REFINEMENT-UNBOUNDED INTERVAL FUNCTIONS AND ABSOLUTE CONTINUITY

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1. Introduction. In this paper we prove the following characterization theorem (Section 3):

THEOREM 1. *If each of g and m is a real-valued non-decreasing function on the number interval* $[a, b]$, *then the following two statements are equivalent:*

(1) *If R is a real-valued, refinement-unbounded* (Section 3) *function of subintervals of* $[a, b]$, *then the integral* (Section 2)

$$\int_{[a,b]} \min\{dg, R(I)dm\}$$

exists and is equal to $g(b) - g(a)$, *and*

(2) *g is absolutely continuous with respect to m.*

2. Preliminary theorems and definitions. Suppose $[a, b]$ is a number interval.

Throughout this paper all integrals discussed are Hellinger **(1)** type limits (with respect to refinements) of the appropriate sums. Thus, if H is a real-valued function of subintervals of $[a, b]$, then $\int_{[a,b]} H(I)$ exists if and only if for each subinterval I of $[a, b]$, $\int_I H(J)$ exists, so that if for $a \leqslant c \leqslant b$, $\int_{[c,c]} H(I)$ denotes 0, then, for a $\leqslant p \leqslant q \leqslant r \leqslant b$,

$$\int_{[p,q]} H(I) + \int_{[q,r]} H(I) = \int_{[p,r]} H(I).$$

We see that if each of x, y, z, and w is a number, then

$$\min\{x, y\} + \min\{z, w\} \leqslant \min\{x + z, y + w\}.$$

This implies that if each of u and v is a real-valued non-decreasing function on $[a, b]$ and E is a refinement of the subdivision D of $[a, b]$, then

$$0 \leqslant \sum_E \min\{\Delta u, \Delta v\} \leqslant \sum_D \min\{\Delta u, \Delta v\},$$

so that

$$\int_{[a,b]} \min\{du, dv\}$$

exists.

We state a lemma whose proof follows by conventional methods.

LEMMA A. *If each of g and m is a real-valued non-decreasing function on* $[a, b]$, *and for* $a \leqslant x \leqslant b$,

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$$h(x) = \sup \int_{[a,x]} \min\{dg, Kdm\}$$

for $0 < K$, *then*

$$\int_{[a,b]} |dh - \int_I \min\{dg, Kdm\}| \to 0 \qquad \text{as } K \to \infty,$$

and h *is absolutely continuous with respect to* m.

3. The characterization theorem. Suppose that $(a, b]$ is a number interval.

Definition. If R is a real-valued function of subintervals of $[a, b]$, then the statement that R is *refinement-unbounded* means that if K is a positive number, then there is a subdivision D of $[a, b]$ such that if I is an interval of a refinement of D, then $K < R(I)$.

We now prove Theorem 1, as quoted in the Introduction. We first show that (2) implies (1).

Suppose (2) is true and R is a real-valued, refinement-unbounded function of subintervals of $[a, b]$.

Suppose c is a positive number. There is a positive number k such that if E is a subset of a subdivision of $[a, b]$ and $\sum_E \Delta m < k$, then $\sum_E \Delta g < c$.

There is a subdivision D of $[a, b]$ such that if I is an interval of a refinement of D, then $(g(b) - g(a) + 1)/k < R(I)$.

Suppose E is a refinement of D. Then

$$\begin{aligned} 0 &\leqslant g(b) - g(a) - \textstyle\sum_E \min\{\Delta g, R(I)\Delta m\} \\ &= \textstyle\sum_E [\Delta g - \min\{\Delta g, R(I)\Delta m\}] = \sum_{E^*} [\Delta g - R(I)\Delta m] \\ &\leqslant \textstyle\sum_{E^*} \Delta g, \end{aligned}$$

where E^* is the set (if any) of all I in E such that $R(I)\Delta m \leqslant \Delta g$. We see that

$$[(g(b) - g(a) + 1)/k]\textstyle\sum_{E^*} \Delta m \leqslant \sum_{E^*} R(I)\Delta m \leqslant \sum_{E^*} \Delta g \leqslant g(b) - g(a),$$

so that

$$\textstyle\sum_{E^*} \Delta m \leqslant [k/(g(b) - g(a) + 1)][g(b) - g(a)] < k,$$

and therefore $\sum_{E^*} \Delta g < c$. Therefore

$$0 \leqslant g(b) - g(a) - \textstyle\sum_E \min\{\Delta g, R(I)\Delta m\} < c.$$

Therefore $\int_{[a,b]} \min\{dg, R(I)dm\}$ exists and is equal to $g(b) - g(a)$. Thus (2) implies (1).

We now show that (1) implies (2).

Suppose (1) is true. For $a \leqslant x \leqslant b$, let

$$h(x) = \sup \int_{[a,x]} \min\{dg, Kdm\} \qquad \text{for } 0 < K.$$

By Lemma A,

$$\int_{[a,b]} |dh - \int_I \min\{dg, Kdm\}| \to 0 \qquad \text{as } K \to \infty$$

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and h is absolutely continuous with respect to m. Furthermore, $\Delta h \leqslant \Delta g$ for each subinterval I of $[a, b]$.

We see that if I is a subinterval of $[a, b]$ and $\Delta m = 0$, then $\Delta g = 0$. We let $\Delta g/\Delta m = 0$ if $\Delta m = 0$, and have the usual meaning otherwise.

If I is a subinterval $[p, q]$ of $[a, b]$, then we let Δx denote $q - p$.

We now state and prove a lemma.

LEMMA B. *If s is in $[a, b]$ and c is a positive number, then there is a segment T containing s such that if I is a subinterval of each of T and $[a, b]$, and s is an end number of I, then $\Delta g \leqslant \Delta h + c$.*

Proof. We assume s to be a right end-number of the interval I. A similar argument holds in case s is a left end-number of I.

