expressed as a convergent power series in some neighbourhood of each point $p \in \mathscr{M}$, and so is zero everywhere if it is zero on any open neighbourhood).

Finally, the *Cartesian product* $\mathscr{A} \times \mathscr{B}$ of manifolds $\mathscr{A}$, $\mathscr{B}$ is a manifold with a natural structure defined by the manifold structures of $\mathscr{A}$, $\mathscr{B}$: for arbitrary points $p \in \mathscr{A}$, $q \in \mathscr{B}$, there exist coordinate neighbourhoods $\mathscr{U}, \mathscr{V}$ containing p, q respectively, so the point $(p, q) \in \mathscr{A} \times \mathscr{B}$ is contained in the coordinate neighbourhood $\mathscr{U} \times \mathscr{V}$ in $\mathscr{A} \times \mathscr{B}$ which assigns to it the coordinates (x^i, y^j), where x^i are the coordinates of p in $\mathscr{U}$ and y^j are the coordinates of q in $\mathscr{V}$.

2.2 Vectors and tensors

Tensor fields are the set of geometric objects on a manifold defined in a natural way by the manifold structure. A tensor field is equivalent to a tensor defined at each point of the manifold, so we first define tensors at a point of the manifold, starting from the basic concept of a vector at a point.

A C^k *curve* $\lambda(t)$ in $\mathscr{M}$ is a C^k map of an interval of the real line R^1 into $\mathscr{M}$. The *vector* (contravariant vector) $(\partial/\partial t)_\lambda|_{t_0}$ tangent to the C^1 curve $\lambda(t)$ at the point $\lambda(t_0)$ is the operator which maps each C^1 function f at $\lambda(t_0)$ into the number $(\partial f/\partial t)_\lambda|_{t_0}$; that is, $(\partial f/\partial t)_\lambda$ is the derivative of f in the direction of $\lambda(t)$ with respect to the parameter t. Explicitly,

$$\left(\frac{\partial f}{\partial t}\right)_\lambda\bigg|_t = \lim_{s\to 0} \frac{1}{s}\{f(\lambda(t+s)) - f(\lambda(t))\}. \tag{2.1}$$

The curve parameter t clearly obeys the relation $(\partial/\partial t)_\lambda t = 1$.

If $(x^1, \ldots, x^n)$ are local coordinates in a neighbourhood of p,

$$\left(\frac{\partial f}{\partial t}\right)_\lambda\bigg|_{t_0} = \sum_{j=1}^{n} \frac{\mathrm{d}x^j(\lambda(t))}{\mathrm{d}t}\bigg|_{t=t_0} \cdot \frac{\partial f}{\partial x^j}\bigg|_{\lambda(t_0)} = \frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\partial f}{\partial x^j}\bigg|_{\lambda(t_0)}.$$

(Here and throughout this book, we adopt the *summation convention* whereby a repeated index implies summation over all values of that index.) Thus every tangent vector at a point p can be expressed as a linear combination of the coordinate derivatives

$$(\partial/\partial x^1)|_p, \ldots, (\partial/\partial x^n)|_p.$$

Conversely, given a linear combination $V^j(\partial/\partial x^j)|_p$ of these operators, where the V^j are any numbers, consider the curve $\lambda(t)$ defined by

$x^j(\lambda(t)) = x^j(p) + tV^j$, for t in some interval $[-\epsilon, \epsilon]$; the tangent vector to this curve at p is $V^j(\partial/\partial x^j)|_p$. Thus the tangent vectors at p form a vector space over R^1 spanned by the coordinate derivatives $(\partial/\partial x^j)|_p$, where the vector space structure is defined by the relation

$$(\alpha X + \beta Y)f = \alpha(Xf) + \beta(Yf)$$

which is to hold for all vectors X, Y, numbers α, β and functions f. The vectors $(\partial/\partial x^j)_p$ are independent (for if they were not, there would exist numbers V^j such that $V^j(\partial/\partial x^j)|_p = 0$ with at least one V^j non-zero; applying this relation to each coordinate x^k shows

$$V^j \partial x^k/\partial x^j = V^k = 0,$$

a contradiction), so the space of all tangent vectors to $\mathscr{M}$ at p, denoted by $T_p(\mathscr{M})$ or simply T_p, is an n-dimensional vector space. This space, representing the set of all directions at p, is called the *tangent vector space* to $\mathscr{M}$ at p. One may think of a vector $\mathbf{V} \in T_p$ as an arrow at p, pointing in the direction of a curve $\lambda(t)$ with tangent vector $\mathbf{V}$ at p, the 'length' of $\mathbf{V}$ being determined by the curve parameter t through the relation $V(t) = 1$. (As $\mathbf{V}$ is an operator, we print it in bold type; its components V^j, and the number $V(f)$ obtained by $\mathbf{V}$ acting on a function f, are numbers, and so are printed in italics.)

If $\{\mathbf{E}_a\}$ ($a = 1$ to n) are any set of n vectors at p which are linearly independent, then any vector $\mathbf{V} \in T_p$ can be written $\mathbf{V} = V^a \mathbf{E}_a$ where the numbers $\{V^a\}$ are the components of $\mathbf{V}$ with respect to the basis $\{\mathbf{E}_a\}$ of vectors at p. In particular one can choose the $\mathbf{E}_a$ as the coordinate basis $(\partial/\partial x^i)|_p$; then the components $V^i = V(x^i) = (\mathrm{d}x^i/\mathrm{d}t)|_p$ are the derivatives of the coordinate functions x^i in the direction $\mathbf{V}$.

A *one-form* (covariant vector) $\boldsymbol{\omega}$ at p is a real valued linear function on the space T_p of vectors at p. If $\mathbf{X}$ is a vector at p, the number into which $\boldsymbol{\omega}$ maps $\mathbf{X}$ will be written $\langle \boldsymbol{\omega}, \mathbf{X} \rangle$; then the linearity implies that

$$\langle \boldsymbol{\omega}, \alpha\mathbf{X} + \beta\mathbf{Y} \rangle = \alpha\langle \boldsymbol{\omega}, \mathbf{X} \rangle + \beta\langle \boldsymbol{\omega}, \mathbf{Y} \rangle$$

holds for all $\alpha, \beta \in R^1$ and $\mathbf{X}, \mathbf{Y} \in T_p$. The subspace of T_p defined by $\langle \boldsymbol{\omega}, \mathbf{X} \rangle =$ (constant) for a given one-form $\boldsymbol{\omega}$, is linear. One may therefore think of a one-form at p as a pair of planes in T_p such that if $\langle \boldsymbol{\omega}, \mathbf{X} \rangle = 0$ the arrow $\mathbf{X}$ lies in the first plane, and if $\langle \boldsymbol{\omega}, \mathbf{X} \rangle = 1$ it touches the second plane.

Given a basis $\{\mathbf{E}_a\}$ of vectors at p, one can define a unique set of n one-forms $\{\mathbf{E}^a\}$ by the condition: $\mathbf{E}^i$ maps any vector $\mathbf{X}$ to the number X^i (the ith component of $\mathbf{X}$ with respect to the basis $\{\mathbf{E}_a\}$).

Then in particular, $\langle \mathbf{E}^a, \mathbf{E}_b \rangle = \delta^a{}_b$. Defining linear combinations of one-forms by the rules

$$\langle \alpha\boldsymbol{\omega} + \beta\boldsymbol{\eta}, \mathbf{X} \rangle = \alpha\langle \boldsymbol{\omega}, \mathbf{X} \rangle + \beta\langle \boldsymbol{\eta}, \mathbf{X} \rangle$$

for any one-forms $\boldsymbol{\omega}$, $\boldsymbol{\eta}$ and any α, $\beta \in R^1$, $\mathbf{X} \in T_p$, one can regard $\{\mathbf{E}^a\}$ as a basis of one-forms since any one-form $\boldsymbol{\omega}$ at p can be expressed as $\boldsymbol{\omega} = \omega_i \mathbf{E}^i$ where the numbers ω_i are defined by $\omega_i = \langle \boldsymbol{\omega}, \mathbf{E}_i \rangle$. Thus the set of all one forms at p forms an n-dimensional vector space at p, the *dual space* $T^*{}_p$ of the tangent space T_p. The basis $\{\mathbf{E}^a\}$ of one-forms is the *dual basis* to the basis $\{\mathbf{E}_a\}$ of vectors. For any $\boldsymbol{\omega} \in T^*{}_p$, $\mathbf{X} \in T_p$ one can express the number $\langle \boldsymbol{\omega}, \mathbf{X} \rangle$ in terms of the components $\boldsymbol{\omega}_i$, $\mathbf{X}^i$ of $\boldsymbol{\omega}$, $\mathbf{X}$ with respect to dual bases $\{\mathbf{E}^a\}$, $\{\mathbf{E}_a\}$ by the relations

$$\langle \boldsymbol{\omega}, \mathbf{X} \rangle = \langle \omega_i \mathbf{E}^i, X^j \mathbf{E}_j \rangle = \omega_i X^i.$$

Each function f on $\mathcal{M}$ defines a one-form $\mathrm{d}f$ at p by the rule: for each vector $\mathbf{X}$,

$$\langle \mathrm{d}f, \mathbf{X} \rangle = Xf.$$

$\mathrm{d}f$ is called the *differential* of f. If $(x^1, \ldots, x^n)$ are local coordinates, the set of differentials $(\mathrm{d}x^1, \mathrm{d}x^2, \ldots, \mathrm{d}x^n)$ at p form the basis of one-forms dual to the basis $(\partial/\partial x^1, \partial/\partial x^2, \ldots, \partial/\partial x^n)$ of vectors at p, since

$$\langle \mathrm{d}x^i, \partial/\partial x^j \rangle = \partial x^i/\partial x^j = \delta^i{}_j.$$

In terms of this basis, the differential $\mathrm{d}f$ of an arbitrary function f is given by

$$\mathrm{d}f = (\partial f/\partial x^i)\, \mathrm{d}x^i.$$

If $\mathrm{d}f$ is non-zero, the surfaces $\{f = \text{constant}\}$ are $(n-1)$-dimensional manifolds. The subspace of T_p consisting of all vectors $\mathbf{X}$ such that $\langle \mathrm{d}f, \mathbf{X} \rangle = 0$ consists of all vectors tangent to curves lying in the surface $\{f = \text{constant}\}$ through p. Thus one may think of $\mathrm{d}f$ as a normal to the surface $\{f = \text{constant}\}$ at p. If $\alpha \neq 0$, $\alpha\, \mathrm{d}f$ will also be a normal to this surface.

From the space T_p of vectors at p and the space $T^*{}_p$ of one-forms at p, we can form the Cartesian product

$$\Pi^s_r = \underbrace{T^*{}_p \times T^*{}_p \times \ldots \times T^*{}_p}_{r \text{ factors}} \times \underbrace{T_p \times T_p \times \ldots \times T_p}_{s \text{ factors}},$$

i.e. the ordered set of vectors and one-forms $(\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^r, \mathbf{Y}_1, \ldots, \mathbf{Y}_s)$ where the $\mathbf{Y}$s and $\boldsymbol{\eta}$s are arbitrary vectors and one-forms respectively.

A *tensor of type* (r, s) *at* p is a function on Π^s_r which is linear in each argument. If $\mathbf{T}$ is a tensor of type (r, s) at p, we write the number into which $\mathbf{T}$ maps the element $(\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^r, \mathbf{Y}_1, \ldots, \mathbf{Y}_s)$ of Π^s_r as

$$T(\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^r, \mathbf{Y}_1, \ldots, \mathbf{Y}_s).$$

Then the linearity implies that, for example,

$$T(\eta^1, \ldots, \eta^r, \alpha\mathbf{X} + \beta\mathbf{Y}, \mathbf{Y}_2, \ldots, \mathbf{Y}_s) = \alpha \,.\, T(\eta^1, \ldots, \eta^r, \mathbf{X}, \mathbf{Y}_2, \ldots, \mathbf{Y}_s) + \beta \,.\, T(\eta^1, \ldots, \eta^r, \mathbf{Y}, \mathbf{Y}_2, \ldots, \mathbf{Y}_s)$$

holds for all $\alpha, \beta \in R^1$ and $\mathbf{X}, \mathbf{Y} \in T_p$.

The space of all such tensors is called the *tensor product*

$$T^r_s(p) = \underbrace{T_p \otimes \ldots \otimes T_p}_{r \text{ factors}} \otimes \underbrace{T^*_p \otimes \ldots \otimes T^*_p}_{s \text{ factors}}.$$

In particular, $T^1_0(p) = T_p$ and $T^0_1(p) = T^*_p$.

Addition of tensors of type (r, s) is defined by the rule: $(\mathbf{T} + \mathbf{T}')$ is the tensor of type (r, s) at p such that for all $\mathbf{Y}_i \in T_p$, $\eta^j \in T^*_p$,

$$(T + T')(\eta^1, \ldots, \eta^r, \mathbf{Y}_1, \ldots, \mathbf{Y}_s) = T(\eta^1, \ldots, \eta^r, \mathbf{Y}_1, \ldots, \mathbf{Y}_s) + T'(\eta^1, \ldots, \eta^r, \mathbf{Y}_1, \ldots, \mathbf{Y}_s).$$

Similarly, *multiplication of a tensor by a scalar* $\alpha \in R^1$ is defined by the rule: $(\alpha\mathbf{T})$ is the tensor such that for all $\mathbf{Y}_i \in T_p$, $\eta^j \in T^*_p$,

$$(\alpha T)(\eta^1, \ldots, \eta^r, \mathbf{Y}_1, \ldots, \mathbf{Y}_s) = \alpha \,.\, T(\eta^1, \ldots, \eta^r, \mathbf{Y}_1, \ldots, \mathbf{Y}_s).$$

With these rules of addition and scalar multiplication, the tensor product $T^r_s(p)$ is a vector space of dimension n^{r+s} over R^1.

Let $\mathbf{X}_i \in T_p$ $(i = 1$ to $r)$ and $\boldsymbol{\omega}^j \in T^*_p$ $(j = 1$ to $s)$. Then we shall denote by $\mathbf{X}_1 \otimes \ldots \otimes \mathbf{X}_r \otimes \boldsymbol{\omega}^1 \otimes \ldots \otimes \boldsymbol{\omega}^s$ that element of $T^r_s(p)$ which maps the element $(\eta^1, \ldots, \eta^r, \mathbf{Y}_1, \ldots, \mathbf{Y}_s)$ of Π^s_r into

$$\langle \eta^1, \mathbf{X}_1 \rangle \langle \eta^2, \mathbf{X}_2 \rangle \ldots \langle \eta^r, \mathbf{X}_r \rangle \langle \boldsymbol{\omega}^1, \mathbf{Y}_1 \rangle \ldots \langle \boldsymbol{\omega}^s, \mathbf{Y}_s \rangle.$$

Similarly, if $\mathbf{R} \in T^r_s(p)$ and $\mathbf{S} \in T^p_q(p)$, we shall denote by $\mathbf{R} \otimes \mathbf{S}$ that element of $T^{r+p}_{s+q}(p)$ which maps the element $(\eta^1, \ldots, \eta^{r+p}, \mathbf{Y}_1, \ldots, \mathbf{Y}_{s+q})$ of Π^{s+q}_{r+p} into the number

$$R(\eta^1, \ldots, \eta^s, \mathbf{Y}_1, \ldots, \mathbf{Y}_r)\, S(\eta^{s+1}, \ldots, \eta^{s+q}, \mathbf{Y}_{r+1}, \ldots, \mathbf{Y}_{r+p}).$$

With the product $\otimes$, the tensor spaces at p form an algebra over R.

If $\{\mathbf{E}_a\}$, $\{\mathbf{E}^a\}$ are dual bases of T_p, T^*_p respectively, then

$$\{\mathbf{E}_{a_1} \otimes \ldots \otimes \mathbf{E}_{a_r} \otimes \mathbf{E}^{b_1} \otimes \ldots \otimes \mathbf{E}^{b_s}\}, \quad (a_i, b_j \text{ run from 1 to } n),$$

will be a basis for $T^r_s(p)$. An arbitrary tensor $\mathbf{T} \in T^r_s(p)$ can be expressed in terms of this basis as

$$\mathbf{T} = T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} \mathbf{E}_{a_1} \otimes \ldots \otimes \mathbf{E}_{a_r} \otimes \mathbf{E}^{b_1} \otimes \ldots \otimes \mathbf{E}^{b_s}$$

where $\{T^{a_1\ldots a_r}{}_{b_1\ldots b_s}\}$ are the *components* of $\mathbf{T}$ with respect to the dual bases $\{\mathbf{E}_a\}$, $\{\mathbf{E}^a\}$ and are given by

$$T^{a_1\ldots a_r}{}_{b_1\ldots b_s} = T(\mathbf{E}^{a_1}, \ldots, \mathbf{E}^{a_r}, \mathbf{E}_{b_1}, \ldots, \mathbf{E}_{b_s}).$$

Relations in the tensor algebra at p can be expressed in terms of the components of tensors. Thus

$$(T+T')^{a_1\ldots a_r}{}_{b_1\ldots b_r} = T^{a_1\ldots a_r}{}_{b_1\ldots b_s} + T'^{a_1\ldots a_r}{}_{b_1\ldots b_s},$$

$$(\alpha T)^{a_1\ldots a_r}{}_{b_1\ldots b_s} = \alpha \,.\, T^{a_1\ldots a_r}{}_{b_1\ldots b_s},$$

$$(T\otimes T')^{a_1\ldots a_{r+p}}{}_{b_1\ldots b_{s+q}} = T^{a_1\ldots a_r}{}_{b_1\ldots b_s} T'^{a_{r+1}\ldots a_{r+p}}{}_{b_{s+1}\ldots b_{s+q}}.$$

Because of its convenience, we shall usually represent tensor relations in this way.

If $\{\mathbf{E}_{a'}\}$ and $\{\mathbf{E}^{a'}\}$ are another pair of dual bases for T_p and $T^*{}_p$, they can be represented in terms of $\{\mathbf{E}_a\}$ and $\{\mathbf{E}^a\}$ by

$$\mathbf{E}_{a'} = \Phi_{a'}{}^a \mathbf{E}_a \qquad (2.2)$$

where $\Phi_{a'}{}^a$ is an $n\times n$ non-singular matrix. Similarly

$$\mathbf{E}^{a'} = \Phi^{a'}{}_a \mathbf{E}^a \qquad (2.3)$$

where $\Phi^{a'}{}_a$ is another $n\times n$ non-singular matrix. Since $\{\mathbf{E}_{a'}\}$, $\{\mathbf{E}^{a'}\}$ are dual bases,

$$\delta^{b'}{}_{a'} = \langle \mathbf{E}^{b'}, \mathbf{E}_{a'}\rangle = \langle \Phi^{b'}{}_b \mathbf{E}^b, \Phi_{a'}{}^a \mathbf{E}_a\rangle = \Phi_{a'}{}^a \Phi^{b'}{}_b \delta_a{}^b = \Phi_{a'}{}^a \Phi^{b'}{}_a,$$

i.e. $\Phi_{a'}{}^a$, $\Phi^{a'}{}_a$ are inverse matrices, and $\delta^a{}_b = \Phi^a{}_{b'} \Phi^{b'}{}_b$.

