

RESEARCH ARTICLE

Remark on complements on surfaces

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Abstract

We give an explicit characterization on the singularities of exceptional pairs in any dimension. In particular, we show that any exceptional Fano surface is $\frac{1}{42}$ -lc. As corollaries, we show that any \mathbb{R} -complementary surface X has an n -complement for some integer $n \leq 192 \cdot 84^{128 \cdot 42^5} \approx 10^{10^{10.5}}$, and Tian’s alpha invariant for any surface is $\leq 3\sqrt{2} \cdot 84^{64 \cdot 42^5} \approx 10^{10^{10.2}}$. Although the latter two values are expected to be far from being optimal, they are the first explicit upper bounds of these two algebraic invariants for surfaces.

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1. Introduction

We work over the field of complex number \mathbb{C} .

Birkar famously proved the boundedness of n -complements for \mathbb{R} -complementary varieties and pairs with hyperstandard coefficients [3], which was later generalized to arbitrary DCC coefficients [20] and arbitrary coefficients [40] under milder conditions. It is interesting to ask whether we can give an explicit bound of n , as such an explicit bound is expected to be useful for the moduli of log surfaces (cf. [1, 26, 28]) and threefold minimal log discrepancies [19]. [39] shows that an \mathbb{R} -complementary surface pair (X, B) is n -complementary for some $n \in \{1, 2, 3, 4, 6\}$ when B has standard coefficients and (X, B) is not exceptional, but the question remained open in general for surfaces. In this paper, we provide an explicit upper bound of n for surfaces.

Theorem 1.1. *Let $X/Z \ni z$ be an \mathbb{R} -complementary surface. Then, $X/Z \ni z$ has an n -complement for some $n \leq 192 \cdot 84^{128 \cdot 42^5}$. In particular, if Z is a point, then $h^0(-nK_X) > 0$.*

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The key ingredient of the proof of Theorem 1.1 is the following result which provides an explicit characterization of the singularities of exceptional pairs in any dimension. Recall that $\text{lct}(d, \Gamma)$ is the set of lc thresholds for effective Weil divisors with respect to pairs of dimension d with coefficients in Γ .

Theorem 1.2. *Let d be a positive integer and $\Gamma \subset [0, 1]$ a DCC set. Let*

$$\epsilon_1(d, \Gamma) := \inf \left\{ 1 - t \mid \begin{array}{l} t < 1, \text{ there exists a pair } (W, \Delta + t\Psi) \text{ of dimension } d \text{ such that} \\ (W, \Delta + t\Psi) \text{ is lc, } K_W + \Delta + t\Psi \equiv 0, \Delta \in \Gamma, \text{ and } 0 \neq \Psi \in \mathbb{N}^+ \end{array} \right\},$$

$$\epsilon_2(d, \Gamma) := \inf\{1 - t \mid t < 1, t \in \text{lct}(d, \Gamma)\},$$

and $\epsilon(d, \Gamma) := \min\{\epsilon_1(d, \Gamma), \epsilon_2(d, \Gamma)\}$. Then, for any exceptional Fano type pair (X, B) of dimension d such that $B \in \Gamma$ and any $0 \leq G \sim_{\mathbb{R}} -(K_X + B)$, $(X, B + G)$ is $\epsilon(d, \Gamma)$ -lc. In particular, (X, B) is $\epsilon(d, \Gamma)$ -lc.

We have the following corollary, which implies Theorem 1.1.

Corollary 1.3. *Exceptional Fano type surfaces are $\frac{1}{42}$ -lc.*

With Corollary 1.3, we also provide an explicit upper bound of Tian’s α -invariant for surfaces:

Corollary 1.4. *Tian’s α -invariant for any surface is $\leq 3\sqrt{2} \cdot 84^{64 \cdot 42^5}$ (when it is well-defined).*

Although the bounds in Theorem 1.1 and Corollary 1.4 are expected to be far from being optimal, these are the first precise upper bounds of these two algebraic invariants for surfaces. Similar topics and alternative directions include the estimation of the lower bound n (cf. [15, 42]), the boundedness of the anti-canonical volume of Fano varieties (cf. [35, 36, 11, 37, 12, 38, 13, 14, 23, 22, 24, 4]), estimation of (ϵ, n) -complement [9], the explicit M^cKernan-Shokurov conjecture [18], precise bounds of mlds [21, 33, 31], etc.

