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THE NORM OF THE PRODUCT OF POLYNOMIALS IN INFINITE DIMENSIONS

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Abstract Given a Banach space E and positive integers k and l we investigate the smallest constant C that satisfies $\|P\|\,\|Q\| \leqslant C\|PQ\|$ for all k-homogeneous polynomials P and l-homogeneous polynomials Q on E. Our estimates are obtained using multilinear maps, the principle of local reflexivity and ideas from the geometry of Banach spaces (type and uniform convexity). We also examine the analogous problem for general polynomials on Banach spaces.

Keywords: polynomials; geometry of Banach spaces; norm inequalities

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1. Introduction

Let k and l be positive integers. In [**7**] it is shown that given any complex Banach space E and any two polynomials P of degree k and Q of degree l on E we have

$$\|P\|\,\|Q\| \leqslant \frac{n^n}{k^k l^l}\|PQ\|,$$

where $n = k + l$. Results of this nature have long been studied for finite dimensions (see [**6**], [**8**], [**9**], [**13**], [**16**], [**17**] and [**18**]). Given a suitable function F of n complex variables, the Mahler measure of F, $M(F)$, is defined by

$$M(F) = \exp\biggl\{\int_0^1 \cdots \int_0^1 \log|F(\mathrm{e}^{2\pi \mathrm{i} t_1}, \dots, \mathrm{e}^{2\pi \mathrm{i} t_n})|\,\mathrm{d}t_1 \cdots \mathrm{d}t_n\biggr\}.$$

It is shown in [**8**] that if $f = gh$ is a polynomial of degree n in one complex variable, then

$$\|g\|\,\|h\| \leqslant \delta^n \|f\|,$$

where δ is the Mahler measure of $F(x, y) = 1 + x + y - xy$. Furthermore, this inequality is shown to be asymptotically sharp as $n \to \infty$. In infinite dimensions inequalities for the norms of products of linear functionals have been presented in [**1**], [**10**] and [**21**].

In this paper, we will show that for certain spaces the constant $n^n/k^k l^l$ can be improved. Our approach is to use the geometry of the underlying Banach space (type and uniform convexity) to estimate the 'weighted' distance between the norming points of P and Q. Using symmetric multilinear maps we show that if P and Q are k and l homogeneous polynomials, respectively, on a Hilbert space, then

$$\|P\|\,\|Q\| \leqslant \frac{n!}{k!l!}\|PQ\|.$$

Given a Banach space E we shall use $\mathcal{P}(^nE)$ to denote the space of all bounded n-homogeneous polynomials on E and $\mathcal{L}^s(^nE)$ to denote the space of all bounded symmetric n-linear mappings on E^n. The space $\mathcal{P}(^nE)$ becomes a Banach space when given the norm $\|P\| = \sup_{\|x\|<1}|P(x)|$, while $\|L\| = \sup_{\|x_i\|<1}|L(x_1,\dots,x_n)|$ makes $\mathcal{L}^s(^nE)$ into a Banach space. For further reading on polynomials on infinite-dimensional Banach spaces we refer the reader to [**12**].

2. Estimates using symmetric multilinear maps and biduals

Given x and y in a complex Banach space E, $P \in \mathcal{P}(^kE)$ and $Q \in \mathcal{P}(^lE)$, Benítez, Sarantopoulos and Tonge [**7**] give the formula

$$P(x)\overline{Q(y)} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k\theta} P(\mathrm{e}^{\mathrm{i}\theta}x + y)\overline{Q(\mathrm{e}^{\mathrm{i}\theta}x + y)}\,\mathrm{d}\theta. \tag{2.1}$$

Using this identity they obtain the following theorem.

Theorem 2.1. *Let E be a complex Banach space and let k and l be positive integers. Let $n = k + l$. If P and Q are polynomials on E of degree k and l, respectively, then*

$$\|P\|\,\|Q\| \leqslant \frac{n^n}{k^k l^l}\|PQ\|. \tag{2.2}$$

The existence of a universal constant which satisfies (2.2) in Theorem 2.1 can also be established using ultrapowers in much the same way as in Proposition 2.3.1 of [**10**]. This proof, however, gives no idea of the size of the constant.

As pointed out in [**7**] the constant $n^n/k^k l^l$ in Equation (2.2) of Theorem 2.1 is sharp. To see this take $E = \ell_1^n$, $P(z) = z_1 \cdots z_k$, $Q(z) = z_{k+1}\cdots z_n$. By Lemma 3.1 of [**20**] we have

$$\|P\| = \frac{1}{k^k}, \quad \|Q\| = \frac{1}{l^l} \quad \text{and} \quad \|PQ\| = \frac{1}{n^n}.$$