The components $T^{a'_1\ldots a'_r}{}_{b'_1\ldots b'_s}$ of a tensor $\mathbf{T}$ with respect to the dual bases $\{\mathbf{E}_{a'}\}$, $\{\mathbf{E}^{a'}\}$ are given by

$$T^{a'_1\ldots a'_r}{}_{b'_1\ldots b'_s} = T(\mathbf{E}^{a_1'}, \ldots, \mathbf{E}^{a'_r}, \mathbf{E}_{b'_1}, \ldots, \mathbf{E}_{b'_s}).$$

They are related to the components $T^{a_1\ldots a_r}{}_{b_1\ldots b_s}$ of $\mathbf{T}$ with respect to the bases $\{\mathbf{E}_a\}$, $\{\mathbf{E}^a\}$ by

$$T^{a'_1\ldots a'_r}{}_{b'_1\ldots b'_s} = T^{a_1\ldots a_r}{}_{b_1\ldots b_s} \Phi^{a'_1}{}_{a_1} \ldots \Phi^{a'_r}{}_{a_r} \Phi_{b'_1}{}^{b_1} \ldots \Phi_{b'_s}{}^{b_s}. \qquad (2.4)$$

The *contraction* of a tensor $\mathbf{T}$ of type (r,s), with components $T^{ab\ldots d}{}_{ef\ldots g}$ with respect to bases $\{\mathbf{E}_a\}$, $\{\mathbf{E}^a\}$, on the first contravariant and first covariant indices is defined to be the tensor $C^1_1(\mathbf{T})$ of type $(r-1, s-1)$ whose components with respect to the same basis are $T^{ab\ldots d}{}_{af\ldots g}$, i.e.

$$C^1_1(\mathbf{T}) = T^{ab\ldots d}{}_{af\ldots g} \mathbf{E}_b \otimes \ldots \otimes \mathbf{E}_d \otimes \mathbf{E}^f \otimes \ldots \otimes \mathbf{E}^g.$$

If $\{\mathbf{E}_{a'}\}$, $\{\mathbf{E}^{a'}\}$ are another pair of dual bases, the contraction $C^1_1(\mathbf{T})$ defined by them is

$$\begin{aligned} C'^1_1(\mathbf{T}) &= T^{a'b'\dots d'}{}_{a'f'\dots g'}\mathbf{E}_{b'}\otimes\dots\otimes\mathbf{E}_{d'}\otimes\mathbf{E}^{f'}\otimes\dots\otimes\mathbf{E}^{g'} \\ &= \Phi^{a'}{}_a\,\Phi^a{}_{h'}\,T^{h'b'\dots d'}{}_{a'f'\dots g'}\,\Phi_{b'}{}^b\dots\Phi_{d'}{}^d\,\Phi^{f'}{}_f\dots\Phi^{g'}{}_g \\ &\qquad .\mathbf{E}_b\otimes\dots\otimes\mathbf{E}_d\otimes\mathbf{E}^f\dots\otimes\mathbf{E}^g \\ &= T^{ab\dots d}{}_{af\dots g}\mathbf{E}_b\otimes\dots\otimes\mathbf{E}_d\otimes\mathbf{E}^f\otimes\dots\otimes\mathbf{E}^g = C^1_1(\mathbf{T}), \end{aligned}$$

so the contraction C^1_1 of a tensor is independent of the basis used in its definition. Similarly, one could contract $\mathbf{T}$ over any pair of contravariant and covariant indices. (If we were to contract over two contravariant or covariant indices, the resultant tensor would depend on the basis used.)

The symmetric part of a tensor $\mathbf{T}$ of type $(2, 0)$ is the tensor $S(\mathbf{T})$ defined by

$$S(\mathbf{T})(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2) = \frac{1}{2!}\{T(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2)+T(\boldsymbol{\eta}_2,\boldsymbol{\eta}_1)\}$$

for all $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in T^*{}_p$. We shall denote the components $S(\mathbf{T})^{ab}$ of $S(\mathbf{T})$ by $T^{(ab)}$; then

$$T^{(ab)} = \frac{1}{2!}\{T^{ab}+T^{ba}\}.$$

Similarly, the components of the skew-symmetric part of $\mathbf{T}$ will be denoted by

$$T^{[ab]} = \frac{1}{2!}\{T^{ab}-T^{ba}\}.$$

In general, the components of the symmetric or antisymmetric part of a tensor on a given set of covariant or contravariant indices will be denoted by placing round or square brackets around the indices. Thus

$$T_{(a_1\dots a_r)}{}^{b\dots f} = \frac{1}{r!}\{\text{sum over all permutations of the indices } a_1 \text{ to } a_r (T_{a_1\dots a_r}{}^{b\dots f})\}$$

and

$$T_{[a_1\dots a_r]}{}^{b\dots f} = \frac{1}{r!}\{\text{alternating sum over all permutations of the indices } a_1 \text{ to } a_r\ (T_{a_1\dots a_r}{}^{b\dots f})\}.$$

For example,

$$K^a{}_{[bcd]} = \tfrac{1}{6}\{K^a{}_{bcd}+K^a{}_{dbc}+K^a{}_{cdb}-K^a{}_{bdc}-K^a{}_{cbd}-K^a{}_{dcb}\}.$$

A tensor is *symmetric* in a given set of contravariant or covariant indices if it is equal to its symmetrized part on these indices, and is *antisymmetric* if it is equal to its antisymmetrized part. Thus, for example, a tensor $\mathbf{T}$ of type (0, 2) is symmetric if $T_{ab} = \frac{1}{2}(T_{ab} + T_{ba})$, (which we can also express in the form: $T_{[ab]} = 0$).

A particularly important subset of tensors is the set of tensors of type $(0, q)$ which are antisymmetric on all q positions (so $q \leqslant n$); such a tensor is called a *q-form*. If $\mathbf{A}$ and $\mathbf{B}$ are p- and q-forms respectively, one can define a $(p+q)$-form $\mathbf{A} \wedge \mathbf{B}$ from them, where $\wedge$ is the skew-symmetrized tensor product $\otimes$; that is, $\mathbf{A} \wedge \mathbf{B}$ is the tensor of type $(0, p+q)$ with components determined by

$$(A \wedge B)_{a \ldots bc \ldots f} = A_{[a \ldots b} B_{c \ldots f]}.$$

This rule implies $(\mathbf{A} \wedge \mathbf{B}) = (-)^{pq} (\mathbf{B} \wedge \mathbf{A})$. With this product, the space of forms (i.e. the space of all p-forms for all p, including one-forms and defining scalars as zero-forms) constitutes the Grassmann algebra of forms. If $\{\mathbf{E}^a\}$ is a basis of one-forms, then the forms $\mathbf{E}^{a_1} \wedge \ldots \wedge \mathbf{E}^{a_p}$ (a_i run from 1 to n) are a basis of p-forms, as any p-form $\mathbf{A}$ can be written $\mathbf{A} = A_{a \ldots b} \mathbf{E}^a \wedge \ldots \wedge \mathbf{E}^b$, where $A_{a \ldots b} = A_{[a \ldots b]}$.

So far, we have considered the set of tensors defined at a point on the manifold. A set of local coordinates $\{x^i\}$ on an open set $\mathscr{U}$ in $\mathscr{M}$ defines a basis $\{(\partial/\partial x^i)|_p\}$ of vectors and a basis $\{(\mathrm{d}x^i)|_p\}$ of one-forms at each point p of $\mathscr{U}$, and so defines a basis of tensors of type (r, s) at each point of $\mathscr{U}$. Such a basis of tensors will be called a coordinate basis. A C^k *tensor field* $\mathbf{T}$ *of type* (r, s) on a set $\mathscr{V} \subset \mathscr{M}$ is an assignment of an element of $T^r_s(p)$ to each point $p \in \mathscr{V}$ such that the components of $\mathbf{T}$ with respect to any coordinate basis defined on an open subset of $\mathscr{V}$ are C^k functions.

In general one need not use a coordinate basis of tensors, i.e. given any basis of vectors $\{\mathbf{E}_a\}$ and dual basis of forms $\{\mathbf{E}^a\}$ on $\mathscr{V}$, there will not necessarily exist any open set in $\mathscr{V}$ on which there are local coordinates $\{x^a\}$ such that $\mathbf{E}_a = \partial/\partial x^a$ and $\mathbf{E}^a = \mathrm{d}x^a$. However if one does use a coordinate basis, certain specializations will result; in particular for any function f, the relations $E_a(E_b f) = E_b(E_a f)$ are satisfied, being equivalent to the relations $\partial^2 f/\partial x^a \partial x^b = \partial^2 f/\partial x^b \partial x^a$. If one changes from a coordinate basis $\mathbf{E}_a = \partial/\partial x^a$ to a coordinate basis $\mathbf{E}_{a'} = \partial/\partial x^{a'}$, applying (2.2), (2.3) to x^a, $x^{a'}$ shows that

$$\Phi_{a'}{}^{a} = \frac{\partial x^a}{\partial x^{a'}}, \quad \Phi^{a'}{}_{a} = \frac{\partial x^{a'}}{\partial x^a}.$$

Clearly a general basis $\{\mathbf{E}_a\}$ can be obtained from a coordinate basis

$\{\partial/\partial x^i\}$ by giving the functions $E_a{}^i$ which are the components of the $\mathbf{E}_a$ with respect to the basis $\{\partial/\partial x^i\}$; then (2.2) takes the form $\mathbf{E}_a = E_a{}^i \partial/\partial x^i$ and (2.3) takes the form $\mathbf{E}^a = E^a{}_i \mathrm{d}x^i$, where the matrix $E^a{}_i$ is dual to the matrix $E_a{}^i$.

2.3 Maps of manifolds

In this section we define, via the general concept of a C^k manifold map, the concepts of 'imbedding', 'immersion', and of associated tensor maps, the first two being useful later in the study of submanifolds, and the last playing an important role in studying the behaviour of families of curves as well as in studying symmetry properties of manifolds.

A map ϕ from a C^k n-dimensional manifold $\mathscr{M}$ to a $C^{k'}$ n'-dimensional manifold $\mathscr{M}'$ is said to be a C^r *map* ($r \leqslant k$, $r \leqslant k'$) if, for any local coordinate systems in $\mathscr{M}$ and $\mathscr{M}'$, the coordinates of the image point $\phi(p)$ in $\mathscr{M}'$ are C^r functions of the coordinates of p in $\mathscr{M}$. As the map will in general be many–one rather than one–one (e.g. it cannot be one–one if $n > n'$), it will in general not have an inverse; and if a C^r map does have an inverse, this inverse will in general not be C^r (e.g. if ϕ is the map $R^1 \to R^1$ given by $x \to x^3$, then ϕ^{-1} is not differentiable at the point $x = 0$).

If f is a function on $\mathscr{M}'$, the mapping ϕ defines the function $\phi^* f$ on $\mathscr{M}$ as the function whose value at the point p of $\mathscr{M}$ is the value of f at $\phi(p)$, i.e.

$$\phi^* f(p) = f(\phi(p)). \tag{2.5}$$

Thus when ϕ maps points from $\mathscr{M}$ to $\mathscr{M}'$, ϕ^* maps functions linearly from $\mathscr{M}'$ to $\mathscr{M}$.

If $\lambda(t)$ is a curve through the point $p \in \mathscr{M}$, then the image curve $\phi(\lambda(t))$ in $\mathscr{M}'$ passes through the point $\phi(p)$. If $r \geqslant 1$, the tangent vector to this curve at $\phi(p)$ will be denoted by $\phi_*(\partial/\partial t)_\lambda|_{\phi(p)}$; one can regard it as the image, under the map ϕ, of the vector $(\partial/\partial t)_\lambda|_p$. Clearly ϕ_* is a linear map of $T_p(\mathscr{M})$ into $T_{\phi(p)}(\mathscr{M}')$. From (2.5) and the definition (2.1) of a vector as a directional derivative, the vector map ϕ_* can be characterized by the relation: for each C^r ($r \geqslant 1$) function f at $\phi(p)$ and vector $\mathbf{X}$ at p,

$$X(\phi^* f)|_p = \phi_* X(f)|_{\phi(p)}. \tag{2.6}$$

Using the vector mapping ϕ_* from $\mathscr{M}$ to $\mathscr{M}'$, we can if $r \geqslant 1$ define a linear one-form mapping ϕ^* from $T^*{}_{\phi(p)}(\mathscr{M}')$ to $T^*{}_p(\mathscr{M})$ by the condition: vector–one-form contractions are to be preserved under the maps. Then the one-form $\mathbf{A}\in T^*{}_{\phi(p)}$ is mapped into the one-form $\phi^*\mathbf{A}\in T^*{}_p$ where, for arbitrary vectors $\mathbf{X}\in T_p$,

$$\langle\phi^*\mathbf{A},\mathbf{X}\rangle|_p = \langle\mathbf{A},\phi_*\mathbf{X}\rangle|_{\phi(p)}.$$

A consequence of this is that

$$\phi^*(\mathrm{d}f) = \mathrm{d}(\phi^*f). \tag{2.7}$$

The maps ϕ_* and ϕ^* can be extended to maps of contravariant tensors from $\mathscr{M}$ to $\mathscr{M}'$ and covariant tensors from $\mathscr{M}'$ to $\mathscr{M}$ respectively, by the rules $\phi_*\colon \mathbf{T}\in T^r_0(p)\to\phi_*\mathbf{T}\in T^r_0(\phi(p))$ where for any $\boldsymbol{\eta}^i\in T^*{}_{\phi(p)}$,

$$T(\phi^*\boldsymbol{\eta}^1,\ldots,\phi^*\boldsymbol{\eta}^r)|_p = \phi_*T(\boldsymbol{\eta}^1,\ldots,\boldsymbol{\eta}^r)|_{\phi(p)}$$

and

$$\phi^*\colon \mathbf{T}\in T^0_s(\phi(p))\to\phi^*\mathbf{T}\in T^0_s(p),$$

where for any $\mathbf{X}_i\in T_p$,

$$\phi^*T(\mathbf{X}_1,\ldots,\mathbf{X}_s)|_p = T(\phi_*\mathbf{X}_1,\ldots,\phi_*\mathbf{X}_s)|_{\phi(p)}.$$

When $r\geqslant 1$, the C^r map ϕ from $\mathscr{M}$ to $\mathscr{M}'$ is said to be of *rank* s at p if the dimension of $\phi_*(T_p(\mathscr{M}))$ is s. It is said to be *injective* at p if $s = n$ (and so $n\leqslant n'$) at p; then no vector in T_p is mapped to zero by ϕ_*. It is said to be *surjective* if $s = n'$ (so $n\geqslant n'$).

A C^r map ϕ ($r\geqslant 0$) is said to be an *immersion* if it and its inverse are C^r maps, i.e. if for each point $p\in\mathscr{M}$ there is a neighbourhood $\mathscr{U}$ of p in $\mathscr{M}$ such that the inverse ϕ^{-1} restricted to $\phi(\mathscr{U})$ is also a C^r map. This implies $n\leqslant n'$. By the implicit function theorem (Spivak (1965), p. 41), when $r\geqslant 1$, ϕ will be an immersion if and only if it is injective at every point $p\in\mathscr{M}$; then ϕ_* is an isomorphism of T_p into the image $\phi_*(T_p)\subset T_{\phi(p)}$. The image $\phi(\mathscr{M})$ is then said to be an n-dimensional *immersed submanifold* in $\mathscr{M}'$. This submanifold may intersect itself, i.e. ϕ may not be a one–one map from $\mathscr{M}$ to $\phi(\mathscr{M})$ although it is one–one when restricted to a sufficiently small neighbourhood of $\mathscr{M}$. An immersion is said to be an *imbedding* if it is a homeomorphism onto its image in the induced topology. Thus an imbedding is a one–one immersion; however not all one–one immersions are imbeddings, cf. figure 6. A map ϕ is said to be a proper map if the inverse image $\phi^{-1}(\mathscr{K})$ of any compact set $\mathscr{K}\subset\mathscr{M}'$ is compact. It can be shown that a proper one–one immersion is an imbedding. The image $\phi(\mathscr{M})$ of $\mathscr{M}$ under an imbedding ϕ is said to be an n-dimensional *imbedded submanifold* of $\mathscr{M}'$.

The map ϕ from $\mathscr{M}$ to $\mathscr{M}'$ is said to be a C^r *diffeomorphism* if it is a one–one C^r map and the inverse ϕ^{-1} is a C^r map from $\mathscr{M}'$ to $\mathscr{M}$. In this case, $n = n'$, and ϕ is both injective and surjective if $r \geqslant 1$; conversely, the implicit function theorem shows that if ϕ_* is both injective and surjective at p, then there is an open neighbourhood $\mathscr{U}$ of p such that $\phi: \mathscr{U} \to \phi(\mathscr{U})$ is a diffeomorphism. Thus ϕ is a local diffeomorphism near p if ϕ_* is an isomorphism from T_p to $T_{\phi(p)}$.

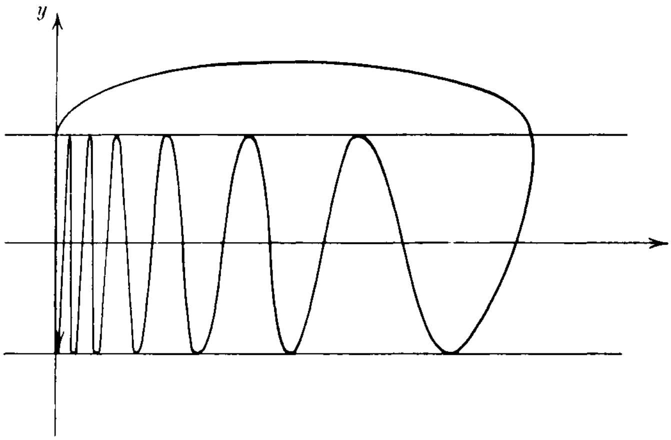

FIGURE 6. A one–one immersion of R^1 in R^2 which is not an imbedding, obtained by joining smoothly part of the curve $y = \sin(1/x)$ to the curve

$$\{(y, 0);\ -\infty < y < 1\}.$$

When the map ϕ is a C^r ($r \geqslant 1$) diffeomorphism, ϕ_* maps $T_p(\mathscr{M})$ to $T_{\phi(p)}(\mathscr{M}')$ and $(\phi^{-1})^*$ maps $T^*{}_p(\mathscr{M})$ to $T^*{}_{\phi(p)}(\mathscr{M}')$. Thus we can define a map ϕ_* of $T^r_s(p)$ to $T^r_s(\phi(p))$ for any r, s, by

$$T(\boldsymbol{\eta}^1, \ldots, \boldsymbol{\eta}^s, \mathbf{X}_1, \ldots, \mathbf{X}_r)|_p$$
$$= \phi_* T((\phi^{-1})^*\boldsymbol{\eta}^1, \ldots, (\phi^{-1})^*\boldsymbol{\eta}^s, \phi_*\mathbf{X}_1, \ldots, \phi_*\mathbf{X}_r)|_{\phi(p)}$$

for any $\mathbf{X}_i \in T_p$, $\boldsymbol{\eta}^i \in T^*{}_p$. This map of tensors of type (r, s) on $\mathscr{M}$ to tensors of type (r, s) on $\mathscr{M}'$ preserves symmetries and relations in the tensor algebra; e.g. the contraction of $\phi_*\mathbf{T}$ is equal to ϕ_* (the contraction of $\mathbf{T}$).