Postscript

After the first version of this paper, 1) Totaro [41] conjectured that the smallest Tian’s alpha invariant for del Pezzo surfaces is equal to $\frac{21}{2}$, given by $X_{154} \subset \mathbb{P}(77, 45, 19, 14)$, and 2) The author and Shokurov [32] prove that $\epsilon(2, \{0\}) = \frac{1}{13}$. This result allows us to get better explicit bounds of the n -complements and the α -invariants. Nevertheless, in order to make the paper self-contained, we will not use any results in [32].

2. Preliminaries

We adopt the standard notation and definitions in [29, 6] and will freely use them. For the notation of (relative) pairs $(X/Z \ni z, B)$ and complements, we refer the reader to [9].

Definition 2.1. Let $(X/Z \ni z, B)$ be an \mathbb{R} -complementary pair. We say that $(X/Z \ni z, B)$ is *exceptional* if $(X/Z \ni z, B + G)$ is klt for any $G \geq 0$ such that $K_X + B + G \sim_{\mathbb{R}} 0$ over a neighborhood of z .

Definition 2.2. Let d be a positive integer and $\Gamma \subset [0, 1]$ a set. We let

$$\epsilon_1(d, \Gamma) := \inf \left\{ 1 - t \mid \begin{array}{l} t < 1, \text{ there exists a pair } (X, B + tC) \text{ of dimension } d \text{ such that} \\ (X, B + tC) \text{ is lc, } K_X + B + tC \equiv 0, B \in \Gamma, \text{ and } 0 \neq C \in \mathbb{N}^+ \end{array} \right\},$$

$$\epsilon_2(d, \Gamma) := \inf\{1 - t \mid t < 1, t \in \text{lct}(d, \Gamma)\},$$

and $\epsilon(d, \Gamma) := \min\{\epsilon_1(d, \Gamma), \epsilon_2(d, \Gamma)\}$. By [17, Theorem 1.5], $\epsilon(d, \Gamma) > 0$ when Γ is DCC.

Remark 2.3. Usually, $\epsilon_1(d, \Gamma) < \epsilon_2(d, \Gamma)$, (e.g., when $\Gamma = D(\Gamma)$ [17, Lemma 11.2, Proposition 11.5]) and in such cases, $\epsilon(d, \Gamma) = \epsilon_1(d, \Gamma)$.

Lemma 2.4 [26, 5.3 Theorem, 5.4 Theorem]. *Let $\Gamma := \{1 - \frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{1\}$. Then,*

1. $\epsilon_2(2, \Gamma) = \epsilon_2(2, \{0\}) = \frac{1}{6}$.
2. $\epsilon_1(2, \Gamma) = \frac{1}{42} \leq \epsilon_1(2, \{0\})$.

Remark 2.5. It was expected that $\epsilon_1(2, \{0\}) = \frac{1}{13}$ (cf. [1, Notation 4.1], [28, 40]). After the first version of this paper, the author and Shokurov prove this result in [32]. It is interesting to ask whether $\epsilon_1(d, \Gamma)$ is equal to the 1-gap of mlds for pairs with coefficients in Γ in dimension $d + 1$ (cf. [21, 33, 31]).