We first show that if $a < s \leqslant b$ and m is continuous from the left at s, then so is g. Suppose that $m(s) - m(p) \to 0$ as $p \to s$ for $a \leqslant p < s$, but that for some positive number k, $g(s) - g(p) \geqslant k$ for $a \leqslant p < s$.

$$[g(s) - g(p)]/[m(s) - m(p)] \to \infty \text{ as } p \to s \text{ for } a \leqslant p < s.$$

There is a function R of subintervals of $[a, b]$ such that

$$R(I) = \begin{cases} (1/2)(\Delta g/\Delta m) \text{ if } I \text{ is } [p, s] \text{ for } a \leqslant p < s, \\ 1/\Delta x \text{ otherwise.} \end{cases}$$

We see that R is refinement-unbounded.

If E is a subdivision of $[a, b]$ such that $[u, s]$ is in E, then

$$\begin{aligned} \textstyle\sum_E \min\{\Delta g, R(I)\Delta m\} &= [\textstyle\sum_{\{E-[u,s]\}} \min\{\Delta g, R(I)\Delta m\}] + [g(s) - g(u)]/2 \\ &\leqslant [\textstyle\sum_{\{E-[u,s]\}} \Delta g] + g(s) - g(u) - k/2 = g(b) - g(a) - k/2, \end{aligned}$$

so that

$$g(b) - g(a) = \int_{[a,b]} \min\{dg, R(I)dm\} \leqslant g(b) - g(a) - k/2,$$

a contradiction. Therefore g is continuous from the left at s.

Next, suppose that $a < s \leqslant b$ and m is not continuous from the left at s. Suppose c is a positive number. There is a number t and a positive number K such that $a \leqslant t < s$ and such that if $t \leqslant u \leqslant v < s$, then $g(v) - g(u) < c$ and $g(s) - g(v) \leqslant K[m(s) - m(v)]$.

Suppose that $t \leqslant r < s$ and D is a subdivision of $[r, s]$ such that $[u, s]$ is in D.

$$\begin{aligned} g(s) - g(r) &= g(s) - g(u) + g(u) - g(r) \leqslant g(s) - g(u) + c \leqslant g(s) \\ &\qquad - g(u) + c + \textstyle\sum_{\{D-[u,s]\}} \min\{\Delta g, K\Delta m\} \\ &= \min\{g(s) - g(u), K[m(s) - m(u)]\} + c + \textstyle\sum_{\{D-[u,s]\}} \min\{\Delta g, K\Delta m\} \\ &= c + \textstyle\sum_D \min\{\Delta g, K\Delta m\}, \end{aligned}$$

so that

$$g(s) - g(r) \leqslant \int_{[r,s]} \min\{dg, Kdm\} + c \leqslant h(s) - h(r) + c.$$

This proves Lemma B.

There is an increasing unbounded sequence $\{K_i\}_{i=1}^{\infty}$ of positive numbers such that for each positive integer n,

$$\int_{[a,b]} |dh - \int_I \min\{dg, K_n dm\}| < 1/n.$$

There is a sequence $\{D_i\}_{i=1}^{\infty}$ of subdivisions of $[a, b]$ such that for each positive integer n,

(1) every interval of D_{n+1} is a proper subset of an interval of D_n,

(2) if I is in D_n, then $\Delta x < 1/n$, and

(3) $\sum_{D_n} [\min\{\Delta g, K_n \Delta m\} - \int_I \min\{dg, K_n dm\}] < 1/n$.

There is a real-valued function R of subintervals of $[a, b]$ such that for each subinterval I of $[a, b]$,

$$R(I) = \begin{cases} K_n \text{ if } I \text{ is in } D_n \text{ for some } n, \\ 1/\Delta x \text{ otherwise.} \end{cases}$$

We see that R is refinement-unbounded.

Suppose j is a positive number. Suppose D is a subdivision of $[a, b]$. Let W denote the number of intervals in D. By Lemma B there is a positive integer V such that

(1) $2/V < j/2$,

(2) if I is in D_V, then no interval of D is a subset of I, and

(3) if I is an interval of a refinement of D_V containing an end number of an interval of D, then $\Delta g \leqslant \Delta h + j/[8(W + 1)]$.

There is a common refinement E of D and D_V such that every end number of an interval of E is an end number of an interval of D or D_V.

Letting E^* denote the set (if any) of all I in E and D_V, we see that

$$\begin{aligned}\sum_E \min\{\Delta g, R(I)\Delta m\} &= \sum_{E^*} \min\{\Delta g, R(I)\Delta m\} + \sum_{\{E-E^*\}} \min\{\Delta g, R(I)\Delta m\} \\ &\leqslant \sum_{E^*} \min\{\Delta g, K_V \Delta m\} + \sum_{\{E-E^*\}} [\Delta h + j/[8(W + 1)] \\ &\leqslant [\sum_{E^*} \Delta h] + 2/V + [\sum_{\{E-E^*\}} \Delta h] + j/4 \\ &\leqslant \sum_E \Delta h + j/2 + j/4 = h(b) - h(a) + 3j/4.\end{aligned}$$

Therefore

$$g(b) - g(a) = \int_{[a,b]} \min\{dg, R(I)dm\} \leqslant h(b) - h(a) + 3j/4,$$

so that $g(b) - g(a) = h(b) - h(a)$, which implies that $\Delta g = \Delta h$ for each subinterval I of $[a, b]$. Therefore g is absolutely continuous with respect to m. Hence (1) implies (2).

Thus (1) and (2) are equivalent.

Reference

1. E. Hellinger, *Die Orthogonalinvarianten quadratischer Formen von unendlichvielen Variablen*, Diss. (Göttingen, 1907).

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