2.4 Exterior differentiation and the Lie derivative

We shall study three differential operators on manifolds, the first two being defined purely by the manifold structure while the third is defined (see § 2.5) by placing extra structure on the manifold.

The *exterior differentiation* operator d maps r-form fields linearly to $(r+1)$-form fields. Acting on a zero-form field (i.e. a function) f, it gives the one-form field $\mathrm{d}f$ defined by (cf. §2.2)

$$\langle \mathrm{d}f, \mathbf{X}\rangle = Xf \text{ for all vector fields } \mathbf{X} \tag{2.8}$$

and acting on the r-form field

$$\mathbf{A} = A_{ab\ldots d}\,\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \ldots \wedge \mathrm{d}x^d$$

it gives the $(r+1)$-form field $\mathrm{d}\mathbf{A}$ defined by

$$\mathrm{d}\mathbf{A} = \mathrm{d}A_{ab\ldots d} \wedge \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \ldots \wedge \mathrm{d}x^d. \tag{2.9}$$

To show that this $(r+1)$-form field is independent of the coordinates $\{x^a\}$ used in its definition, consider another set of coordinates $\{x^{a'}\}$. Then

$$\mathbf{A} = A_{a'b'\ldots d'}\,\mathrm{d}x^{a'} \wedge \mathrm{d}x^{b'} \wedge \ldots \wedge \mathrm{d}x^{d'},$$

where the components $A_{a'b'\ldots d'}$ are given by

$$A_{a'b'\ldots d'} = \frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}\cdots\frac{\partial x^d}{\partial x^{d'}}A_{ab\ldots d}.$$

Thus the $(r+1)$-form $\mathrm{d}\mathbf{A}$ defined by these coordinates is

$$\begin{aligned}
\mathrm{d}\mathbf{A} &= \mathrm{d}A_{a'b'\ldots d'}\,dx^{a'} \wedge \mathrm{d}x^{b'} \wedge \ldots \wedge \mathrm{d}x^{d'} \\
&= \mathrm{d}\left(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}\cdots\frac{\partial x^d}{\partial x^{d'}}A_{ab\ldots d}\right) \wedge \mathrm{d}x^{a'} \wedge \mathrm{d}x^{b'} \wedge \ldots \wedge \mathrm{d}x^{d'} \\
&= \frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}\cdots\frac{\partial x^d}{\partial x^{d'}}\mathrm{d}A_{ab\ldots d} \wedge \mathrm{d}x^{a'} \wedge \mathrm{d}x^{b'} \wedge \ldots \wedge \mathrm{d}x^{d'} \\
&\quad + \frac{\partial^2 x^a}{\partial x^{a'}\,\partial x^{e'}}\frac{\partial x^b}{\partial x^{b'}}\cdots\frac{\partial x^d}{\partial x^{d'}}A_{ab\ldots d}\,\mathrm{d}x^{e'} \wedge \mathrm{d}x^{a'} \wedge \mathrm{d}x^{b'} \wedge \ldots \wedge \mathrm{d}x^{d'} + \ldots + \ldots \\
&= \mathrm{d}A_{ab\ldots d} \wedge \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \ldots \wedge \mathrm{d}x^d
\end{aligned}$$

as $\partial^2 x^a/\partial x^{a'}\,\partial x^{e'}$ is symmetric in a' and e', but $\mathrm{d}x^{e'} \wedge \mathrm{d}x^{a'}$ is skew. Note that this definition only works for *forms*; it would not be independent of the coordinates used if the $\wedge$ product were replaced by a tensor product. Using the relation $\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g$, which holds for arbitrary functions f, g, it follows that for any r-form $\mathbf{A}$ and form $\mathbf{B}$, $\mathrm{d}(\mathbf{A} \wedge \mathbf{B}) = \mathrm{d}\mathbf{A} \wedge \mathbf{B} + (-)^r \mathbf{A} \wedge \mathrm{d}\mathbf{B}$. Since (2.8) implies that the local coordinate expression for $\mathrm{d}f$ is $\mathrm{d}f = (\partial f/\partial x^i)\,\mathrm{d}x^i$, it follows that $\mathrm{d}(\mathrm{d}f) = (\partial^2 f/\partial x^i\,\partial x^j)\,\mathrm{d}x^i \wedge \mathrm{d}x^j = 0$, as the first term is symmetric and the second skew-symmetric. Similarly it follows from (2.9) that

$$\mathrm{d}(\mathrm{d}\mathbf{A}) = 0$$

holds for any r-form field $\mathbf{A}$.

The operator d commutes with manifold maps, in the sense: if $\phi: \mathscr{M} \to \mathscr{M}'$ is a C^r ($r \geq 2$) map and **A** is a C^k ($k \geq 2$) form field on $\mathscr{M}'$, then (by (2.7))

$$\mathrm{d}(\phi^*\mathbf{A}) = \phi^*(\mathrm{d}\mathbf{A})$$

(which is equivalent to the chain rule for partial derivatives).

The operator d occurs naturally in the general form of Stokes' theorem on a manifold. We first define integration of n-forms: let $\mathscr{M}$ be a compact, orientable n-dimensional manifold with boundary $\partial\mathscr{M}$ and let $\{f_\alpha\}$ be a partition of unity for a finite oriented atlas $\{\mathscr{U}_\alpha, \phi_\alpha\}$. Then if **A** is an n-form field on $\mathscr{M}$, the integral of **A** over $\mathscr{M}$ is defined as

$$\int_{\mathscr{M}} \mathbf{A} = (n!)^{-1} \sum_\alpha \int_{\phi_\alpha(\mathscr{U}_\alpha)} f_\alpha A_{12\ldots n} \mathrm{d}x^1 \mathrm{d}x^2 \ldots \mathrm{d}x^n, \qquad (2.10)$$

where $A_{12\ldots n}$ are the components of **A** with respect to the local coordinates in the coordinate neighbourhood $\mathscr{U}_\alpha$, and the integrals on the right-hand side are ordinary multiple integrals over open sets $\phi_\alpha(\mathscr{U}_\alpha)$ of R^n. Thus integration of forms on $\mathscr{M}$ is defined by mapping the form, by local coordinates, into R^n and performing standard multiple integrals there, the existence of the partition of unity ensuring the global validity of this operation.

The integral (2.10) is well-defined, since if one chose another atlas $\{\mathscr{V}_\beta, \psi_\beta\}$ and partition of unity $\{g_\beta\}$ for this atlas, one would obtain the integral

$$(n!)^{-1} \sum_\beta \int_{\psi_\beta(\mathscr{V}_\beta)} g_\beta A_{1'2'\ldots n'} \,\mathrm{d}x^{1'} \,\mathrm{d}x^{2'} \ldots \mathrm{d}x^{n'},$$

where $x^{i'}$ are the corresponding local coordinates. Comparing these two quantities in the overlap $(\mathscr{U}_\alpha \cap \mathscr{V}_\beta)$ of coordinate neighbourhoods belonging to two atlases, the first expression can be written

$$(n!)^{-1} \sum_\alpha \sum_\beta \int_{\phi_\alpha(\mathscr{U}_\alpha \cap \mathscr{V}_\beta)} f_\alpha g_\beta A_{12\ldots n} \,\mathrm{d}x^1 \,\mathrm{d}x^2 \ldots \mathrm{d}x^n,$$

and the second can be written

$$(n!)^{-1} \sum_\alpha \sum_\beta \int_{\psi_\beta(\mathscr{U}_\alpha \cap \mathscr{V}_\beta)} f_\alpha g_\beta A_{1'2'\ldots n'} \,\mathrm{d}x^{1'} \,\mathrm{d}x^{2'} \ldots \mathrm{d}x^{n'}.$$

Comparing the transformation laws for the form **A** and the multiple integrals in R^n, these expressions are equal at each point, so $\int_{\mathscr{M}} \mathbf{A}$ is independent of the atlas and partition of unity chosen.

Similarly, one can show that this integral is invariant under diffeomorphisms:

$$\int_{\mathscr{M}'} \phi_* \mathbf{A} = \int_{\mathscr{M}} \mathbf{A}$$

if ϕ is a C^r diffeomorphism ($r \geqslant 1$) from $\mathscr{M}$ to $\mathscr{M}'$.

Using the operator d, the *generalized Stokes' theorem* can now be written in the form: if $\mathbf{B}$ is an $(n-1)$-form field on $\mathscr{M}$, then

$$\int_{\partial\mathscr{M}} \mathbf{B} = \int_{\mathscr{M}} \mathrm{d}\mathbf{B},$$

which can be verified (see e.g. Spivak (1965)) from the definitions above; it is essentially a general form of the fundamental theorem of calculus. To perform the integral on the left, one has to define an orientation on the boundary $\partial\mathscr{M}$ of $\mathscr{M}$. This is done as follows: if $\mathscr{U}_\alpha$ is a coordinate neighbourhood from the oriented atlas of $\mathscr{M}$ such that $\mathscr{U}_\alpha$ intersects $\partial\mathscr{M}$, then from the definition of $\partial\mathscr{M}$, $\phi_\alpha(\mathscr{U}_\alpha \cap \partial\mathscr{M})$ lies in the plane $x^1 = 0$ in R^n and $\phi_\alpha(\mathscr{U}_\alpha \cap \mathscr{M})$ lies in the lower half $x^1 \leqslant 0$. The coordinates $(x^2, x^3, \ldots, x^n)$ are then oriented coordinates in the neighbourhood $\mathscr{U}_\alpha \cap \partial\mathscr{M}$ of $\partial\mathscr{M}$. It may be verified that this gives an oriented atlas on $\partial\mathscr{M}$.

The other type of differentiation defined naturally by the manifold structure is *Lie differentiation*. Consider any C^r ($r \geqslant 1$) vector field $\mathbf{X}$ on $\mathscr{M}$. By the fundamental theorem for systems of ordinary differential equations (Burkill (1956)) there is a unique maximal curve $\lambda(t)$ through each point p of $\mathscr{M}$ such that $\lambda(0) = p$ and whose tangent vector at the point $\lambda(t)$ is the vector $\mathbf{X}|_{\lambda(t)}$. If $\{x^i\}$ are local coordinates, so that the curve $\lambda(t)$ has coordinates $x^i(t)$ and the vector $\mathbf{X}$ has components X^i, then this curve is locally a solution of the set of differential equations

$$\mathrm{d}x^i/\mathrm{d}t = X^i(x^1(t), \ldots, x^n(t)).$$

This curve is called the *integral curve* of $\mathbf{X}$ with initial point p. For each point q of $\mathscr{M}$, there is an open neighbourhood $\mathscr{U}$ of q and an $\epsilon > 0$ such that $\mathbf{X}$ defines a family of diffeomorphisms ϕ_t: $\mathscr{U} \to \mathscr{M}$ whenever $|t| < \epsilon$, obtained by taking each point p in $\mathscr{U}$ a parameter distance t along the integral curves of $\mathbf{X}$ (in fact, the ϕ_t form a one-parameter local group of diffeomorphisms, as $\phi_{t+s} = \phi_t \circ \phi_s = \phi_s \circ \phi_t$ for $|t|$, $|s|$, $|t+s| < \epsilon$, so $\phi_{-t} = (\phi_t)^{-1}$ and ϕ_0 is the identity). This diffeomorphism maps each tensor field $\mathbf{T}$ at p of type (r,s) into $\phi_{t*}\mathbf{T}|_{\phi_t(p)}$.

The *Lie derivative* $L_\mathbf{X}\mathbf{T}$ of a tensor field $\mathbf{T}$ with respect to $\mathbf{X}$ is defined to be minus the derivative with respect to t of this family of tensor fields, evaluated at $t = 0$, i.e.

$$L_{\mathbf{X}}\mathbf{T}|_p = \lim_{t\to 0}\frac{1}{t}\{\mathbf{T}|_p - \phi_{t*}\mathbf{T}|_p\}.$$

From the properties of ϕ_*, it follows that

(1) $L_{\mathbf{X}}$ preserves tensor type, i.e. if $\mathbf{T}$ is a tensor field of type (r, s), then $L_{\mathbf{X}}\mathbf{T}$ is also a tensor field of type (r, s);

(2) $L_{\mathbf{X}}$ maps tensors linearly and preserves contractions.

As in ordinary calculus, one can prove Leibniz' rule:

(3) For arbitrary tensors $\mathbf{S}, \mathbf{T}$, $L_{\mathbf{X}}(\mathbf{S}\otimes\mathbf{T}) = L_{\mathbf{X}}\mathbf{S}\otimes\mathbf{T} + \mathbf{S}\otimes L_{\mathbf{X}}\mathbf{T}$.

Direct from the definitions:

(4) $L_{\mathbf{X}}f = Xf$, where f is any function.

Under the map ϕ_t, the point $q = \phi_{-t}(p)$ is mapped into p. Therefore ϕ_{t*} is a map from T_q to T_p. Thus, by (2.6),

$$(\phi_{t*}Y)f|_p = Y(\phi_t{}^*f)|_q.$$

If $\{x^i\}$ are local coordinates in a neighbourhood of p, the coordinate components of $\phi_{t*}\mathbf{Y}$ at p are

$$(\phi_{t*}Y)^i|_p = \phi_{t*}Y|_p x^i = Y^j|_q\frac{\partial}{\partial x^j(q)}(x^i(p))$$
$$= \frac{\partial x^i(\phi_t(q))}{\partial x^j(q)}Y^j|_q.$$

Now
$$\frac{\mathrm{d}x^i(\phi_t(q))}{\mathrm{d}t} = X^i|_{\phi_t(q)},$$

therefore
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x^i(\phi_t(q))}{\partial x^j(q)}\right)\bigg|_{t=0} = \frac{\partial X^i}{\partial x^j}\bigg|_p,$$

so
$$(L_{\mathbf{X}}Y)^i = -\frac{\mathrm{d}}{\mathrm{d}t}(\phi_{t*}Y)^i|_{t=0} = \frac{\partial Y^i}{\partial x^j}X^j - \frac{\partial X^i}{\partial x^j}Y^j. \tag{2.11}$$

One can rewrite this in the form

$$(L_{\mathbf{X}}Y)f = X(Yf) - Y(Xf)$$

for all C^2 functions f. We shall sometimes denote $L_{\mathbf{X}}\mathbf{Y}$ by $[\mathbf{X}, \mathbf{Y}]$, i.e.

$$L_{\mathbf{X}}\mathbf{Y} = -L_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}].$$

If the Lie derivative of two vector fields $\mathbf{X}$, $\mathbf{Y}$ vanishes, the vector fields are said to commute. In this case, if one starts at a point p, goes a parameter distance t along the integral curves of $\mathbf{X}$ and then a parameter distance s along the integral curves of $\mathbf{Y}$, one arrives at the same point as if one first went a distance s along the integral curves of **Y** and then a parameter distance t along the integral curves of **X** (see figure 7). Thus the set of all points which can be reached along integral curves of **X** and **Y** from a given point p will then form an immersed two-dimensional submanifold through p.

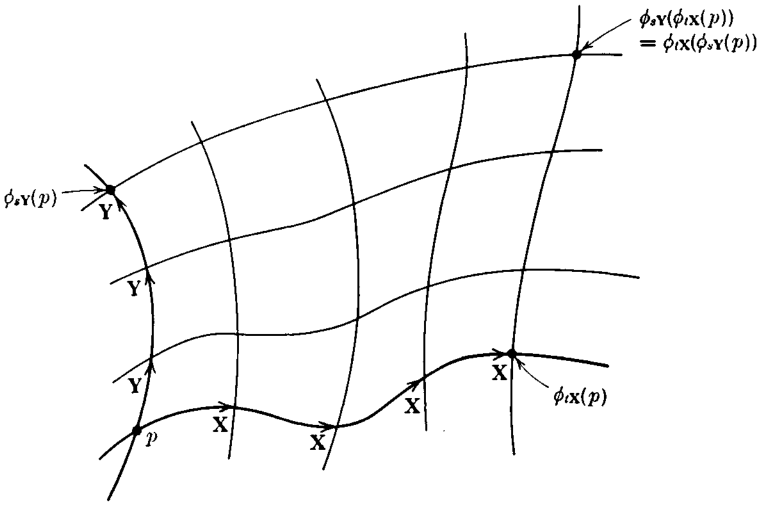

FIGURE 7. The transformations generated by commuting vector fields **X**, **Y** move a point p to points $\phi_{t\mathbf{X}}(p)$, $\phi_{s\mathbf{Y}}(p)$ respectively. By successive applications of these transformations, p is moved to the points of a two-surface.