Remark 2.6. When $\Gamma = \{1 - \frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{1\}$ is the standard set, $\epsilon(d, \Gamma) \leq \frac{1}{N_{d+2-1}}$ by considering the example $(\mathbb{P}^d, \sum_{i=1}^{d+1} (1 - \frac{1}{N_i})H_i + (1 - \frac{1}{N_{d+2-1}})H_{d+2})$, where $\{N_i\}_{i=1}^{+\infty}$ is the Sylvester sequence 2, 3, 7, 43, ... and H_i are general hyperplanes of degree 1. It is also expected that $\epsilon(d, \Gamma) = \frac{1}{N_{d+2-1}}$ [27, 8.16].

Lemma 2.7 [(cf. [2, Proof of Lemma 3.7], [30, After Theorem A], [4, Lemma 2.2)].] *Let ϵ be a positive real number and X an ϵ -lc Fano surface. Then, IK_X is Cartier for some positive integer $I \leq 2\left(\frac{2}{\epsilon}\right)^{\frac{128}{\epsilon^5}}$.*

We will frequently use the following result to run minimal model programs:

Theorem 2.8 [6, Corollary 1.3.2]. *Fano type varieties are Mori dream spaces. In particular, for any Fano type variety X and any \mathbb{R} -Cartier \mathbb{R} -divisor D on X , any sequence of D -MMP terminates with either a good minimal model or a Mori fiber space.*

3. The nonexceptional case

Lemma 3.1. *Let $X/Z \ni z$ be an \mathbb{R} -complementary surface that is not exceptional. Then, $X/Z \ni z$ has an n -complement for some $n \in \{1, 2, 3, 4, 6\}$.*

Proof. There exists an lc but not klt pair $(X/Z \ni z, B)$ such that $K_X + B \sim_{\mathbb{R}, Z} 0$ over a neighborhood of z . Let $f : Y \rightarrow X$ be a dlt modification of $(X/Z \ni z, B)$ and let $K_Y + B_Y := f^*(K_X + B)$. Then, $[B_Y] \neq 0$. By [39, 2.3 Inductive Theorem], $(Y/Z \ni z, [B_Y])$ has a monotonic n -complement $(Y/Z \ni z, B_Y^+)$ for some $n \in \{1, 2, 3, 4, 6\}$ for any $z \in Z$. Hence, $X/Z \ni z$ has an n -complement $(X/Z \ni z, f_*B_Y^+)$ for some $n \in \{1, 2, 3, 4, 6\}$. □

Corollary 3.2. *Let $X/Z \ni z$ be an \mathbb{R} -complementary surface and $\dim Z > 0$. Then, $X/Z \ni z$ has an n -complement for some $n \in \{1, 2, 3, 4, 6\}$.*

Proof. For any pair $(X/Z \ni z, B)$ such that $K_X + B \sim_{\mathbb{R}, Z} 0$ over a neighborhood of z , $(X/Z \ni z, B + t f^*z)$ is an lc but not klt pair such that $K_X + B + t f^*z \sim_{\mathbb{R}, Z} 0$ over a neighborhood of z , where $t := \text{lct}(X, B; f^*z)$. The corollary follows from Lemma 3.1. □

4. The exceptional case

4.1. Proof of Theorem 1.2 and Corollary 1.3

Lemma 4.1. *Let d be a positive integer, $\Gamma \subset [0, 1]$ a DCC set and $\epsilon := \epsilon(d, \Gamma)$. Let $(X/Z, T + aS)$ be a pair such that X is of Fano type over Z , $-(K_X + T + aS)$ is nef/ Z , $T \in \Gamma$, $S \neq 0$ is a reduced divisor and $a \in (1 - \epsilon, 1)$. Then, we may run a $-(K_X + T + S)$ -MMP/ Z which consists of a sequence of divisorial contractions and flips*

$$(X, T + S) := (X_0, T_0 + S_0) \dashrightarrow (X_1, T_1 + S_1) \dashrightarrow \cdots \dashrightarrow (X_n, T_n + S_n),$$

such that

1. $(X_i, T_i + S_i)$ is lc for each i ,
2. $S_n \neq 0$ and
3. $-(K_{X_n} + T_n + S_n)$ is nef/ Z .