Thus

$$\|P\|\,\|Q\| = \frac{n^n}{k^k l^l}\|PQ\|.$$

For many spaces, however, it is possible to improve on the constants given in Theorem 2.1. Our first approach is to obtain a version of (2.1) for symmetric n-linear maps. Given A in $\mathcal{L}^s(^kE)$ and B in $\mathcal{L}^s(^lE)$ we define functions $(AB)_s$ and $(A\bar{B})_s$ of n variables by

$$(AB)_s(x_1,\dots,x_n) = \frac{1}{n!}\sum_{\sigma\in S_n} A(x_{\sigma(1)},\dots,x_{\sigma(k)})B(x_{\sigma(k+1)},\dots,x_{\sigma(n)})$$

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and

$$(A\bar{B})_s(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} A(x_{\sigma(1)}, \dots, x_{\sigma(k)}) \overline{B(x_{\sigma(k+1)}, \dots, x_{\sigma(n)})}.$$

Note that $(AB)_s \in \mathcal{L}^s(^nE)$, while $(A\bar{B})_s$ is a symmetric real n-linear function on E.

Proposition 2.2. *Let E be a complex Banach space and let k and l be positive integers. Let $n = k + l$. For $A \in \mathcal{L}^s(^kE)$ and $B \in \mathcal{L}^s(^lE)$ we have*

$$\|A\| \, \|B\| \leqslant \frac{n!}{k!l!} \|(A\bar{B})_s\|. \tag{2.3}$$

Proof. This follows immediately from the identity

$$A(x_1, \dots, x_k)\overline{B(x_{k+1}, \dots, x_n)} = \frac{1}{2\pi} \frac{n!}{k!l!} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}k\theta} (A\bar{B})_s(\mathrm{e}^{\mathrm{i}\theta} x_1, \dots, \mathrm{e}^{\mathrm{i}\theta} x_k, x_{k+1}, \dots, x_n) \, \mathrm{d}\theta.$$

□

Again, this inequality is sharp. Take E equal to ℓ_1^n,

$$A(x^1, x^2, \dots, x^k) = \frac{1}{k!} \sum_{\sigma \in S_k} x^1_{\sigma(1)} x^2_{\sigma(2)} \cdots x^k_{\sigma(k)},$$

$$B(x^1, x^2, \dots, x^l) = \frac{1}{l!} \sum_{\tau \in S_l} x^1_{\tau(k+1)} \cdots x^l_{\tau(n)}.$$

Then

$$\|A\| \, \|B\| = \frac{n!}{k!l!} \|(A\bar{B})_s\|.$$

Given a complex Banach space E, Benítez, Sarantopulos and Tonge define the nth (linear) polarization constant $c_n(E)$ as

$$c_n(E) = \inf\{M > 0 : \|f_1\| \, \|f_2\| \cdots \|f_n\| \leqslant M \|f_1 \cdots f_n\| \text{ for all } f_1, \dots, f_n \in E'\}.$$

Clearly, by [**7**] we have that $c_n(E) \leqslant n^n$. If H is a complex Hilbert space of dimension at least n, then Arias-de-Reyna [**2**] shows that $c_n(H) = n^{n/2}$. Estimates for $c_n(L_p(\mu))$ are given in [**19**, Proposition 16].

If $P \in \mathcal{P}(^nE)$, we shall use $\check{P}$ to denote the (unique) symmetric n-linear map associated with P. Given a complex Banach space E we let $C_E(n)$ denote the polarization constant of degree n of E and we let $R_E(n)$ denote the polarization constant of degree n of the underlying real Banach space E_R. That is,

$$C_E(n) = \inf\{C : \|\check{P}\| \leqslant C\|P\| \text{ for all } P \in \mathcal{P}(^nE)\},$$

whereas

$$R_E(n) = \inf\{C : \|\check{P}\| \leqslant C\|P\| \text{ for all } P \in \mathcal{P}(^nE_R)\}.$$

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We note that, in general, $C_E(n) \leqslant R_E(n)$. For positive integers k and l we define $F_E(k,l)$, $G_E(k,l)$ and $H_E(k,l)$ to be the infimum of all positive reals that satisfy, respectively,

$$\|P\|\,\|Q\| \leqslant F_E(k,l)\|PQ\|$$

for all $P \in \mathcal{P}(^kE)$ and $Q \in \mathcal{P}(^lE)$, and

$$\|A\|\,\|B\| \leqslant G_E(k,l)\|(AB)_s\|$$

and

$$\|A\|\,\|B\| \leqslant H_E(k,l)\|(A\bar{B})_s\|$$

for all $A \in \mathcal{L}^s(^kE)$ and $B \in \mathcal{L}^s(^lE)$. If k and l are integers, we denote by $C_E(k,l)$ the infimum of all positive reals that satisfy

$$\sup_{x,y \in B_E} \|\check{P}(x)^k(y)^l\| \leqslant C_E(k,l)\|P\|$$

for all $P \in \mathcal{P}(^nE)$ and by $R_E(k,l)$ the infimum of all positive reals that satisfy

$$\sup_{x,y \in B_{E_R}} \|\check{P}(x)^k(y)^l\| \leqslant R_E(k,l)\|P\|$$

for all $P \in \mathcal{P}(^nE_R)$. We have the following relationship between these constants.