The components of the Lie derivative of a one-form $\boldsymbol{\omega}$ may be found by contracting the relation

$$L_{\mathbf{X}}(\boldsymbol{\omega} \otimes \mathbf{Y}) = L_{\mathbf{X}}\boldsymbol{\omega} \otimes \mathbf{Y} + \boldsymbol{\omega} \otimes L_{\mathbf{X}}\mathbf{Y}$$

(Lie derivative property (3)) to obtain

$$L_{\mathbf{X}}\langle \boldsymbol{\omega}, \mathbf{Y}\rangle = \langle L_{\mathbf{X}}\boldsymbol{\omega}, \mathbf{Y}\rangle + \langle \boldsymbol{\omega}, L_{\mathbf{X}}\mathbf{Y}\rangle$$

(by property (2) of Lie derivatives), where **X**, **Y** are arbitrary C^1 vector fields, and then choosing **Y** as a basis vector $\mathbf{E}_i$. One finds the coordinate components (on choosing $\mathbf{E}_i = \partial/\partial x^i$) to be

$$(L_{\mathbf{X}}\boldsymbol{\omega})_i = (\partial \omega_i/\partial x^j)\, X^j + \omega_j(\partial X^j/\partial x^i)$$

because (2.11) implies

$$(L_{\mathbf{X}}(\partial/\partial x^i))^j = -\partial X^j/\partial x^i.$$

Similarly, one can find the components of the Lie derivative of any C^r ($r \geqslant 1$) tensor field **T** of type (r, s) by using Leibniz' rule on

$$L_{\mathbf{X}}(\mathbf{T} \otimes \mathbf{E}^a \otimes \ldots \otimes \mathbf{E}^d \otimes \mathbf{E}_e \otimes \ldots \otimes \mathbf{E}_g),$$

and then contracting on all positions. One finds the coordinate components to be

$$(L_{\mathbf{X}}T)^{ab\ldots d}{}_{ef\ldots g} = (\partial T^{ab\ldots d}{}_{ef\ldots g}/\partial x^i)X^i - T^{ib\ldots d}{}_{ef\ldots g}\,\partial X^a/\partial x^i$$
$$-\,(\text{all upper indices}) + T^{ab\ldots d}{}_{if\ldots g}\,\partial X^i/\partial x^e + (\text{all lower indices}). \tag{2.12}$$

Because of (2.7), any Lie derivative commutes with d, i.e. for any p-form field $\boldsymbol{\omega}$,

$$\mathrm{d}(L_{\mathbf{X}}\boldsymbol{\omega}) = L_{\mathbf{X}}(\mathrm{d}\boldsymbol{\omega}).$$

From these formulae, as well as from the geometrical interpretation, it follows that the Lie derivative $L_{\mathbf{X}}\mathbf{T}|_p$ of a tensor field $\mathbf{T}$ of type (r, s) depends not only on the direction of the vector field $\mathbf{X}$ at the point p, but also on the direction of $\mathbf{X}$ at neighbouring points. Thus the two differential operators defined by the manifold structure are too limited to serve as the generalization of the concept of a partial derivative one needs in order to set up field equations for physical quantities on the manifold; d operates only on forms, while the ordinary partial derivative is a directional derivative depending only on a direction at the point in question, unlike the Lie derivative. One obtains such a generalized derivative, the covariant derivative, by introducing extra structure on the manifold. We do this in the next section.

2.5 Covariant differentiation and the curvature tensor

The extra structure we introduce is a (affine) connection on $\mathscr{M}$. A *connection* ∇ at a point p of $\mathscr{M}$ is a rule which assigns to each vector field $\mathbf{X}$ at p a differential operator $\nabla_{\mathbf{X}}$ which maps an arbitrary C^r $(r \geqslant 1)$ vector field $\mathbf{Y}$ into a vector field $\nabla_{\mathbf{X}}\mathbf{Y}$, where:

(1) $\nabla_{\mathbf{X}}\mathbf{Y}$ is a tensor in the argument $\mathbf{X}$, i.e. for any functions f, g, and C^1 vector fields $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$,

$$\nabla_{f\mathbf{X}+g\mathbf{Y}}\mathbf{Z} = f\nabla_{\mathbf{X}}\mathbf{Z} + g\nabla_{\mathbf{Y}}\mathbf{Z};$$

(this is equivalent to the requirement that the derivative $\nabla_{\mathbf{X}}$ at p depends only on the direction of $\mathbf{X}$ at p);

(2) $\nabla_{\mathbf{X}}\mathbf{Y}$ is linear in $\mathbf{Y}$, i.e. for any C^1 vector fields $\mathbf{Y}$, $\mathbf{Z}$ and $\alpha, \beta \in R^1$,

$$\nabla_{\mathbf{X}}(\alpha\mathbf{Y} + \beta\mathbf{Z}) = \alpha\nabla_{\mathbf{X}}\mathbf{Y} + \beta\nabla_{\mathbf{X}}\mathbf{Z};$$

(3) for any C^1 function f and C^1 vector field $\mathbf{Y}$,

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = X(f)\mathbf{Y} + f\nabla_{\mathbf{X}}\mathbf{Y}.$$

Then $\nabla_{\mathbf{X}}\mathbf{Y}$ is the *covariant derivative* (with respect to ∇) of $\mathbf{Y}$ *in the direction* $\mathbf{X}$ at p. By (1), we can define $\nabla\mathbf{Y}$, the *covariant derivative* of $\mathbf{Y}$, as that tensor field of type (1, 1) which, when contracted with $\mathbf{X}$, produces the vector $\nabla_{\mathbf{X}}\mathbf{Y}$. Then we have

$$(3) \Leftrightarrow \nabla(f\mathbf{Y}) = \mathrm{d}f \otimes \mathbf{Y} + f\nabla\mathbf{Y}.$$

A C^r *connection* ∇ on a C^k manifold $\mathscr{M}$ ($k \geqslant r+2$) is a rule which assigns a connection ∇ to each point such that if $\mathbf{Y}$ is a C^{r+1} vector field on $\mathscr{M}$, then $\nabla\mathbf{Y}$ is a C^r tensor field.

Given any C^{r+1} vector basis $\{\mathbf{E}_a\}$ and dual one-form basis $\{\mathbf{E}^a\}$ on a neighbourhood $\mathscr{U}$, we shall write the components of $\nabla\mathbf{Y}$ as $Y^a{}_{;b}$, so

$$\nabla\mathbf{Y} = Y^a{}_{;b}\,\mathbf{E}^b \otimes \mathbf{E}_a.$$

The connection is determined on $\mathscr{U}$ by n^3 C^r functions $\Gamma^a{}_{bc}$ defined by

$$\Gamma^a{}_{bc} = \langle \mathbf{E}^a, \nabla_{\mathbf{E}_b}\mathbf{E}_c \rangle \Leftrightarrow \nabla\mathbf{E}_c = \Gamma^a{}_{bc}\,\mathbf{E}^b \otimes \mathbf{E}_a.$$

For any C^1 vector field $\mathbf{Y}$,

$$\nabla\mathbf{Y} = \nabla(Y^c\mathbf{E}_c) = \mathrm{d}Y^c \otimes \mathbf{E}_c + Y^c\Gamma^a{}_{bc}\,\mathbf{E}^b \otimes \mathbf{E}_a.$$

Thus the components of $\nabla\mathbf{Y}$ with respect to coordinate bases $\{\partial/\partial x^a\}$, $\{\mathrm{d}x^b\}$ are

$$Y^a{}_{;b} = \partial Y^a/\partial x^b + \Gamma^a{}_{bc}\,Y^c.$$

The transformation properties of the functions $\Gamma^a{}_{bc}$ are determined by connection properties (1), (2), (3); for

$$\begin{aligned}\Gamma^{a'}{}_{b'c'} &= \langle \mathbf{E}^{a'}, \nabla_{\mathbf{E}_{b'}}\mathbf{E}_{c'} \rangle = \langle \Phi^{a'}{}_a\,\mathbf{E}^a, \nabla_{\Phi_{b'}{}^b\mathbf{E}_b}(\Phi_{c'}{}^c\mathbf{E}_c)\rangle \\ &= \Phi^{a'}{}_a\,\Phi_{b'}{}^b(E_b(\Phi_{c'}{}^a) + \Phi_{c'}{}^c\,\Gamma^a{}_{bc})\end{aligned}$$

if $\mathbf{E}_{a'} = \Phi_{a'}{}^a\,\mathbf{E}_a$, $\mathbf{E}^{a'} = \Phi^{a'}{}_a\,\mathbf{E}^a$. One can rewrite this as

$$\Gamma^{a'}{}_{b'c'} = \Phi^{a'}{}_a(E_{b'}(\Phi_{c'}{}^a) + \Phi_{b'}{}^b\,\Phi_{c'}{}^c\,\Gamma^a{}_{bc}).$$

In particular, if the bases are coordinate bases defined by coordinates $\{x^a\}$, $\{x^{a'}\}$, the transformation law is

$$\Gamma^{a'}{}_{b'c'} = \frac{\partial x^{a'}}{\partial x^a}\left(\frac{\partial^2 x^a}{\partial x^{b'}\,\partial x^{c'}} + \frac{\partial x^b}{\partial x^{b'}}\frac{\partial x^c}{\partial x^{c'}}\,\Gamma^a{}_{bc}\right).$$

Because of the term $E_{b'}(\Phi_{c'}{}^a)$, the $\Gamma^a{}_{bc}$ do not transform as the components of a tensor. However if $\nabla\mathbf{Y}$ and $\hat{\nabla}\mathbf{Y}$ are covariant derivatives obtained from two different connections, then

$$\nabla\mathbf{Y} - \hat{\nabla}\mathbf{Y} = (\Gamma^a{}_{bc} - \hat{\Gamma}^a{}_{bc})\,Y^c\mathbf{E}^b \otimes \mathbf{E}_a$$

will be a tensor. Thus the difference terms $(\Gamma^a{}_{bc} - \hat{\Gamma}^a{}_{bc})$ will be the components of a tensor.

The definition of a covariant derivative can be extended to any C^r tensor field if $r \geqslant 1$ by the rules (cf. the Lie derivative rules):

(1) if $\mathbf{T}$ is a C^r tensor field of type (q, s), then $\nabla\mathbf{T}$ is a C^{r-1} tensor field of type $(q, s+1)$;

(2) ∇ is linear and commutes with contractions;

(3) for arbitrary tensor fields $\mathbf{S}$, $\mathbf{T}$, Liebniz' rule holds, i.e.

$$\nabla(\mathbf{S}\otimes\mathbf{T}) = \nabla\mathbf{S}\otimes\mathbf{T}+\mathbf{S}\otimes\nabla\mathbf{T};$$

(4) $\nabla f = \mathrm{d}f$ for any function f.

We write the components of $\nabla\mathbf{T}$ as $(\nabla_{\mathbf{E}_h}T)^{a\ldots d}{}_{e\ldots g} = T^{a\ldots d}{}_{e\ldots g;h}$. As a consequence of (2) and (3),

$$\nabla_{\mathbf{E}_b}\mathbf{E}^c = -\Gamma^c{}_{ba}\mathbf{E}^a,$$

where $\{\mathbf{E}^a\}$ is the dual basis to $\{\mathbf{E}_a\}$, and methods similar to those used in deriving (2.12) show that the coordinate components of $\nabla\mathbf{T}$ are

$$T^{ab\ldots d}{}_{ef\ldots g;h} = \partial T^{ab\ldots d}{}_{ef\ldots g}/\partial x^h + \Gamma^a{}_{hj}T^{jb\ldots d}{}_{ef\ldots g}$$
$$+\text{(all upper indices)} - \Gamma^j{}_{he}T^{ab\ldots d}{}_{jf\ldots g} - \text{(all lower indices)}. \quad (2.13)$$

As a particular example, the unit tensor $\mathbf{E}_a\otimes\mathbf{E}^a$, which has components $\delta^a{}_b$, has vanishing covariant derivative, and so the generalized unit tensors with components $\delta^{(a_1}{}_{b_1}\delta^{a_2}{}_{b_2}\ldots\delta^{a_s)}{}_{b_s}$, $\delta^{[a_1}{}_{b_1}\delta^{a_2}{}_{b_2}\ldots\delta^{a_p]}{}_{b_p}$ $(p \leqslant n)$ also have vanishing covariant derivatives.

If $\mathbf{T}$ is a C^r $(r \geqslant 1)$ tensor field defined along a C^r curve $\lambda(t)$, one can define $\mathrm{D}\mathbf{T}/\partial t$, the *covariant derivative of* $\mathbf{T}$ *along* $\lambda(t)$, as $\nabla_{\partial/\partial t}\bar{\mathbf{T}}$ where $\bar{\mathbf{T}}$ is any C^r tensor field extending $\mathbf{T}$ onto an open neighbourhood of λ. $\mathrm{D}\mathbf{T}/\partial t$ is a C^{r-1} tensor field defined along $\lambda(t)$, and is independent of the extension $\bar{\mathbf{T}}$. In terms of components, if $\mathbf{X}$ is the tangent vector to $\lambda(t)$, then $\mathrm{D}T^{a\ldots d}{}_{e\ldots g}/\partial t = T^{a\ldots d}{}_{e\ldots g;h}X^h$. In particular one can choose local coordinates so that $\lambda(t)$ has the coordinates $x^a(t)$, $X^a = \mathrm{d}x^a/\mathrm{d}t$, and then for a vector field $\mathbf{Y}$

$$\mathrm{D}Y^a/\partial t = \partial Y^a/\partial t + \Gamma^a{}_{bc}Y^c\,\mathrm{d}x^b/\mathrm{d}t. \quad (2.14)$$

The tensor $\mathbf{T}$ is said to be *parallelly transported* along λ if $\mathrm{D}\mathbf{T}/\partial t = 0$. Given a curve $\lambda(t)$ with endpoints p, q, the theory of solutions of ordinary differential equations shows that if the connection ∇ is at least C^{1-} one obtains a unique tensor at q by parallelly transferring any given tensor from p along λ. Thus parallel transfer along λ is a linear map from $T^r_s(p)$ to $T^r_s(q)$ which preserves all tensor products and tensor contractions, so in particular if one parallelly transfers a basis of vectors along a given curve from p to q, this determines an isomorphism of T_p to T_q. (If there are self-intersections in the curve, p and q could be the *same* point.)

A particular case is obtained by considering the covariant derivative of the tangent vector itself along λ. The curve $\lambda(t)$ is said to be a *geodesic curve* if

$$\nabla_{\mathbf{X}}\mathbf{X} = \frac{\mathrm{D}}{\partial t}\left(\frac{\partial}{\partial t}\right)_{\lambda}$$

is parallel to $(\partial/\partial t)_{\lambda}$, i.e. if there is a function f (perhaps zero) such that $X^a{}_{;b}X^b = fX^a$. For such a curve, one can find a new parameter $v(t)$ along the curve such that

$$\frac{\mathrm{D}}{\partial v}\left(\frac{\partial}{\partial v}\right)_{\lambda} = 0;$$

such a parameter is called an *affine parameter*. The associated tangent vector $\mathbf{V} = (\partial/\partial v)_{\lambda}$ is parallel to $\mathbf{X}$ but has its scale determined by $V(v) = 1$; it obeys the equations

$$V^a{}_{;b}V^b = 0 \Leftrightarrow \frac{\mathrm{d}^2x^a}{\mathrm{d}v^2} + \Gamma^a{}_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}v}\frac{\mathrm{d}x^c}{\mathrm{d}v} = 0, \tag{2.15}$$

the second expression being the local coordinate expression obtainable from (2.14) applied to the vector $\mathbf{V}$. The affine parameter of a geodesic curve is determined up to an additive and a multiplicative constant, i.e. up to transformations $v' = av+b$ where a, b are constants; the freedom of choice of b corresponds to the freedom to choose a new initial point $\lambda(0)$, the freedom of choice in a corresponding to the freedom to renormalize the vector $\mathbf{V}$ by a constant scale factor, $\mathbf{V}' = (1/a)\,\mathbf{V}$. The curve parametrized by any of these affine parameters is said to be a *geodesic*.

Given a C^r ($r \geqslant 0$) connection, the standard existence theorems for ordinary differential equations applied to (2.15) show that for any point p of $\mathscr{M}$ and any vector $\mathbf{X}_p$ at p, there exists a maximal geodesic $\lambda_{\mathbf{X}}(v)$ in $\mathscr{M}$ with starting point p and initial direction $\mathbf{X}_p$, i.e. such that $\lambda_{\mathbf{X}}(0) = p$ and $(\partial/\partial v)_{\lambda}|_{v=0} = \mathbf{X}_p$. If $r \geqslant 1-$, this geodesic is unique and depends continuously on p and $\mathbf{X}_p$. If $r \geqslant 1$, it depends differentiably on p and $\mathbf{X}_p$. This means that if $r \geqslant 1$, one can define a C^r map exp: $T_p \to \mathscr{M}$, where for each $\mathbf{X} \in T_p$, exp ($\mathbf{X}$) is the point in $\mathscr{M}$ a unit parameter distance along the geodesic $\lambda_{\mathbf{X}}$ from p. This map may not be defined for all $\mathbf{X} \in T_p$, since the geodesic $\lambda_{\mathbf{X}}(v)$ may not be defined for all v. If v does take all values, the geodesic $\lambda(v)$ will be said to be a *complete* geodesic. The manifold $\mathscr{M}$ is said to be *geodesically complete* if all geodesics on $\mathscr{M}$ are complete, that is if exp is defined on all T_p for every point p of $\mathscr{M}$.

Whether $\mathscr{M}$ is complete or not, the map $\exp_p$ is of rank n at p. Therefore by the implicit function theorem (Spivak (1965)) there exists an open neighbourhood $\mathscr{N}_0$ of the origin in T_p and an open neighbourhood $\mathscr{N}_p$ of p in $\mathscr{M}$ such that the map exp is a C^r diffeomorphism of $\mathscr{N}_0$ onto $\mathscr{N}_p$. Such a neighbourhood $\mathscr{N}_p$ is called a *normal neighbourhood* of p. Further, one can choose $\mathscr{N}_p$ to be *convex*, i.e. to be such that any point q of $\mathscr{N}_p$ can be joined to any other point r in $\mathscr{N}_p$ by a unique geodesic starting at q and totally contained in $\mathscr{N}_p$. Within a convex normal neighbourhood $\mathscr{N}$ one can define coordinates $(x^1, \ldots, x^n)$ by choosing any point $q \in \mathscr{N}$, choosing a basis $\{\mathbf{E}_a\}$ of T_q, and defining the coordinates of the point r in $\mathscr{N}$ by the relation $r = \exp(x^a \mathbf{E}_a)$ (i.e. one assigns to r the coordinates, with respect to the basis $\{\mathbf{E}_a\}$, of the point $\exp^{-1}(r)$ in T_q.) Then $(\partial/\partial x^i)|_q = \mathbf{E}_i$ and (by (2.15)) $\Gamma^i{}_{(jk)}|_q = 0$. Such coordinates will be called *normal coordinates* based on q. The existence of normal neighbourhoods has been used by Geroch (1968*c*) to prove that a connected C^3 Hausdorff manifold $\mathscr{M}$ with a C^1 connection has a countable basis. Thus one may infer the property of paracompactness of a C^3 manifold from the existence of a C^1 connection on the manifold. The 'normal' local behaviour of geodesics in these neighbourhoods is in contrast to the behaviour of geodesics in the large in a general space, where on the one hand two arbitrary points cannot in general be joined by any geodesic, and on the other hand some of the geodesics through one point may converge to 'focus' at some other point. We shall later encounter examples of both types of behaviour.

Given a C^r connection ∇, one can define a C^{r-1} tensor field $\mathbf{T}$ of type (1, 2) by the relation

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}],$$

where $\mathbf{X}$, $\mathbf{Y}$ are arbitrary C^r vector fields. This tensor is called the *torsion tensor*. Using a coordinate basis, its components are

$$T^i{}_{jk} = \Gamma^i{}_{jk} - \Gamma^i{}_{kj}.$$

We shall deal only with *torsion-free* connections, i.e. we shall assume $\mathbf{T} = 0$. In this case, the coordinate components of the connection obey $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$, so such a connection is often called a symmetric connection. A connection is torsion-free if and only if $f_{;ij} = f_{;ji}$ for all functions f. From the geodesic equation (2.15) it follows that a torsion-free connection is completely determined by a knowledge of the geodesics on $\mathscr{M}$.