Here, T_i and S_i are the strict transforms of T and S on X_i respectively.

Proof. By Theorem 2.8, we may run a $-(K_X + T + S)$ -MMP/ Z .

(1) Since $-(K_X + T + aS)$ is nef/ Z , $(X/Z, T + aS)$ is \mathbb{R} -complementary. Hence, $(X_i/Z, T_i + aS_i)$ is \mathbb{R} -complementary for each i . In particular, $(X_i, T_i + aS_i)$ is lc for each i . By the definition of ϵ , $(X_i, T_i + S_i)$ is lc for each i .

(2) Since $-(K_X + T + aS)$ is nef/ Z , by the negativity lemma, $X \dashrightarrow X_n$ is $-(K_X + T + aS)$ -non-negapositive. Since $X \dashrightarrow X_n$ is a $-(K_X + T + S)$ -MMP, $X \dashrightarrow X_n$ is $-(K_X + T + S)$ -negative. Hence, $X \dashrightarrow X_n$ is S -positive, and we get (2).

(3) Suppose not, then this MMP terminates with a $-(K_{X_n} + T_n + S_n)$ -Mori fiber space $X_n \rightarrow V$. Then, $-(K_{X_n} + T_n + S_n)$ is anti-ample/ V . Since $-(K_X + T + aS)$ is nef/ Z , $-(K_{X_n} + T_n + aS_n)$ is nef/ V and there exists a real number $c \in [a, 1) \subset (1 - \epsilon, 1)$ such that $K_{X_n} + T_n + cS_n \equiv_V 0$. By (1), $(X_n, T_n + cS_n)$ is lc. Let F be a general fiber of $X_n \rightarrow V$, $T_F := T|_F$ and $S_F := S|_F$. Then, $(F, T_F + cS_F)$ is lc and $K_F + T_F + cS_F \equiv 0$. Thus, $\epsilon \leq \epsilon_1(d, \Gamma) \leq \epsilon_1(\dim F, \Gamma) \leq 1 - c$, a contradiction. \square

Proof of Theorem 1.2. Let $a := \text{tml}(X, B + G)$ be the total minimal log discrepancy of $(X, B + G)$, and E a divisor over X such that $a(E, X, B + G) = a$. Suppose that $a < \epsilon := \epsilon(d, \Gamma)$. We let $c := \text{mult}_E G$ and let $e := 1 - a - c$. If E is exceptional over X , then we let $f : Y \rightarrow X$ be a divisorial contraction which extracts E . If E is not exceptional over X , then we let $f : Y \rightarrow X$ be the identity morphism. Then, we have $K_Y + eE + B_Y = f^*(K_X + B)$, where B_Y is the strict transform of B on Y . We let $G_Y := f^*G$.

Claim 4.2. Y is of Fano type.

Proof. If $Y = X$, then it is clear Y is of Fano type. Otherwise, there exists a klt pair (X, Δ) such that $-(K_X + \Delta)$ is big and nef. Let $\Delta_Y := f_*^{-1}\Delta$ and let $a' := a(E, X, \Delta)$. Then, $a' \leq a < \epsilon < 1$, so $(Y, \Delta_Y + (1 - a')E)$ is a klt pair such that $-(K_Y + \Delta_Y + (1 - a')E)$ is big and nef. Thus, Y is of Fano type. \square

Proof of Theorem 1.2 continued. Since $-(K_Y + (1 - a)E + B_Y) \sim_{\mathbb{R}} G_Y - cE \geq 0$, by Claim 4.2 and Theorem 2.8, we may run a $-(K_Y + (1 - a)E + B_Y)$ -MMP which terminates with a model T such that $-(K_T + (1 - a)E_T + B_T)$ is nef, where E_T, B_T are the strict transforms of E, B on T , respectively. Since E is not a component of $G_Y - cE$ and the MMP only contracts divisors that are contained in $\text{Supp}(G_Y - cE)$, $E_T \neq 0$. By Lemma 4.1, we may run a $-(K_T + E_T + B_T)$ -MMP which terminates with a model V , such that $(V, E_V + B_V)$ is lc, $E_V \neq 0$ and $-(K_V + E_V + B_V)$ is nef, where E_V, B_V are the strict transforms of E_T, B_T on V , respectively. Since X is of Fano type, V is of Fano type. It is clear that $(V, E_V + B_V)$ is not exceptional.

For any prime divisor D over X , we have

$$\begin{aligned} a(D, X, B) &\geq a(D, Y, (1 - a)E + B_Y) \geq a(D, T, (1 - a)E_T + B_T) \\ &\geq a(D, T, E_T + B_T) \geq a(D, V, E_V + B_V). \end{aligned}$$

By [3, Lemma 2.17], X is not exceptional, a contradiction. \square

Remark 4.3. The proof of Theorem 1.2 also works for generalized pairs [5]. For simplicity, we omit the proof.

Corollary 4.4. Let $(X, B + G)$ be a pair such that (X, B) is exceptional, X is a Fano type surface, $B \in \{1 - \frac{1}{n} \mid n \in \mathbb{N}^+\} \cup \{1\}$ and $0 \leq G \sim_{\mathbb{R}} -(K_X + B)$. Then, $(X, B + G)$ is $\frac{1}{42}$ -lc. In particular, (X, B) and X are $\frac{1}{42}$ -lc.

Proof. It immediately follows from Theorem 1.2 and Lemma 2.4. □

Proof of Corollary 1.3. It follows from Corollary 4.4. □

Corollary 4.5. *For any exceptional Fano surface X , there exists $I \leq 2 \cdot 84^{128 \cdot 42^5}$ such that IK_X is Cartier. In particular, $K_X^2 \geq \frac{1}{I}$.*

Proof. It follows from Corollary 1.3 and Lemma 2.7. □

4.2. Exceptional surface complements

Lemma 4.6. *Let $(X := \mathbb{P}^1, B)$ be a pair such that $\deg(K_X + B) \leq 0$ and $B \in \{\frac{k}{12} \mid k \in \mathbb{N}^+, 0 \leq k \leq 12\} \cup \{1 - \frac{1}{n} \mid n \in \mathbb{N}^+\}$. Then, (X, B) has a monotonic n -complement such that $12 \mid n$ and $n \leq 276$.*

Proof. We may write $B = C + D$ where $C, D \geq 0$, $C \in \{1 - \frac{1}{n} \mid n \in \mathbb{N}^+, 12 \nmid n\}$, $12D$ is integral and $C \wedge D = 0$. Then, the coefficients of C are $\geq \frac{4}{5}$. In particular, C has at most 2 irreducible components. Possibly adding divisors of the form $\frac{1}{12}p$ to D where p are general points on X , we may assume that $0 \geq \deg(K_X + B) > -\frac{1}{12}$. We have the following cases.

Case 1. $C = 0$. Then, (X, B) is a 12-complement of itself.

Case 2. C has 1 irreducible component C_1 . Then, $C = aC_1$ for some $a \in (0, 1)$ and $(X, [\frac{12a}{12}]C_1 + D)$ is a monotonic 12-complement of (X, B) .

Case 3. C has 2 irreducible components C_1, C_2 . We have $C = a_1C_1 + a_2C_2$. Possibly switching C_1, C_2 , we may assume that $a_1 \leq a_2$. If $D = 0$, then $(X, C_1 + C_2)$ is a monotonic 1-complement of (X, B) . If $D \neq 0$, then $a_1 \leq \frac{23}{24}$. Let m be the denominator of a_1 . Then, $m \leq 24$ and $(X, a_1C_1 + (2 - \deg D - a_1)C_2 + D)$ is a monotonic $\text{lcm}(12, m)$ -complement of (X, B) . Since $12 \mid \text{lcm}(12, m)$ and $\text{lcm}(12, m) \leq 276$, we are done. □