When the torsion vanishes, the covariant derivatives of arbitrary C^1 vector fields $\mathbf{X}$, $\mathbf{Y}$ are related to their Lie derivative by

$$[\mathbf{X}, \mathbf{Y}] = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} \Leftrightarrow (L_{\mathbf{X}}\mathbf{Y})^a = Y^a{}_{;b}X^b - X^a{}_{;b}Y^b, \quad (2.16)$$

and for any C^1 tensor field **T** of type (r,s) one finds

$$(L_{\mathbf{X}}T)^{ab\ldots d}{}_{ef\ldots g} = T^{ab\ldots d}{}_{ef\ldots g;h}X^h - T^{jb\ldots d}{}_{ef\ldots g}X^a{}_{;j}$$
$$-\text{(all upper indices)} + T^{ab\ldots d}{}_{jf\ldots g}X^j{}_{;e} + \text{(all lower indices)}. \quad (2.17)$$

One can also easily verify that the exterior derivative is related to the covariant derivative by

$$\mathrm{d}\mathbf{A} = A_{a\ldots c;\,d}\,\mathrm{d}x^d \wedge \mathrm{d}x^a \wedge \ldots \wedge \mathrm{d}x^c \Leftrightarrow (\mathrm{d}A)_{a\ldots cd} = (-)^p A_{[a\ldots c;\,d]},$$

where **A** is any p-form. Thus equations involving the exterior derivative or Lie derivative can always be expressed in terms of the covariant derivative. However, because of their definitions, the Lie derivative and exterior derivative are independent of the connection.

If one starts from a given point p and parallelly transfers a vector $\mathbf{X}_p$ along a curve γ that ends at p again, one will obtain a vector $\mathbf{X}'_p$ which is in general different from $\mathbf{X}_p$; if one chooses a different curve γ', the new vector one obtains at p will in general be different from $\mathbf{X}_p$ and $\mathbf{X}'_p$. This non-integrability of parallel transfer corresponds to the fact that the covariant derivatives do not generally commute. The *Riemann* (*curvature*) *tensor* gives a measure of this non-commutation. Given C^{r+1} vector fields **X**, **Y**, **Z**, a C^{r-1} vector field $\mathbf{R}(\mathbf{X},\mathbf{Y})\,\mathbf{Z}$ is defined by a C^r connection ∇ by

$$\mathbf{R}(\mathbf{X},\mathbf{Y})\,\mathbf{Z} = \nabla_{\mathbf{X}}(\nabla_{\mathbf{Y}}\mathbf{Z}) - \nabla_{\mathbf{Y}}(\nabla_{\mathbf{X}}\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}. \quad (2.18)$$

Then $\mathbf{R}(\mathbf{X},\mathbf{Y})\,\mathbf{Z}$ is linear in **X**, **Y**, **Z** and it may be verified that the value of $\mathbf{R}(\mathbf{X},\mathbf{Y})\,\mathbf{Z}$ at p depends only on the values of **X**, **Y**, **Z** at p, i.e. it is a C^{r-1} tensor field of type (3, 1). To write (2.18) in component form, we define the second covariant derivative $\nabla\nabla\mathbf{Z}$ of the vector **Z** as the covariant derivative $\nabla(\nabla\mathbf{Z})$ of $\nabla\mathbf{Z}$; it has components

$$Z^a{}_{;\,bc} = (Z^a{}_{;\,b})_{;\,c}.$$

Then (2.18) can be written

$$\begin{aligned} R^a{}_{bcd}X^cY^dZ^b &= (Z^a{}_{;\,d}Y^d)_{;\,c}X^c - (Z^a{}_{;\,d}X^d)_{;\,c}Y^c \\ &\qquad - Z^a{}_{;\,d}(Y^d{}_{;\,c}X^c - X^d{}_{;\,c}Y^c) \\ &= (Z^a{}_{;\,dc} - Z^a{}_{;\,cd})\,X^cY^d, \end{aligned}$$

where the Riemann tensor components $R^a{}_{bcd}$ with respect to dual bases $\{\mathbf{E}_a\}$, $\{\mathbf{E}^a\}$ are defined by $R^a{}_{bcd} = \langle \mathbf{E}^a, \mathbf{R}(\mathbf{E}_c,\mathbf{E}_d)\,\mathbf{E}_b\rangle$. As **X**, **Y** are arbitrary vectors,

$$Z^a{}_{;\,dc} - Z^a{}_{;\,cd} = R^a{}_{bcd}Z^b \quad (2.19)$$

expresses the non-commutation of second covariant derivatives of **Z** in terms of the Riemann tensor.

Since

$$\nabla_{\mathbf{X}}(\boldsymbol{\eta}\otimes\nabla_{\mathbf{Y}}\mathbf{Z}) = \nabla_{\mathbf{X}}\boldsymbol{\eta}\otimes\nabla_{\mathbf{Y}}\mathbf{Z}+\boldsymbol{\eta}\otimes\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z}$$
$$\Rightarrow \langle\boldsymbol{\eta}, \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z}\rangle = X(\langle\boldsymbol{\eta}, \nabla_{\mathbf{Y}}\mathbf{Z}\rangle) - \langle\nabla_{\mathbf{X}}\boldsymbol{\eta}, \nabla_{\mathbf{Y}}\mathbf{Z}\rangle$$

holds for any C^2 one-form field $\boldsymbol{\eta}$ and vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$, (2.18) implies

$$\langle\mathbf{E}^a, \mathbf{R}(\mathbf{E}_c, \mathbf{E}_d)\,\mathbf{E}_b\rangle = E_c(\langle\mathbf{E}^a, \nabla_{\mathbf{E}_d}\mathbf{E}_b\rangle) - E_d(\langle\mathbf{E}^a, \nabla_{\mathbf{E}_c}\mathbf{E}_b\rangle)$$
$$- \langle\nabla_{\mathbf{E}_c}\mathbf{E}^a, \nabla_{\mathbf{E}_d}\mathbf{E}_b\rangle + \langle\nabla_{\mathbf{E}_d}\mathbf{E}^a, \nabla_{\mathbf{E}_c}\mathbf{E}_b\rangle - \langle\mathbf{E}^a, \nabla_{[\mathbf{E}_c,\mathbf{E}_d]}\mathbf{E}_b\rangle.$$

Choosing the bases as coordinate bases, one finds the expression

$$R^a{}_{bcd} = \partial\Gamma^a{}_{db}/\partial x^c - \partial\Gamma^a{}_{cb}/\partial x^d + \Gamma^a{}_{cf}\Gamma^f{}_{db} - \Gamma^a{}_{df}\Gamma^f{}_{cb} \qquad (2.20)$$

for the coordinate components of the Riemann tensor, in terms of the coordinate components of the connection.

It can be verified from these definitions that in addition to the symmetry

$$R^a{}_{bcd} = -R^a{}_{bdc} \Leftrightarrow R^a{}_{b(cd)} = 0 \qquad (2.21a)$$

the curvature tensor has the symmetry

$$R^a{}_{[bcd]} = 0 \Leftrightarrow R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = 0. \qquad (2.21b)$$

Similarly the first covariant derivatives of the Riemann tensor satisfy *Bianchi's identities*

$$R^a{}_{b[cd;e]} = 0 \Leftrightarrow R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} = 0. \qquad (2.22)$$

It now turns out that parallel transfer of an arbitrary vector along an arbitrary closed curve is locally integrable (i.e. $\mathbf{X}'_p$ is necessarily the same as $\mathbf{X}_p$ for each $p\in\mathscr{M}$) only if $R^a{}_{bcd} = 0$ at all points of $\mathscr{M}$; in this case we say that the connection is *flat*.

By contracting the curvature tensor, one can define the *Ricci tensor* as the tensor of type (0, 2) with components

$$R_{bd} = R^a{}_{bad}.$$

2.6 The metric

A *metric tensor* $\mathbf{g}$ at a point $p\in\mathscr{M}$ is a symmetric tensor of type (0, 2) at p, so a C^r metric on $\mathscr{M}$ is a C^r symmetric tensor field $\mathbf{g}$. The metric $\mathbf{g}$ at p assigns a 'magnitude' $(|g(\mathbf{X},\mathbf{X})|)^{\frac{1}{2}}$ to each vector $\mathbf{X}\in T_p$ and defines the 'cos angle'

$$\frac{g(\mathbf{X},\mathbf{Y})}{(|g(\mathbf{X},\mathbf{X})\,.\,g(\mathbf{Y},\mathbf{Y})|)^{\frac{1}{2}}}$$

between any vectors $\mathbf{X},\mathbf{Y}\in T_p$ such that $g(\mathbf{X},\mathbf{X})\,.\,g(\mathbf{Y},\mathbf{Y}) \neq 0$; vectors $\mathbf{X},\mathbf{Y}$ will be said to be *orthogonal* if $g(\mathbf{X},\mathbf{Y}) = 0$.

The components of $\mathbf{g}$ with respect to a basis $\{\mathbf{E}_a\}$ are

$$g_{ab} = g(\mathbf{E}_a, \mathbf{E}_b) = g(\mathbf{E}_b, \mathbf{E}_a),$$

i.e. the components are simply the scalar products of the basis vectors $\mathbf{E}_a$. If a coordinate basis $\{\partial/\partial x^a\}$ is used, then

$$\mathbf{g} = g_{ab}\, \mathrm{d}x^a \otimes \mathrm{d}x^b. \tag{2.23}$$

Tangent space magnitudes defined by the metric are related to magnitudes on the manifold by the definition: the *path length* between points $p = \gamma(a)$ and $q = \gamma(b)$ along a C^0, piecewise C^1 curve $\gamma(t)$ with tangent vector $\partial/\partial t$ such that $g(\partial/\partial t, \partial/\partial t)$ has the same sign at all points along $\gamma(t)$, is the quantity

$$L = \int_a^b (|g(\partial/\partial t, \partial/\partial t)|)^{\frac{1}{2}}\, \mathrm{d}t. \tag{2.24}$$

We may symbolically express the relations (2.23), (2.24) in the form

$$\mathrm{d}s^2 = g_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$$

used in classical textbooks to represent the length of the 'infinitesimal' arc determined by the coordinate displacement $x^i \to x^i + \mathrm{d}x^i$.

The metric is said to be *non-degenerate* at p if there is no non-zero vector $\mathbf{X} \in T_p$ such that $g(\mathbf{X}, \mathbf{Y}) = 0$ for all vectors $\mathbf{Y} \in T_p$. In terms of components, the metric is non-degenerate if the matrix (g_{ab}) of components of $\mathbf{g}$ is non-singular. We shall from now on always assume the metric tensor is non-degenerate. Then we can define a unique symmetric tensor of type (2, 0) with components g^{ab} with respect to the basis $\{\mathbf{E}_a\}$ dual to the basis $\{\mathbf{E}^a\}$, by the relations

$$g^{ab}g_{bc} = \delta^a{}_c,$$

i.e. the matrix (g^{ab}) of components is the inverse of the matrix (g_{ab}). It follows that the matrix (g^{ab}) is also non-singular, so the tensors g^{ab}, g_{ab} can be used to give an isomorphism between any covariant tensor argument and any contravariant argument, or to 'raise and lower indices'. Thus, if X^a are the components of a contravariant vector, then X_a are the components of a uniquely associated covariant vector, where $X_a = g_{ab}X^b$, $X^a = g^{ab}X_b$; similarly, to a tensor T_{ab} of type $(0, 2)$ we can associate unique tensors $T^a{}_b = g^{ac}T_{cb}$, $T_a{}^b = g^{bc}T_{ac}$, $T^{ab} = g^{ac}g^{bd}T_{cd}$. We shall in general regard such associated covariant and contravariant tensors as representations of the same geometric object (so in particular, g_{ab}, $\delta_a{}^b$ and g^{ab} may be thought of as representations (with respect to dual bases) of the same geometric object $\mathbf{g}$), although in some cases where we have more than one metric we shall have to distinguish carefully which metric is used to raise or lower indices.

The *signature* of $\mathbf{g}$ at p is the number of positive eigenvalues of the matrix (g_{ab}) at p, minus the number of negative ones. If $\mathbf{g}$ is non-degenerate and continuous, the signature will be constant on $\mathscr{M}$; by suitable choice of the basis $\{\mathbf{E}_a\}$, the metric components can at any point p be brought to the form

$$g_{ab} = \operatorname{diag}(\underbrace{+1, +1, \ldots, +1}_{\frac{1}{2}(n+s)\text{ terms}}, \underbrace{-1, \ldots, -1}_{\frac{1}{2}(n-s)\text{ terms}}),$$

where s is the signature of $\mathbf{g}$ and n is the dimension of $\mathscr{M}$. In this case the basis vectors $\{\mathbf{E}_a\}$ form an orthonormal set at p, i.e. each is a unit vector orthogonal to every other basis vector.

A metric whose signature is n is called a *positive definite metric*; for such a metric, $g(\mathbf{X}, \mathbf{X}) = 0 \Rightarrow \mathbf{X} = 0$, and the canonical form is

$$g_{ab} = \operatorname{diag}(\underbrace{+1, \ldots, +1}_{n\text{ terms}}).$$

A positive definite metric is a 'metric' on the space, in the topological sense of the word.

A metric whose signature is $(n-2)$ is called a *Lorentz metric*; the canonical form is

$$g_{ab} = \operatorname{diag}(\underbrace{+1, \ldots, +1}_{(n-1)\text{ terms}}, -1).$$

With a Lorentz metric on $\mathscr{M}$, the non-zero vectors at p can be divided into three classes: a vector $\mathbf{X} \in T_p$ being said to be *timelike*, *null*, or *spacelike* according to whether $g(\mathbf{X}, \mathbf{X})$ is negative, zero, or positive, respectively. The null vectors form a double cone in T_p which separates the timelike from the spacelike vectors (see figure 8). If $\mathbf{X}$, $\mathbf{Y}$ are any two non-spacelike (i.e. timelike or null) vectors in the same half of the light cone at p, then $g(\mathbf{X}, \mathbf{Y}) \leqslant 0$, and equality can only hold if $\mathbf{X}$ and $\mathbf{Y}$ are parallel null vectors (i.e. if $\mathbf{X} = \alpha\mathbf{Y}$, $g(\mathbf{X}, \mathbf{X}) = 0$).

Any paracompact C^r manifold admits a C^{r-1} positive definite metric (that is, one defined on the whole of $\mathscr{M}$). To see this, let $\{f_\alpha\}$ be a partition of unity for a locally finite atlas $\{\mathscr{U}_\alpha, \phi_\alpha\}$. Then one can define g by

$$g(\mathbf{X}, \mathbf{Y}) = \sum_\alpha f_\alpha \langle (\phi_\alpha)_* \mathbf{X}, (\phi_\alpha)_* \mathbf{Y} \rangle,$$

where $\langle\ ,\ \rangle$ is the natural scalar product in Euclidean space R^n; thus one uses the atlas to determine the metric by mapping the

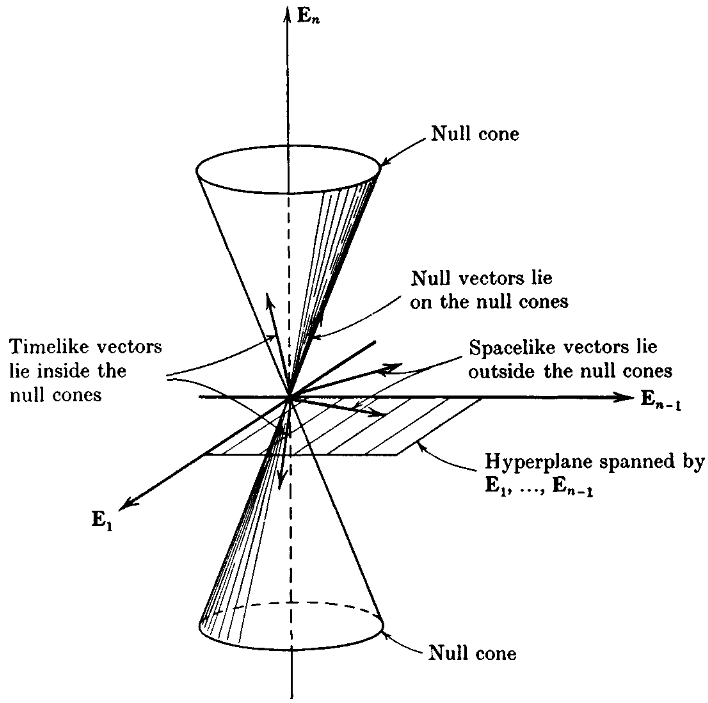

FIGURE 8. The null cones defined by a Lorentz metric.

Euclidean metric into $\mathscr{M}$. This is clearly not invariant under change of atlas, so there are many such positive definite metrics on $\mathscr{M}$.

In contrast to this, a C^r paracompact manifold admits a C^{r-1} Lorentz metric if and only if it admits a non-vanishing C^{r-1} line element field; by a line element field is meant an assignment of a pair of equal and opposite vectors $(\mathbf{X}, -\mathbf{X})$ at each point p of $\mathscr{M}$, i.e. a line element field is like a vector field but with undetermined sign. To see this, let $\hat{\mathbf{g}}$ be a C^{r-1} positive definite metric defined on the manifold. Then one can define a Lorentz metric $\mathbf{g}$ by

$$g(\mathbf{Y},\mathbf{Z}) = \hat{g}(\mathbf{Y},\mathbf{Z}) - 2\frac{\hat{g}(\mathbf{X},\mathbf{Y})\,\hat{g}(\mathbf{X},\mathbf{Z})}{\hat{g}(\mathbf{X},\mathbf{X})}$$

at each point p, where $\mathbf{X}$ is one of the pair $(\mathbf{X}, -\mathbf{X})$ at p. (Note that as $\mathbf{X}$ appears an even number of times, it does not matter whether $\mathbf{X}$ or $-\mathbf{X}$ is chosen.) Then $g(\mathbf{X},\mathbf{X}) = -\hat{g}(\mathbf{X},\mathbf{X})$, and if $\mathbf{Y}$, $\mathbf{Z}$ are orthogonal to $\mathbf{X}$ with respect to $\hat{\mathbf{g}}$, they are also orthogonal to $\mathbf{X}$ with respect to $\mathbf{g}$ and $g(\mathbf{Y},\mathbf{Z}) = \hat{g}(\mathbf{Y},\mathbf{Z})$. Thus an orthonormal basis for $\hat{\mathbf{g}}$ is also an orthonormal basis for $\mathbf{g}$. As $\hat{\mathbf{g}}$ is not unique, there are in fact many

Lorentz metrics on $\mathscr{M}$ if there is one. Conversely, if $\mathbf{g}$ is a given Lorentz metric, consider the equation $g_{ab}X^b = \lambda \hat{g}_{ab} X^b$ where $\hat{\mathbf{g}}$ is any positive definite metric. This will have one negative and $(n-1)$ positive eigenvalues. Thus the eigenvector field $\mathbf{X}$ corresponding to the negative eigenvalue will locally be a vector field determined up to a sign and a normalizing factor; one can normalize it by $g_{ab}X^aX^b = -1$, so defining a line element field on $\mathscr{M}$.