Theorem 4.7. *Let X be an \mathbb{R} -complementary exceptional surface.*

1. *If $\kappa(-K_X) = 0$, then X has an n -complement for some $n \leq 21$.*
2. *If $\kappa(-K_X) = 1$, then X has an n -complement for some n such that $12 \mid n$ and $n \leq 276$.*
3. *If $\kappa(-K_X) = 2$, then X has an n -complement for some $n \leq 192 \cdot 84^{128 \cdot 42^5}$.*

Proof. There exists a klt pair (X, B) such that $K_X + B \sim_{\mathbb{R}} 0$. Thus, $(X, (1 + \delta)B)$ is klt for some $0 < \delta \ll 1$, so we may run a $(K_X + (1 + \delta)B)$ -MMP which terminates with good minimal model X' . Since $K_X + (1 + \delta)B \sim_{\mathbb{R}} -\delta K_X$, this is also a $-K_X$ -MMP. By abundance for klt surfaces, $-K_{X'}$ is semi-ample. Possibly replacing X with X' , we may assume that $-K_X$ is semi-ample.

If $\kappa(-K_X) = 0$, then $K_X \equiv 0$. Hence, $nK_X \sim 0$ for some positive integer $n \leq 21$ [7, 43, 44], and X is an n -complement of itself for some $n \leq 21$.

If $\kappa(-K_X) = 1$, then $-K_X$ defines a contraction $f : X \rightarrow Z$. By Kodaira’s canonical bundle formula, we have

$$12K_X \sim 12f^*(K_Z + B_Z + M_Z)$$

such that $12M_Z$ is an integral divisor, $B_Z \in \{1 - \frac{1}{n} \mid n \in \mathbb{N}^+\}$ and

$$B_Z = \sum_{z \in Z} (1 - \text{lct}(X, 0; f^*z))z.$$

We may choose M_Z such that $B_Z \wedge M_Z = 0$ and $(Z, B_Z + M_Z)$ is klt. Since $-K_X$ is semi-ample, $\deg(K_Z + B_Z + M_Z) \leq 0$. Hence, Z is either an elliptic curve or \mathbb{P}^1 . If Z is an elliptic curve, then $B_Z = M_Z = 0$ and $12K_X$ is base-point-free. Hence, X is a 12-complement of itself. If $Z = \mathbb{P}^1$, by Lemma 4.6, there exists an integer $n \leq 276$ such that $12 \mid n$ and $(Z, B_Z + M_Z)$ has a monotonic

n -complement $(Z, B_Z + G + M_Z)$. By the construction of B_Z , (X, f^*G) lc. Hence, (X, f^*G) is an n -complement of X .

If $\kappa(-K_X) = 2$, then $-K_X$ defines a birational morphism $f : X \rightarrow Y$. We have $K_X = f^*K_Y$. Possibly replacing X with Y , we may assume that X is Fano. By Corollary 1.3, X is $\frac{1}{42}$ -lc. By Lemma 2.7, IK_X is Cartier for some $I \leq 2 \cdot 84^{128 \cdot 42^5}$. By the effective base-point-freeness theorem ([16, Theorem 1.1, Remark 1.2] [25, 1.1 Theorem]), $|-96IK_X|$ is base-point-free. In particular, X has a $96I$ -complement. \square

5. Proof of the main theorems

Proof of Theorem 1.1. If $\dim Z > 0$ or $\dim Z = 0$ and X is not exceptional, the theorem follows from Lemma 3.1 and Corollary 3.2. Otherwise, $\dim Z = 0$ and X is exceptional, and the theorem follows from Theorem 4.7. \square

Proof of Corollary 1.4. Recall that Tian’s α -invariant for a variety X is defined as

$$\alpha(X) := \inf\{t \geq 0 \mid \text{lct}(X, 0; D) \mid D \in |-K_X|_{\mathbb{Q}}\}.$$