In fact, any non-compact manifold admits a line element field, while a compact manifold does so if and only if its Euler invariant is zero (e.g. the torus T^2 does, but the sphere S^2 does not, admit a line element field). It will later turn out that a manifold can be a reasonable model of space–time only if it is non-compact, so there will exist many Lorentz metrics on $\mathscr{M}$.

So far, the metric tensor and connection have been introduced as separate structures on $\mathscr{M}$. However given a metric $\mathbf{g}$ on $\mathscr{M}$, there is a unique torsion-free connection on $\mathscr{M}$ defined by the condition: the covariant derivative of $\mathbf{g}$ is zero, i.e.

$$g_{ab;c} = 0. \tag{2.25}$$

With this connection, parallel transfer of vectors preserves scalar products defined by $\mathbf{g}$, so in particular magnitudes of vectors are invariant. For example if $\partial/\partial t$ is the tangent vector to a geodesic, then $g(\partial/\partial t, \partial/\partial t)$ is constant along the geodesic.

From (2.25) it follows that

$$\begin{aligned} X(g(\mathbf{Y},\mathbf{Z})) = \nabla_{\mathbf{X}}(g(\mathbf{Y},\mathbf{Z})) &= \nabla_{\mathbf{X}} g(\mathbf{Y},\mathbf{Z}) + g(\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}) \\ &+ g(\mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}) = g(\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}) + g(\mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}) \end{aligned}$$

holds for arbitrary C^1 vector fields $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$. Adding the similar expression for $Y(g(\mathbf{Z},\mathbf{X}))$ and subtracting that for $Z(g(\mathbf{X},\mathbf{Y}))$ shows

$$\begin{aligned} g(\mathbf{Z},\nabla_{\mathbf{X}}\mathbf{Y}) = \tfrac{1}{2}\{&-Z(g(\mathbf{X},\mathbf{Y})) + Y(g(\mathbf{Z},\mathbf{X})) + X(g(\mathbf{Y},\mathbf{Z})) \\ &+ g(\mathbf{Z},[\mathbf{X},\mathbf{Y}]) + g(\mathbf{Y},[\mathbf{Z},\mathbf{X}]) - g(\mathbf{X},[\mathbf{Y},\mathbf{Z}])\}. \end{aligned}$$

Choosing $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ as basis vectors, one obtains the connection components

$$\Gamma_{abc} = g(\mathbf{E}_a, \nabla_{\mathbf{E}_b}\mathbf{E}_c) = g_{ad}\Gamma^d{}_{bc}$$

in terms of the derivatives of the metric components $g_{ab} = g(\mathbf{E}_a, \mathbf{E}_b)$, and the Lie derivatives of the basis vectors. In particular, on using a coordinate basis these Lie derivatives vanish, so one obtains the usual Christoffel relations

$$\Gamma_{abc} = \tfrac{1}{2}\{\partial g_{ab}/\partial x^c + \partial g_{ac}/\partial x^b - \partial g_{bc}/\partial x^a\} \tag{2.26}$$

for the coordinate components of the connection.

From now on we will assume that the connection on $\mathscr{M}$ is the unique C^{r-1} torsion-free connection determined by the C^r metric $\mathbf{g}$. Using this connection, one can define normal coordinates (§2.5) in a neighbourhood of a point q using an orthonormal basis of vectors at q. In these coordinates the components g_{ab} of $\mathbf{g}$ at q will be $\pm\delta_{ab}$ and the components $\Gamma^a{}_{bc}$ of the connection will vanish at q. By 'normal coordinates', we shall in future mean normal coordinates defined using an orthonormal basis.

The Riemann tensor of the connection defined by the metric is a C^{r-2} tensor with the symmetry

$$R_{(ab)cd} = 0 \Leftrightarrow R_{abcd} = -R_{bacd} \tag{2.27a}$$

in addition to the symmetries (2.21); as a consequence of (2.21) and (2.27a), the Riemann tensor is also symmetric in the pairs of indices $\{ab\}$, $\{cd\}$, i.e.

$$R_{abcd} = R_{cdab}. \tag{2.27b}$$

This implies that the Ricci tensor is symmetric:

$$R_{ab} = R_{ba}. \tag{2.27c}$$

The *curvature scalar* R is the contraction of the Ricci tensor:

$$R = R^a{}_a = R^a{}_{bad}\, g^{bd}.$$

With these symmetries, there are $\frac{1}{12}n^2(n^2-1)$ algebraically independent components of R_{abcd}, where n is the dimension of M; $\frac{1}{2}n(n+1)$ of them can be represented by the components of the Ricci tensor. If $n = 1$, $R_{abcd} = 0$; if $n = 2$ there is one independent component of R_{abcd}, which is essentially the function R. If $n = 3$, the Ricci tensor completely determines the curvature tensor; if $n > 3$, the remaining components of the curvature tensor can be represented by the *Weyl tensor* C_{abcd}, defined by

$$C_{abcd} = R_{abcd} + \frac{2}{n-2}\{g_{a[d}R_{c]b} + g_{b[c}R_{d]a}\} + \frac{2}{(n-1)(n-2)} R g_{a[c}g_{d]b}.$$

As the last two terms on the right-hand side have the curvature tensor symmetries (2.21), (2.27), it follows that C_{abcd} also has these symmetries. One can easily verify that in addition,

$$C^a{}_{bad} = 0,$$

i.e. one can think of the Weyl tensor as that part of the curvature tensor such that all contractions vanish.

An alternative characterization of the Weyl tensor is given by the fact that it is a conformal invariant. The metrics $\mathbf{g}$ and $\hat{\mathbf{g}}$ are said to be *conformal* if

$$\hat{\mathbf{g}} = \Omega^2 \mathbf{g} \tag{2.28}$$

for some non-zero suitably differentiable function Ω. Then for any vectors $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{V}$, $\mathbf{W}$ at a point p,

$$\frac{g(\mathbf{X},\mathbf{Y})}{g(\mathbf{V},\mathbf{W})} = \frac{\hat{g}(\mathbf{X},\mathbf{Y})}{\hat{g}(\mathbf{V},\mathbf{W})},$$

so angles and ratios of magnitudes are preserved under conformal transformations; in particular, the null cone structure in T_p is preserved by conformal transformations, since

$$g(\mathbf{X},\mathbf{X}) > 0,\ = 0,\ < 0 \Rightarrow \hat{g}(\mathbf{X},\mathbf{X}) > 0,\ = 0,\ < 0,$$

respectively. As the metric components are related by

$$\hat{g}_{ab} = \Omega^2 g_{ab}, \quad \hat{g}^{ab} = \Omega^{-2} g^{ab},$$

the coordinate components of the connections defined by the metrics (2.28) are related by

$$\hat{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + \Omega^{-1}\left(\delta^a{}_b \frac{\partial \Omega}{\partial x^c} + \delta^a{}_c \frac{\partial \Omega}{\partial x^b} - g_{bc} g^{ad} \frac{\partial \Omega}{\partial x^d}\right). \tag{2.29}$$

Calculating the Riemann tensor of $\hat{\mathbf{g}}$, one finds

$$\hat{R}^{ab}{}_{cd} = \Omega^{-2} R^{ab}{}_{cd} + \delta^{[a}{}_{[c} \Omega^{b]}{}_{d]},$$

where $\Omega^a{}_b := 4\Omega^{-1}(\Omega^{-1})_{;be} g^{ae} - 2(\Omega^{-1})_{;c}(\Omega^{-1})_{;d} g^{cd} \delta^a{}_b$;

the covariant derivatives in this equation are those determined by the metric $\mathbf{g}$. Then (assuming $n > 2$)

$$\hat{R}^b{}_d = \Omega^{-2} R^b{}_d + (n-2)\,\Omega^{-1}(\Omega^{-1})_{;dc} g^{bc} - (n-2)^{-1} \Omega^{-n} (\Omega^{n-2})_{;ac} g^{ac} \delta^b{}_d$$

and $$\hat{C}^a{}_{bcd} = C^a{}_{bcd},$$

the last equation expressing the fact that the Weyl tensor is conformally invariant. These relations imply

$$\hat{R} = \Omega^{-2} R - 2(n-1)\,\Omega^{-3} \Omega_{;cd} g^{cd} - (n-1)(n-4)\,\Omega^{-4} \Omega_{;c} \Omega_{;d} g^{cd}. \tag{2.30}$$

Having split the Riemann tensor into a part represented by the Ricci tensor and a part represented by the Weyl tensor, one can use the Bianchi identities (2.22) to obtain differential relations between the Ricci tensor and the Weyl tensor: contracting (2.22) one obtains

$$R^a{}_{bcd;a} = R_{bd;c} - R_{bc;d} \tag{2.31}$$

and contracting again one obtains

$$R^a{}_{c;a} = \tfrac{1}{2}R_{;c}.$$

From the definition of the Weyl tensor, one can (if $n > 3$) rewrite (2.31) in the form

$$C^a{}_{bcd;a} = 2\frac{n-3}{n-2}\left(R_{b[d;c]} - \frac{1}{2(n-1)}g_{b[d}R_{;c]}\right). \tag{2.32}$$

If $n \leqslant 4$, (2.31) contain all the information in the Bianchi identities (2.22), so if $n = 4$, (2.32) are equivalent to these identities.

A diffeomorphism $\phi: \mathscr{M} \to \mathscr{M}$ will be said to be an *isometry* if it carries the metric into itself, that is, if the mapped metric $\phi_* \mathbf{g}$ is equal to $\mathbf{g}$ at every point. Then the map $\phi_*: T_p \to T_{\phi(p)}$ preserves scalar products, as

$$g(\mathbf{X}, \mathbf{Y})|_p = \phi_* g(\phi_* \mathbf{X}, \phi_* \mathbf{Y})|_{\phi(p)} = g(\phi_* \mathbf{X}, \phi_* \mathbf{Y})|_{\phi(p)}.$$

If the local one-parameter group of diffeomorphisms ϕ_t generated by a vector field $\mathbf{K}$ is a group of isometries (i.e. for each t, the transformation ϕ_t is an isometry) we call the vector field $\mathbf{K}$ a *Killing vector field*. The Lie derivative of the metric with respect to $\mathbf{K}$ is

$$L_{\mathbf{K}}\mathbf{g} = \lim_{t \to 0} \frac{1}{t}(\mathbf{g} - \phi_{t*}\mathbf{g}) = 0,$$

since $\mathbf{g} = \phi_{t*}\mathbf{g}$ for each t. But from (2.17), $L_{\mathbf{K}} g_{ab} = 2K_{(a;b)}$, so a Killing vector field $\mathbf{K}$ satisfies Killing's equation

$$K_{a;b} + K_{b;a} = 0. \tag{2.33}$$

Conversely, if $\mathbf{K}$ is a vector field which satisfies Killing's equation, then $L_{\mathbf{K}}\mathbf{g} = 0$, so

$$\begin{aligned}
\phi_{t*}\mathbf{g}|_p &= \mathbf{g}|_p + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t'}(\phi_{t'*}\mathbf{g})|_p \,\mathrm{d}t' \\
&= \mathbf{g}|_p + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(\phi_{t'*}\phi_{s*}\mathbf{g})_{s=0}|_p \mathrm{d}t' \\
&= \mathbf{g}|_p + \int_0^t \left(\phi_{t'*}\frac{\mathrm{d}}{\mathrm{d}s}\phi_{s*}\mathbf{g}\right)_{s=0}\bigg|_p \mathrm{d}t' \\
&= \mathbf{g}|_p - \int_0^t \phi_{t'*}(L_{\mathbf{K}}\mathbf{g}|_{\phi_{-t'}(p)})\,\mathrm{d}t' = \mathbf{g}|_p.
\end{aligned}$$

Thus $\mathbf{K}$ is a Killing vector field if and only if it satisfies Killing's equation. Then one can locally choose coordinates $x^a = (x^\nu, t)$ ($\nu = 1$ to $n-1$)

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such that $K^a = \partial x^a/\partial t = \delta^a{}_n$; in these coordinates Killing's equation takes the form

$$\partial g_{ab}/\partial t = 0 \Leftrightarrow g_{ab} = g_{ab}(x^\nu).$$

A general space will not have any symmetries, and so will not admit any Killing vector fields. However a special space may admit r linearly independent Killing vector fields $\mathbf{K}_\alpha$ ($\alpha = 1, \ldots, r$). It can be shown that the set of all Killing vector fields on such a space forms a Lie algebra of dimension r over R, with the algebra product given by the Lie bracket [,] (see (2.16)), where $0 \leqslant r \leqslant \frac{1}{2}n(n+1)$. (The upper limit may be lessened if the metric is degenerate.) The local group of diffeomorphisms generated by these vector fields is an r-dimensional Lie group of isometries of the manifold $\mathscr{M}$. The full group of isometries of $\mathscr{M}$ may include some discrete isometries (such as reflections in a plane) which are not generated by Killing vector fields; the symmetry properties of the space are completely characterized by this full group of isometries.

2.7 Hypersurfaces

If $\mathscr{S}$ is an $(n-1)$-dimensional manifold and $\theta: \mathscr{S} \to \mathscr{M}$ is an imbedding, the image $\theta(\mathscr{S})$ of $\mathscr{S}$ is said to be a *hypersurface* in $\mathscr{M}$. If $p \in \mathscr{S}$, the image of T_p in $T_{\theta(p)}$ under the map θ_* will be a $(n-1)$-dimensional plane through the origin. Thus there will be some non-zero form $\mathbf{n} \in T^*{}_{\theta(p)}$ such that for any vector $\mathbf{X} \in T_p$, $\langle \mathbf{n}, \theta_* \mathbf{X} \rangle = 0$. The form $\mathbf{n}$ is unique up to a sign and a normalizing factor, and if $\theta(\mathscr{S})$ is given locally by the equation $f = 0$ where $\mathrm{d}f \neq 0$ then $\mathbf{n}$ may be taken locally as $\mathrm{d}f$. If $\theta(\mathscr{S})$ is two-sided in $\mathscr{M}$, one can choose $\mathbf{n}$ to be a nowhere zero one-form field on $\theta(\mathscr{S})$. This will be the situation if $\mathscr{S}$ and $\mathscr{M}$ are both orientable manifolds. In this case, the choice of a direction of $\mathbf{n}$ will relate the orientations of $\theta(\mathscr{S})$ and of $\mathscr{M}$: if $\{x^i\}$ are local coordinates from the oriented atlas of $\mathscr{M}$ such that locally $\theta(\mathscr{S})$ has the equation $x^1 = 0$ and $\mathbf{n} = \alpha\, \mathrm{d}x^1$ where $\alpha > 0$, then $(x^2, \ldots, x^n)$ are oriented local coordinates for $\theta(\mathscr{S})$.

If $\mathbf{g}$ is a metric on $\mathscr{M}$, the imbedding will induce a metric $\theta^*\mathbf{g}$ on $\mathscr{S}$, where if $\mathbf{X}, \mathbf{Y} \in T_p$, $\theta^*g(\mathbf{X}, \mathbf{Y})|_p = g(\theta_* \mathbf{X}, \theta_* \mathbf{Y})|_{\theta(p)}$. This metric is sometimes called the first fundamental form of $\mathscr{S}$. If $\mathbf{g}$ is positive definite the metric $\theta^*\mathbf{g}$ will be positive definite, while if $\mathbf{g}$ is Lorentz, $\theta^*\mathbf{g}$ will be

(*a*) Lorentz if $g^{ab} n_a n_b > 0$ (in this case, $\theta(\mathscr{S})$ will be said to be a *timelike hypersurface*), (*b*) degenerate if $g^{ab}n_a n_b = 0$ (in this case, $\theta(\mathscr{S})$ will be said to be a *null hypersurface*),

(*c*) positive definite if $g^{ab}n_a n_b < 0$ (in this case, $\theta(\mathscr{S})$ will be said to be a *spacelike hypersurface*).

To see this, consider the vector $N^b = n_a g^{ab}$. This will be orthogonal to all the vectors tangent to $\theta(\mathscr{S})$, i.e. to all vectors in the subspace $H = \theta_*(T_p)$ in $T_{\theta(p)}$. Suppose first that $\mathbf{N}$ does not itself lie in this subspace. Then if $(\mathbf{E}_2, \ldots, \mathbf{E}_n)$ are a basis for T_p, $(\mathbf{N}, \theta_*(\mathbf{E}_2), \ldots, \theta_*(\mathbf{E}_n))$ will be linearly independent and so will be a basis for $T_{\theta(p)}$. The components of $\mathbf{g}$ with respect to this basis will be

$$g_{ab} = \begin{pmatrix} g(\mathbf{N},\mathbf{N}) & 0 \\ 0 & g(\theta_*(\mathbf{E}_i), \theta_*(\mathbf{E}_j)) \end{pmatrix} = \begin{pmatrix} g(\mathbf{N},\mathbf{N}) & 0 \\ 0 & \theta^* g(\mathbf{E}_i, \mathbf{E}_j) \end{pmatrix}.$$

As the metric $\mathbf{g}$ is assumed to be non-degenerate, this shows that $g(\mathbf{N},\mathbf{N}) \neq 0$. If $\mathbf{g}$ is positive definite, $g(\mathbf{N},\mathbf{N})$ must be positive and so the induced metric $\theta^*\mathbf{g}$ must also be positive definite. If $\mathbf{g}$ is Lorentz and $g(\mathbf{N},\mathbf{N}) = g^{ab}n_a n_b < 0$, then $\theta^*\mathbf{g}$ must be positive definite since the matrix of the components of $\mathbf{g}$ has only one negative eigenvalue. Similarly if $g(\mathbf{N},\mathbf{N}) = g^{ab}n_a n_b > 0$, then $\theta^*\mathbf{g}$ will be a Lorentz metric. Now suppose that $\mathbf{N}$ is tangent to $\theta(\mathscr{S})$. Then there is some non-zero vector $\mathbf{X} \in T_p$ such that $\theta_*(\mathbf{X}) = \mathbf{N}$. But $g(\mathbf{N}, \theta_*\mathbf{Y}) = 0$ for all $\mathbf{Y} \in T_p$, which implies $\theta^* g(\mathbf{X}, \mathbf{Y}) = 0$. Thus $\theta^*\mathbf{g}$ is degenerate. Also, taking $\mathbf{Y}$ to be $\mathbf{X}$, $g(\mathbf{N},\mathbf{N}) = g^{ab}n_a n_b = 0$.