We may assume that $\alpha(X) > 1$. If $\kappa(-K_X) \leq 1$, then by Theorem 4.7, X has an n -complement (X, G) for some $n \leq 276$. Hence, (X, nG) is not klt, and $\alpha(X) \leq 276$. Thus, we may assume that $\kappa(-K_X) = 2$. We may run a $(-K_X)$ -MMP and replace X with the canonical model of $-K_X$, and assume that $-K_X$ is ample. By Corollary 4.5, $K_X^2 \geq \frac{1}{7}$. Since $\alpha(X)^2 \cdot \text{vol}(-K_X) \leq 9$ (cf. [8, Theorems A,D]), $\alpha(X) \leq 3\sqrt{7} \leq 3\sqrt{2} \cdot 84^{64 \cdot 42^5}$. \square

6. Further remarks

Remark 6.1 (Reasonable and optimal bounds). Recall that we expect $\epsilon_1(2, \{0\}) = \frac{1}{13}$. If we can prove this, then the bound of n in Theorem 1.1 can be improved to $192 \cdot 26^{128 \cdot 13^5} \approx 10^{107.8}$. This is much smaller than the current bound, albeit it is still expected to be far from optimal. However, if one can get a better bound for $I = I(\epsilon)$ in Lemma 2.7, then the bound of n may be greatly improved. For example, [34, Proof of Lemma 4.9] actually implies that the local Cartier index of any $\frac{2}{5}$ -klt weak Fano surface is ≤ 19 . With a little more effort, one can show that the global Cartier index of any $\frac{2}{5}$ -klt weak Fano surface is ≤ 385 . This is much smaller than the bound given by Lemma 2.7 which is $2 \cdot 5^{12500}$. By applying the arguments in this paper, we shall get $n \leq 36960$ and $\alpha(X) \leq 3\sqrt{385} \approx 58.86$ for exceptional $\frac{2}{5}$ -klt surfaces.

Remark 6.2 (Explicit bound for pairs). One may also ask whether we can find an explicit bounded of n for n -complements of surface pairs (X, B) .

For pairs with finite rational coefficients, the bound is computable via the methods introduced in [2], but may be much larger than the case when $B = 0$. This is because the bound $\epsilon(d, \Gamma) = \frac{1}{42}$ in Lemma 2.4 will be changed to a number which is very close to 1 as in [2, 3.5] when Γ is not the standard set. It is very difficult to represent that number in a very explicit function of the common denominator of the coefficient set (even when the coefficient set is $\{0, \frac{1}{3}\}$, for example). We also need to go through the inductive arguments as in [39, 2.3 Inductive Theorem] for nonexceptional complements.

For pairs with finite (maybe irrational) coefficients or DCC coefficients, a theory on ‘explicit uniform rational polytopes’ (cf. [20]) is needed, which is still unknown.

For pairs with coefficients in $[0, 1]$, one needs to go through all the previous simpler cases and check the details of the proof of [40, Theorem 3] and avoid using any inexplicit boundedness result. This is considered to be much more difficult. See [10] for a similar result.

Remark 6.3 (Explicit bound of threefold mlds). By Remark 6.2 and following the details of the proof of [19], one will be able to provide a computable lower bound of the 1-gap of threefold mlds for pairs with finite rational coefficients (or, more generally, hyperstandard rational coefficients). This is because all other constants in the proof of [19] can be explicitly bounded except the t in [19, Lemma 6.4]. Here,

an explicit boundedness of n -complement for threefold singularities is needed, but this just follows from Theorem 1.1 and Remark 6.2. Nevertheless, such bound will, again, be far from being optimal. For example, when we have standard coefficients, the 1-gap is expected to be $\frac{1}{42}$ (by Remark 2.5 and Lemma 2.4), but we can only show that the 1-gap is $\geq \epsilon$ for some $\epsilon \approx \frac{1}{192 \cdot 42^{128 \cdot 42^5}}$.

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