If $g^{ab}n_a n_b \neq 0$, one can normalize the normal form $\mathbf{n}$ to have unit magnitude, i.e. $g^{ab}n_a n_b = \pm 1$. In this case the map $\theta^*: T^*_{\theta(p)} \to T^*_p$ will be one–one on the $(n-1)$-dimensional subspace $H^*_{\theta(p)}$ of $T^*_{\theta(p)}$ consisting of all forms $\boldsymbol{\omega}$ at $\theta(p)$ such that $g^{ab}n_a\omega_b = 0$, because $\theta^*\mathbf{n} = 0$ and $\mathbf{n}$ does not lie in H^*. Therefore the inverse $(\theta^*)^{-1}$ will be a map $\tilde{\theta}_*$ of T^*_p onto $H^*_{\theta(p)}$, and so into $T^*_{\theta(p)}$.

This map can be extended in the usual way to a map of covariant tensors on $\mathscr{S}$ to covariant tensors on $\theta(\mathscr{S})$ in $\mathscr{M}$; as there already is a map θ_* of contravariant tensors on $\mathscr{S}$ to $\theta(\mathscr{S})$, one can extend θ_* to a map $\tilde{\theta}_*$ of arbitrary tensors on $\mathscr{S}$ to $\theta(\mathscr{S})$. This map has the property that $\tilde{\theta}_*\mathbf{T}$ has zero contraction with $\mathbf{n}$ on all indices, i.e.

$$(\tilde{\theta}_* T)^{a\ldots b}{}_{c\ldots d}\, n_a = 0 \quad \text{and} \quad (\tilde{\theta}_* T)^{a\ldots b}{}_{c\ldots d}\, g^{ce} n_e = 0$$

for any tensor $\mathbf{T} \in T^r_s(\mathscr{S})$.

The tensor $\mathbf{h}$ on $\theta(\mathscr{S})$ is defined by $\mathbf{h} = \tilde{\theta}_*(\theta^*\mathbf{g})$. In terms of the normalized form $\mathbf{n}$ (remember $g^{ab}n_a n_b = \pm 1$),

$$h_{ab} = g_{ab} \mp n_a n_b$$

since this implies $\theta^*\mathbf{h} = \theta^*\mathbf{g}$ and $h_{ab}g^{bc}n_c = 0$.

The tensor $h^a{}_b = g^{ac}h_{cb}$ is a projection operator, i.e. $h^a{}_b h^b{}_c = h^a{}_c$. It projects a vector $\mathbf{X} \in T_{\theta(p)}$ into its part lying in the subspace $H = \theta_*(T_p)$ of $T_{\theta(p)}$ tangent to $\theta(\mathscr{S})$,

$$X^a = h^a{}_b X^b \pm n^a n_b X^b,$$

where the second term represents the part of $\mathbf{X}$ orthogonal to $\theta(\mathscr{S})$. Also $h^a{}_b$ projects a form $\boldsymbol{\omega} \in T^*{}_{\theta(p)}$ into its part lying in the subspace $H^*{}_{\theta(p)}$:

$$\omega_a = h^b{}_a \omega_b \pm n_a n^b \omega_b.$$

Similarly one can project any tensor $\mathbf{T} \in T^r_s(\theta(p))$ into its part in

$$H^r_s(\theta(p)) = \underbrace{H_{\theta(p)} \otimes \ldots \otimes H_{\theta(p)}}_{r \text{ factors}} \otimes \underbrace{H^*_{\theta(p)} \otimes \ldots \otimes H^*_{\theta(p)}}_{s \text{ factors}},$$

i.e. its part which is orthogonal to $\mathbf{n}$ on all indices.

The map θ_* is one–one from T_p to $H_{\theta(p)}$. Therefore one can define a map $\tilde{\theta}^*$ from $T_{\theta(p)}$ to T_p by first projecting with $h^a{}_b$ into $H_{\theta(p)}$ and then using the inverse $(\theta)_*{}^{-1}$. As one already has a map θ^* of forms on $\theta(\mathscr{S})$ to forms on $\mathscr{S}$, one can extend the definition of θ^* to a map $\tilde{\theta}^*$ of tensors of any type on $\theta(\mathscr{S})$ to tensors on $\mathscr{S}$. This map has the property that $\tilde{\theta}^*(\tilde{\theta}_*\mathbf{T}) = \mathbf{T}$ for any tensor $\mathbf{T} \in T^r_s(p)$ and $\tilde{\theta}_*(\tilde{\theta}^*\mathbf{T}) = \mathbf{T}$ for any tensor $\mathbf{T} \in H^r_s(\theta(p))$. We shall identify tensors on $\mathscr{S}$ with tensors in H^r_s on $\theta(\mathscr{S})$ if they correspond under the maps $\tilde{\theta}_*$, $\tilde{\theta}^*$. In particular, $\mathbf{h}$ can then be regarded as the induced metric on $\theta(\mathscr{S})$.

If $\bar{\mathbf{n}}$ is any extension of the unit normal $\mathbf{n}$ onto an open neighbourhood of $\theta(\mathscr{S})$ then the tensor $\boldsymbol{\chi}$ defined on $\theta(\mathscr{S})$ by

$$\chi_{ab} = h^c{}_a h^d{}_b \bar{n}_{c;d}$$

is called the *second fundamental form* of $\mathscr{S}$. It is independent of the extension, since the projections by $h^a{}_b$ restrict the covariant derivatives to directions tangent to $\theta(\mathscr{S})$. Locally the field $\bar{\mathbf{n}}$ can be expressed in the form $\bar{\mathbf{n}} = \alpha\, \mathrm{d}f$ where f and α are functions on $\mathscr{M}$ and $f = 0$ on $\theta(\mathscr{S})$. Therefore χ_{ab} must be symmetric, since $f_{;ab} = f_{;ba}$ and $f_{;a} h^a{}_b = 0$.

The induced metric $\mathbf{h} = \theta^*\mathbf{g}$ on $\mathscr{S}$ defines a connection on $\mathscr{S}$. We shall denote covariant differentiation with respect to this connection by a double stroke, $\|$. For any tensor $\mathbf{T} \in H^r_s$,

$$T^{a\ldots b}{}_{c\ldots d\|e} = \bar{T}^{i\ldots j}{}_{k\ldots l;m} h^a{}_i \ldots h^b{}_j h^k{}_c \ldots h^l{}_d h^m{}_e,$$

where $\bar{\mathbf{T}}$ is any extension of $\mathbf{T}$ to a neighbourhood of $\theta(\mathscr{S})$. This definition is independent of the extension, as the hs restrict the covariant differentiation to directions tangential to $\theta(\mathscr{S})$. To see this is the correct formula, one has only to show that the covariant derivative of the induced metric is zero and that the torsion vanishes. This follows because

$$h_{ab\|c} = (g_{ef} \mp \bar{n}_e \bar{n}_f)_{;g} h^e{}_a h^f{}_b h^g{}_c = 0,$$

and $$f_{\|ab} = h^e{}_a h^g{}_b f_{;eg} = h^e{}_a h^g{}_b f_{;ge} = f_{\|ba}.$$

The curvature tensor $R'^a{}_{bcd}$ of the induced metric **h** can be related to the curvature tensor $R^a{}_{bcd}$ on $\theta(\mathscr{S})$ and the second fundamental form χ as follows. If $\mathbf{Y} \in H$ is a vector field on $\theta(\mathscr{S})$, then

$$R'^a{}_{bcd} Y^b = Y^a{}_{\|dc} - Y^a{}_{\|cd}.$$

Now

$$Y^a{}_{\|dc} = (Y^a{}_{\|d})_{\|c} = (\bar{Y}^e{}_{;f} h^g{}_e h^f{}_i)_{;k} h^a{}_g h^i{}_d h^k{}_c$$

$$= \bar{Y}^e{}_{;fk} h^a{}_e h^f{}_d h^k{}_c \mp \bar{Y}^e{}_{;f} \bar{n}_e \bar{n}^g{}_{;k} h^f{}_d h^a{}_g h^k{}_c \mp \bar{Y}^e{}_{;f} \bar{n}^f \bar{n}_{i;k} h^a{}_e h^i{}_d h^k{}_c$$

and $$\bar{Y}^e{}_{;f} \bar{n}_e h^f{}_d = (\bar{Y}^e \bar{n}_e)_{;f} h^f{}_d - \bar{Y}^e \bar{n}_{e;f} h^f{}_d = -\bar{Y}^e \bar{n}_{e;f} h^f{}_d,$$

since $\bar{Y}^e \bar{n}_e = 0$ on $\theta(\mathscr{S})$, therefore

$$R'^a{}_{bcd} Y^b = (R^e{}_{bkf} h^a{}_e h^k{}_c h^f{}_d \pm \chi_{bd} \chi^a{}_c \mp \chi_{bc} \chi^a{}_d) Y^b.$$

Since this holds for all $\mathbf{Y} \in H$,

$$R'^a{}_{bcd} = R^e{}_{fgh} h^a{}_e h^f{}_b h^g{}_c h^h{}_d \pm \chi^a{}_c \chi_{bd} \mp \chi^a{}_d \chi_{bc}. \tag{2.34}$$

This is known as Gauss' equation.

Contracting this equation on a and c and multiplying by h^{bd}, one obtains the curvature scalar R' of the induced metric:

$$R' = R \mp 2R_{ab} n^a n^b \pm (\chi^a{}_a)^2 \mp \chi^{ab} \chi_{ab}. \tag{2.35}$$

One can derive another relation between the second fundamental form and the curvature tensor $R^a{}_{bcd}$ on $\theta(\mathscr{S})$ by subtracting the expressions

$$(\chi^a{}_a)_{\|b} = (\bar{n}^a{}_{;d} h^d{}_a)_{;e} h^e{}_b$$

and $$(\chi^a{}_b)_{\|a} = (\bar{n}^c{}_{;d} h^a{}_c h^d{}_e)_{;f} h^f{}_a h^e{}_b,$$

finding $$\chi^a{}_{b\|a} - \chi^a{}_{a\|b} = R_{ef} n^f h^e{}_b. \tag{2.36}$$

This is known as Codacci's equation.

2.8 The volume element and Gauss' theorem

If $\{\mathbf{E}^a\}$ is a basis of one-forms, one can form from it the n-form

$$\boldsymbol{\epsilon} = n!\mathbf{E}^1 \wedge \mathbf{E}^2 \wedge \ldots \wedge \mathbf{E}^n.$$

If $\{\mathbf{E}^{a'}\}$, related to $\{\mathbf{E}^{a}\}$ by $\mathbf{E}^{a'} = \Phi^{a'}{}_{a}\mathbf{E}^{a}$, is another basis, the n-form $\boldsymbol{\epsilon}'$ defined by this basis will be related to $\boldsymbol{\epsilon}$ by

$$\boldsymbol{\epsilon}' = \det(\Phi^{a'}{}_{a})\,\boldsymbol{\epsilon},$$

so this form is not unique. However, one can use the existence of the metric to define (in a given basis) the form

$$\boldsymbol{\eta} = |g|^{\frac{1}{2}}\boldsymbol{\epsilon}$$

where $g \equiv \det(g_{ab})$. This form has components

$$\eta_{ab\ldots d} = n!\,|g|^{\frac{1}{2}}\,\delta^{1}{}_{[a}\,\delta^{2}{}_{b}\ldots\delta^{n}{}_{d]}.$$

The transformation law for g will just cancel the determinant, $\det(\Phi^{a'}{}_{a})$, provided that $\det(\Phi^{a'}{}_{a}) > 0$. Therefore if $\mathscr{M}$ is orientable the n-forms $\boldsymbol{\eta}$ defined by coordinate bases of an oriented atlas will be identical, i.e. given an orientation of $\mathscr{M}$, one can define a unique n-form field $\boldsymbol{\eta}$, the *canonical n-form*, on $\mathscr{M}$.

The contravariant antisymmetric tensor

$$\eta^{ab\ldots d} = g^{ae}g^{bf}\ldots g^{dh}\eta_{ef\ldots h}$$

has components

$$\eta^{ab\ldots d} = (-)^{\frac{1}{2}(n-s)}n!\,|g|^{\frac{1}{2}}\,\delta^{[a}{}_{1}\,\delta^{b}{}_{2}\ldots\delta^{d]}{}_{n},$$

where s is the signature of $\mathbf{g}$ (so $\frac{1}{2}(n-s)$ is the number of negative eigenvalues of the matrix of metric components (g_{ab})). Therefore these tensors satisfy the relations

$$\eta^{ab\ldots d}\eta_{ef\ldots h} = (-)^{\frac{1}{2}(n-s)}\,n!\,\delta^{a}{}_{[e}\,\delta^{b}{}_{f}\ldots\delta^{d}{}_{h]}. \qquad (2.37)$$

The Christoffel relations imply that the covariant derivatives of $\eta_{ab\ldots d}$ and $\eta^{ab\ldots d}$ with respect to the connection defined by the metric vanish, i.e.

$$\eta^{ab\ldots d}{}_{;e} = 0 = \eta_{ab\ldots d;e}.$$

Using the canonical n-form, one can define the volume (with respect to the metric $\mathbf{g}$) of an n-dimensional submanifold $\mathscr{U}$ as $\dfrac{1}{n!}\displaystyle\int_{\mathscr{U}}\boldsymbol{\eta}$. Thus $\boldsymbol{\eta}$ can be regarded as a positive definite volume measure on $\mathscr{M}$. We shall often use it in this sense, and shall denote it by $\mathrm{d}v$. Note that d is not meant to represent the exterior differential operator here; $\mathrm{d}v$ is simply a measure on $\mathscr{M}$. If f is a function on $\mathscr{M}$, one can define its integral over $\mathscr{U}$ with respect to this volume measure as

$$\int_{\mathscr{U}} f\,\mathrm{d}v = \frac{1}{n!}\int_{\mathscr{U}} f\,\boldsymbol{\eta}.$$

With respect to local oriented coordinates $\{x^a\}$, this can be expressed as the multiple integral

$$\int_{\mathscr{U}} f\,|g|^{\frac{1}{2}}\,\mathrm{d}x^1\,\mathrm{d}x^2 \ldots \mathrm{d}x^n,$$

which is invariant under a change of coordinates.

If $\mathbf{X}$ is a vector field on $\mathscr{M}$, its contraction with $\boldsymbol{\eta}$ will be an $(n-1)$-form field $\mathbf{X}\,.\,\boldsymbol{\eta}$, where

$$(\mathbf{X}\,.\,\boldsymbol{\eta})_{b\ldots d} = X^a \eta_{ab\ldots d}.$$

This $(n-1)$-form may be integrated over any $(n-1)$-dimensional compact orientable submanifold $\mathscr{V}$. We write this integral as

$$\int_{\mathscr{V}} X^a \mathrm{d}\sigma_a = \frac{1}{(n-1)!}\int_{\mathscr{V}} \mathbf{X}\,.\,\boldsymbol{\eta},$$

where the canonical form $\boldsymbol{\eta}$ is regarded as defining a measure-valued form $\mathrm{d}\sigma_a$ on the submanifold $\mathscr{V}$. If the orientation of $\mathscr{V}$ is given by the direction of the normal form n_a, then $\mathrm{d}\sigma_a$ can be expressed as $n_a\,\mathrm{d}\sigma$ where $\mathrm{d}\sigma$ is a positive definite volume measure on the submanifold $\mathscr{V}$. The volume measure $\mathrm{d}\sigma$ is not unique unless the normal n_a is normalized. If n_a is normalized to unit magnitude in a metric $\mathbf{g}$ on $\mathscr{M}$, i.e. $n_a n_b g^{ab} = \pm 1$, then $\mathrm{d}\sigma$ is equal to the volume measure on $\mathscr{V}$ defined by the induced metric on $\mathscr{V}$ (to see this, simply choose an orthonormal basis with $n_a g^{ab}$ as one of the basis vectors).

Using the canonical form, one can derive Gauss' formula from Stokes' theorem: for any compact n-dimensional submanifold $\mathscr{U}$ of $\mathscr{M}$,

$$\int_{\partial\mathscr{U}} X^a \mathrm{d}\sigma_a = \frac{1}{(n-1)!}\int_{\partial\mathscr{U}} \mathbf{X}\,.\,\boldsymbol{\eta} = \frac{1}{(n-1)!}\int_{\mathscr{U}} \mathrm{d}(\mathbf{X}\,.\,\boldsymbol{\eta}).$$

But

$$\begin{aligned}(\mathrm{d}(\mathbf{X}\,.\,\boldsymbol{\eta}))_{a\ldots de} &= (-)^{n-1}(X^g \eta_{g[a\ldots d})_{;e]} \\ &= (-)^{n-1}\delta^s{}_{[a}\ldots\delta^t{}_d\,\delta^u{}_{e]}\,\eta_{gs\ldots t}X^g{}_{;u} \\ &= (-)^{(n-1)-\frac{1}{2}(n-s)}\frac{1}{n!}\eta^{s\ldots tu}\,\eta_{a\ldots de}\,\eta_{gs\ldots t}X^g{}_{;u} \\ &= \eta_{a\ldots de}\,\delta^s{}_{[s}\ldots\delta^t{}_t\,\delta^u{}_{g]}X^g{}_{;u} \\ &= n^{-1}\eta_{a\ldots de}X^g{}_{;g},\end{aligned}$$

on using relation (2.37) twice. Therefore

$$\int_{\partial\mathscr{U}} X^a\,\mathrm{d}\sigma_a = \int_{\mathscr{U}} X^g{}_{;g}\,\mathrm{d}v$$

holds for any vector field $\mathbf{X}$; this is Gauss' theorem. Note that the orientation on $\mathscr{U}$ for which this theorem is valid is that given by the normal form $\boldsymbol{\eta}$ such that $\langle \mathbf{n}, \mathbf{X} \rangle$ is positive if $\mathbf{X}$ is a vector which points out of $\mathscr{U}$. If the metric $\mathbf{g}$ is such that $g^{ab}n_a n_b$ is negative, the vector $g^{ab}n_b$ will point into $\mathscr{U}$.

2.9 Fibre bundles

Some of the geometrical properties of a manifold $\mathscr{M}$ can be most easily examined by constructing a manifold called a fibre bundle, which is locally a direct product of $\mathscr{M}$ and a suitable space. In this section we shall give the definition of a fibre bundle and shall consider four examples that will be used later: the tangent bundle $T(\mathscr{M})$, the tensor bundle $T^r_s(\mathscr{M})$, the bundle of linear frames or bases $L(\mathscr{M})$, and the bundle of orthonormal frames $O(\mathscr{M})$.

A C^k *bundle* over a C^s $(s \geqslant k)$ manifold $\mathscr{M}$ is a C^k manifold $\mathscr{E}$ and a C^k surjective map $\pi: \mathscr{E} \to \mathscr{M}$. The manifold $\mathscr{E}$ is called the total space, $\mathscr{M}$ is called the base space and π the projection. Where no confusion can arise, we will denote the bundle simply by $\mathscr{E}$. In general, the inverse image $\pi^{-1}(p)$ of a point $p \in \mathscr{M}$ need not be homeomorphic to $\pi^{-1}(q)$ for another point $q \in \mathscr{M}$. The simplest example of a bundle is a *product bundle* $(\mathscr{M} \times \mathscr{A}, \mathscr{M}, \pi)$ where $\mathscr{A}$ is some manifold and the projection π is defined by $\pi(p, v) = p$ for all $p \in \mathscr{M}$, $v \in \mathscr{A}$. For example, if one chooses $\mathscr{M}$ as the circle S^1 and $\mathscr{A}$ as the real line R^1, one constructs the cylinder C^2 as a product bundle over S^1.

A bundle which is locally a product bundle is called a fibre bundle. Thus a bundle is a *fibre bundle* with fibre $\mathscr{F}$ if there exists a neighbourhood $\mathscr{U}$ of each point q of $\mathscr{M}$ such that $\pi^{-1}(\mathscr{U})$ is isomorphic with $\mathscr{U} \times \mathscr{F}$, in the sense that for each point $p \in \mathscr{U}$ there is a diffeomorphism ϕ_p of $\pi^{-1}(p)$ onto $\mathscr{F}$ such that the map ψ defined by $\psi(u) = (\pi(u), \phi_{\pi(u)})$ is a diffeomorphism $\psi: \pi^{-1}(\mathscr{U}) \to \mathscr{U} \times \mathscr{F}$. Since $\mathscr{M}$ is paracompact, we can choose a locally finite covering of $\mathscr{M}$ by such open sets $\mathscr{U}_\alpha$. If $\mathscr{U}_\alpha$ and $\mathscr{U}_\beta$ are two members of such a covering, the map

$$(\phi_{\alpha, p}) \circ (\phi_{\beta, p}{}^{-1})$$

is a diffeomorphism of $\mathscr{F}$ onto itself for each $p \in (\mathscr{U}_\alpha \cap \mathscr{U}_\beta)$. The inverse images $\pi^{-1}(p)$ of points $p \in \mathscr{M}$ are therefore necessarily all diffeomorphic to $\mathscr{F}$ (and so to each other). For example, the Möbius strip is a fibre bundle over S^1 with fibre R^1; we need two open sets $\mathscr{U}_1$, $\mathscr{U}_2$

to give a covering by sets of the form $\mathscr{U}_i \times R^1$. This example shows that if a manifold is locally the direct product of two other manifolds, it is nevertheless not, in general, a product manifold; it is for this reason that the concept of a fibre bundle is so useful.

The *tangent bundle* $T(\mathscr{M})$ is the fibre bundle over a C^k manifold $\mathscr{M}$ obtained by giving the set $\mathscr{E} = \bigcup_{p\in\mathscr{M}} T_p$ its natural manifold structure and its natural projection into $\mathscr{M}$. Thus the projection π maps each point of T_p into p. The manifold structure in $\mathscr{E}$ is defined by local coordinates $\{z^A\}$ in the following way. Let $\{x^i\}$ be local coordinates in an open set $\mathscr{U}$ of $\mathscr{M}$. Then any vector $\mathbf{V} \in T_p$ (for any $p \in \mathscr{U}$) can be expressed as $\mathbf{V} = V^i \,\partial/\partial x^i|_p$. The coordinates $\{z^A\}$ are defined in $\pi^{-1}(\mathscr{U})$ by $\{z^A\} = \{x^i, V^a\}$. On choosing a covering of $\mathscr{M}$ by coordinate neighbourhoods $\mathscr{U}_\alpha$, the corresponding charts define a C^{k-1} atlas on $\mathscr{E}$ which turn it into a C^{k-1} manifold (of dimension n^2); to check this, one needs only note that in any overlap $(\mathscr{U}_\alpha \cap \mathscr{U}_\beta)$ the coordinates $\{x^i{}_\alpha\}$ of a point are C^k functions of the coordinates $\{x^i{}_\beta\}$ of the point, and the components $\{V^a{}_\alpha\}$ of a vector field are C^{k-1} functions of the components $\{V^a{}_\beta\}$ of the vector field. Thus in $\pi^{-1}(\mathscr{U}_\alpha \cap \mathscr{U}_\beta)$, the coordinates $\{z^A{}_\alpha\}$ are C^{k-1} functions of the coordinates $\{z^A{}_\beta\}$.

The fibre $\pi^{-1}(p)$ is T_p, and so is a vector space of dimension n. This vector space structure is preserved by the map $\phi_{\alpha,p}: T_p \to R^n$, which is given by $\phi_{\alpha,p}(u) = V^a(u)$, i.e. $\phi_{\alpha,p}$ maps a vector at p into its components with respect to the coordinates $\{x^a{}_\alpha\}$. If $\{x^a{}_\beta\}$ are another set of local coordinates then the map $(\phi_{\alpha,p}) \circ (\phi_{\beta,p}{}^{-1})$ is a linear map of R^n onto itself. Thus it is an element of the general linear group $GL(n, R)$ (the group of all non-singular $n \times n$ matrices).

The *bundle of tensors of type* (r, s) over $\mathscr{M}$, denoted by $T^r_s(\mathscr{M})$, is defined in a very similar way. One forms the set $\mathscr{E} = \bigcup_{p\in\mathscr{M}} T^r_s(p)$, defines the projection π as mapping each point in $T^r_s(p)$ into p, and, for any coordinate neighbourhood $\mathscr{U}$ in $\mathscr{M}$, assigns local coordinates $\{z^A\}$ to $\pi^{-1}(\mathscr{U})$ by $\{z^A\} = \{x^i, T^{a\ldots b}{}_{c\ldots d}\}$ where $\{x^i\}$ are the coordinates of the point p and $\{T^{a\ldots b}{}_{c\ldots d}\}$ are the coordinate components of $\mathbf{T}$ (that is, $\mathbf{T} = T^{a\ldots b}{}_{c\ldots d}\,\partial/\partial x^a \otimes \ldots \otimes \mathrm{d}x^d|_p$). This turns $\mathscr{E}$ into a C^{k-1} manifold of dimension n^{r+s+1}; any point u in $T^r_s(\mathscr{M})$ corresponds to a unique tensor $\mathbf{T}$ of type (r, s) at $\pi(u)$.

The *bundle of linear frames* (or bases) $L(\mathscr{M})$ is a C^{k-1} fibre bundle defined as follows: the total space $\mathscr{E}$ consists of all bases at all points of $\mathscr{M}$, that is all sets of non-zero linearly independent n-tuples of vectors $\{\mathbf{E}_a\}$, $\mathbf{E}_a \in T_p$, for each $p \in \mathscr{M}$ (a runs from 1 to n). The projection

π is the natural one which maps a basis at a point p to the point p. If $\{x^i\}$ are local coordinates in an open set $\mathscr{U} \subset \mathscr{M}$, then

$$\{z^A\} = \{x^a, E_1{}^j, E_2{}^k, \ldots, E_n{}^m\}$$

are local coordinates in $\pi^{-1}(\mathscr{U})$, where $E_a{}^j$ is the jth components of the vector $\mathbf{E}_a$ with respect to the coordinate bases $\partial/\partial x^i$. The general linear group $GL(n, R)$ acts on $L(\mathscr{M})$ in the following way: if $\{\mathbf{E}_a\}$ is a basis at $p \in \mathscr{M}$, then $\mathbf{A} \in GL(n, R)$ maps $u = \{p, \mathbf{E}_a\}$ to

$$A(u) = \{p, A_{ab}\mathbf{E}_b\}.$$

When there is a metric $\mathbf{g}$ of signature s on $\mathscr{M}$, one can define a subbundle of $L(\mathscr{M})$, the *bundle of orthonormal frames* $O(\mathscr{M})$, which consists of orthonormal bases (with respect to $\mathbf{g}$) at all points of $\mathscr{M}$. $O(\mathscr{M})$ is acted on by the subgroup $O(\frac{1}{2}(n+s), \frac{1}{2}(n-s))$ of $GL(n, R)$. This consists of the non-singular real matrices A_{ab} such that

$$A_{ab}G_{bc}A_{dc} = G_{ad},$$

where G_{bc} is the matrix

$$\operatorname{diag}(\underbrace{+1, +1, \ldots, +1}_{\frac{1}{2}(n+s)\text{ terms}}, \underbrace{-1, -1, \ldots, -1}_{\frac{1}{2}(n-s)\text{ terms}}).$$

It maps $(p, \mathbf{E}_a) \in O(\mathscr{M})$ to $(p, A_{ab}\mathbf{E}_b) \in O(\mathscr{M})$. In the case of a Lorentz metric (i.e. $s = n-2$), the group $O(n-1, 1)$ is called the n-dimensional Lorentz group.

A C^r *cross-section* of a bundle is a C^r map $\Phi: \mathscr{M} \to \mathscr{E}$ such that $\pi \circ \Phi$ is the identity map on $\mathscr{M}$; thus a cross-section is a C^r assignment to each point p of $\mathscr{M}$ of an element $\Phi(p)$ of the fibre $\pi^{-1}(p)$. A cross-section of the tangent bundle $T(\mathscr{M})$ is a vector field on $\mathscr{M}$; a cross-section of $T^r_s(\mathscr{M})$ is a tensor field of type (r, s) on $\mathscr{M}$; a cross-section of $L(\mathscr{M})$ is a set of n non-zero vector fields $\{\mathbf{E}_a\}$ which are linearly independent at each point, and a cross-section of $O(\mathscr{M})$ is a set of orthonormal vector fields on $\mathscr{M}$.

Since the zero vectors and tensors define cross-sections in $T(\mathscr{M})$ and $T^r_s(\mathscr{M})$, these fibre bundles will always admit cross-sections. If $\mathscr{M}$ is orientable and non-compact, or is compact with vanishing Euler number, there will exist nowhere zero vector fields, and hence cross-sections of $T(\mathscr{M})$ which are nowhere zero. The bundles $L(\mathscr{M})$ and $O(\mathscr{M})$ may or may not admit cross-sections; for example $L(S^2)$ does not, but $L(R^n)$ does. If $L(\mathscr{M})$ admits a cross-section, $\mathscr{M}$ is said to be *parallelizable*. R. P. Geroch has shown (1968*c*) that a non-compact four-dimensional Lorentz manifold $\mathscr{M}$ admits a spinor structure if and only if it is parallelizable.

One can describe a connection on $\mathscr{M}$ in an elegant geometrical way in terms of the fibre bundle $L(\mathscr{M})$. A connection on $\mathscr{M}$ may be regarded as a rule for parallelly transporting vectors along any curve $\gamma(t)$ in $\mathscr{M}$. Thus if $\{\mathbf{E}_a\}$ is a basis at a point $p = \gamma(t_0)$, i.e. $\{p, \mathbf{E}_a\}$ is a point u in $L(\mathscr{M})$, one can obtain a unique basis at any other point $\gamma(t)$, i.e. a unique point $\bar{\gamma}(t)$ in the fibre $\pi^{-1}(\gamma(t))$, by parallelly transporting $\{\mathbf{E}_a\}$ along $\gamma(t)$. Therefore there is a unique curve $\bar{\gamma}(t)$ in $L(\mathscr{M})$, called the *lift* of $\gamma(t)$, such that:

(1) $\bar{\gamma}(t_0) = u$,

(2) $\pi(\bar{\gamma}(t)) = \gamma(t)$,

(3) the basis represented by the point $\bar{\gamma}(t)$ is parallelly transported along the curve $\gamma(t)$ in $\mathscr{M}$.

In terms of the local coordinates $\{z^A\}$, the curve $\bar{\gamma}(t)$ is given by $\{x^a(\gamma(t)), E_m{}^i(t)\}$, where

$$\frac{\mathrm{d}E_m{}^i(t)}{\mathrm{d}t} + E_m{}^j\,\Gamma^i{}_{aj}\frac{\mathrm{d}x^a(\gamma(t))}{\mathrm{d}t} = 0.$$

Consider the tangent space $T_u(L(\mathscr{M}))$ to the fibre bundle $L(\mathscr{M})$ at the point u. This has a coordinate basis $\{\partial/\partial z^A|_u\}$. The n-dimensional subspace spanned by the tangent vectors $\{(\partial/\partial t)_{\bar{\gamma}(t)}|_u\}$ to the lifts of all curves $\gamma(t)$ through p is called the *horizontal subspace* H_u of $T_u(L(\mathscr{M}))$. In terms of local coordinates,

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)_{\bar{\gamma}} &= \frac{\mathrm{d}x^a(\gamma(t))}{\mathrm{d}t}\frac{\partial}{\partial x^a} + \frac{\mathrm{d}E_m{}^i}{\mathrm{d}t}\frac{\partial}{\partial E_m{}^i}\\ &= \frac{\mathrm{d}x^a(\gamma(t))}{\mathrm{d}t}\left(\frac{\partial}{\partial x^a} - E_m{}^j\,\Gamma^i{}_{aj}\frac{\partial}{\partial E_m{}^i}\right),\end{aligned}$$

so a coordinate basis of H_u is $\{\partial/\partial x^a - E_m{}^j\,\Gamma^i{}_{aj}\,\partial/\partial E_m{}^i\}$. Thus the connection in $\mathscr{M}$ determines the horizontal subspaces in the tangent spaces at each point of $L(\mathscr{M})$. Conversely, a connection in $\mathscr{M}$ may be defined by giving an n-dimensional subspace of $T_u(L(\mathscr{M}))$ for each $u \in L(\mathscr{M})$ with the properties:

(1) If $A \in GL(n, R^1)$, then the map $A_*: T_u(L(\mathscr{M})) \to T_{A(u)}(L(\mathscr{M}))$ maps the horizontal subspace H_u into $H_{A(u)}$;

(2) H_u contains no non-zero vector belonging to the vertical subspace V_u.

Here, the vertical subspace V_u is defined as the n^2-dimensional subspace of $T_u(L(\mathscr{M}))$ spanned by the vectors tangent to curves in the fibre $\pi^{-1}(\pi(u))$; in terms of local coordinates, V_u is spanned by the vectors $\{\partial/\partial E_m{}^i\}$. Property (2) implies that T_u is the direct sum of H_u and V_u.

The projection map $\pi: L(\mathscr{M}) \to \mathscr{M}$ induces a surjective linear map $\pi_*: T_u(L(\mathscr{M})) \to T_{\pi(u)}(\mathscr{M})$, such that $\pi_*(V_u) = 0$ and π_* restricted to H_u is 1–1 onto $T_{\pi(u)}$. Thus the inverse $\pi_*{}^{-1}$ is a linear map of $T_{\pi(u)}(\mathscr{M})$ onto H_u. Therefore for any vector $\mathbf{X} \in T_p(\mathscr{M})$ and point $u \in \pi^{-1}(p)$, there is a unique vector $\bar{\mathbf{X}} \in H_u$, called the *horizontal lift* of $\mathbf{X}$, such that $\pi_*(\bar{\mathbf{X}}) = \mathbf{X}$. Given a curve $\gamma(t)$ in $\mathscr{M}$, and an initial point u in $\pi^{-1}(\gamma(t_0))$, one can construct a unique curve $\bar{\gamma}(t)$ in $L(\mathscr{M})$, where $\bar{\gamma}(t)$ is the curve through u whose tangent vector is the horizontal lift of the tangent vector of $\gamma(t)$ in $\mathscr{M}$. Thus knowing the horizontal subspaces at each point in $L(\mathscr{M})$, one can define parallel propagation of bases along any curve $\gamma(t)$ in $\mathscr{M}$. One can then define the covariant derivative along $\gamma(t)$ of any tensor field $\mathbf{T}$ by taking the ordinary derivatives with respect to t, of the components of $\mathbf{T}$ with respect to a parallelly propagated basis.

If there is a metric $\mathbf{g}$ on $\mathscr{M}$ whose covariant derivative is zero, then orthonormal frames are parallelly propagated into orthonormal frames. Thus the horizontal subspaces are tangent to $O(\mathscr{M})$ in $L(\mathscr{M})$, and define a connection in $O(\mathscr{M})$.

Similarly a connection on $\mathscr{M}$ defines n-dimensional horizontal subspaces in the tangent spaces to the bundles $T(\mathscr{M})$ and $T^r_s(\mathscr{M})$, by parallel propagation of vectors and tensors. These horizontal subspaces have coordinate bases

$$\left\{\frac{\partial}{\partial x^a} - V^e \Gamma^f{}_{ae} \frac{\partial}{\partial V^f}\right\}$$

and

$$\left\{\frac{\partial}{\partial x^e} - \left(T^{f\ldots b}{}_{c\ldots d}\Gamma^a{}_{ef} + \text{(all upper indices)} - T^{a\ldots b}{}_{f\ldots d}\Gamma^f{}_{ec} - \text{(all lower indices)}\right)\frac{\partial}{\partial T^{a\ldots b}{}_{c\ldots d}}\right\}$$

respectively. As with $L(\mathscr{M})$, π_* maps these horizontal subspaces one–one onto $T_{\pi(u)}(\mathscr{M})$; thus again π_* can be inverted to give a unique horizontal lift $\bar{\mathbf{X}} \in T_u$ of any vector $\mathbf{X} \in T_{\pi(u)}$. In the particular case of $T(\mathscr{M})$, u itself corresponds to a unique vector $\mathbf{W} \in T_{\pi(u)}(\mathscr{M})$, and so there is an intrinsic horizontal vector field $\bar{\mathbf{W}}$ defined on $T(\mathscr{M})$ by the connection. In terms of local coordinates $\{x^a, V^b\}$,

$$\bar{\mathbf{W}} = V^a\left(\frac{\partial}{\partial x^a} - V^e\Gamma^f{}_{ae}\frac{\partial}{\partial V^f}\right).$$

This vector field may be interpreted as follows: the integral curve of $\bar{\mathbf{W}}$ through $u = (p, \mathbf{X}) \in T(\mathscr{M})$ is the horizontal lift of the geodesic in $\mathscr{M}$ with tangent vector $\mathbf{X}$ at p. Thus the vector field $\bar{\mathbf{W}}$ represents all geodesics on $\mathscr{M}$. In particular, the family of all geodesics through $p \in \mathscr{M}$ is the family of integral curves of $\bar{\mathbf{W}}$ through the fibre $\pi^{-1}(p) \subset T(\mathscr{M})$; the curves in $\mathscr{M}$ have self intersections at least at p, but the curves in $T(\mathscr{M})$ are non-intersecting everywhere.