Proposition 2.3. *Let k and l be positive integers and let $n = k + l$. Then*

$$\frac{G_E(k,l)}{C_E(k)C_E(l)} \leqslant F_E(k,l) \leqslant G_E(k,l)C_E(k,l) \tag{2.4}$$

and

$$\frac{H_E(k,l)}{C_E(k)C_E(l)} \leqslant F_E(k,l) \leqslant H_E(k,l)R_E(k,l). \tag{2.5}$$

Proof. Take $P \in \mathcal{P}(^kE)$ and $Q \in \mathcal{P}(^lE)$ and let $A = \check{P}$, $B = \check{Q}$. Then we have

$$\begin{aligned}\|P\|\,\|Q\| \leqslant \|A\|\,\|B\| &\leqslant G_E(k,l) \sup_{x,y \in B_E} |(AB)_s(x)^k(y)^l| \\ &\leqslant G_E(k,l)C_E(k,l)\|PQ\|\end{aligned}$$

and

$$\begin{aligned}\|A\|\,\|B\| &\leqslant C_E(k)C_E(l)\|P\|\,\|Q\| \\ &\leqslant C_E(k)C_E(l)F_E(k,l)\|PQ\| \\ &\leqslant C_E(k)C_E(l)F_E(k,l)\|(AB)_s\|,\end{aligned}$$

which gives (2.4).

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As $\|PQ\| = \|P\bar{Q}\|$ we obtain the left-hand side of (2.5) as we obtained the left-hand side of (2.4). Since $(A\bar{B})_s$ is a real n-linear symmetric function, we have

$$\begin{aligned}\|P\|\,\|Q\| \leqslant \|A\|\,\|B\| &\leqslant H_E(k,l) \sup_{x,y\in B_E} \|(A\bar{B})_s(x)^k(y)^l\| \\ &\leqslant H_E(k,l)R_E(k,l)\|P\bar{Q}\| \\ &= H_E(k,l)R_E(k,l)\|PQ\|,\end{aligned}$$

which is (2.5). □

Thus, if $C_E(n) \equiv 1$, which happens if E is a Hilbert space (see [**14**, Theorem 4]) or ℓ^2_∞, then it follows that $C_E(k,l) = 1$ and we see that $F_E(k,l) = G_E(k,l)$.

When E is a complex Hilbert space we have the following corollary.

Corollary 2.4. *Suppose E is a complex Hilbert space and let $P_i \in \mathcal{P}(^{k_i}E)$ for $1 \leqslant i \leqslant l$. Let $n = k_1 + k_2 + \cdots + k_l$. Then*

$$\|P_1\|\,\|P_2\|\cdots\|P_l\| \leqslant \frac{n!}{k_1!\cdots k_l!}\|P_1P_2\cdots P_l\|. \tag{2.6}$$

Proof. Since $R_E(m) \equiv 1$ (see [**4**, **12**]), $F_E(k,l) = H_E(k,l)$ for all k, l and the result will follow by iteration and Proposition 2.2. □

This result has also been established in [**7**] in the special case where $k_1 = k_2 = \cdots = k_l$ and n is a power of 2. The fact that $n!/k!l!$ is an improvement on (2.2) follows from Lemma 3.2 of [**20**].

Proposition 2.3 used with the estimate given in [**14**, Theorem1] gives an alternative proof of [**7**, Lemma 2].

Proposition 2.3 also gives a lower bound of $F_E(k,l)/G_E(k,l)$ for $C_E(n)$.

For estimates on the biduals of a Banach space we have the following result.

Theorem 2.5. *If E is a complex Banach space such that E'' has the metric approximation property, then $F_E(k,l) = F_{E''}(k,l)$ for every k and l.*

Proof. Choose $c > 0$. Then there are $P \in \mathcal{P}(^kE)$ and $Q \in \mathcal{P}(^lE)$ such that

$$(F_E(k,l) - c)\|PQ\| \leqslant \|P\|\,\|Q\| \leqslant F_E(k,l)\|PQ\|.$$

Let $\tilde{R}$ denote the Aron–Berner extension of a homogeneous polynomial R from E to E''. Since this extension process is multiplicative and norm preserving (see [**3**] and [**11**, Theorem 5]), we have

$$(F_E(k,l) - c)\|\tilde{P}\tilde{Q}\| \leqslant \|\tilde{P}\|\,\|\tilde{Q}\|.$$

It follows that

$$F_E(k,l) \leqslant F_{E''}(k,l).$$

On the other hand, given $c > 0$, choose $P \in \mathcal{P}(^kE'')$ and $Q \in \mathcal{P}(^lE'')$ so that

$$(F_{E''}(k,l) - c)\|PQ\| \leqslant \|P\|\,\|Q\|.$$

Now choose $\varepsilon > 0$ and x'', y'' and z'' in $B_{E''}$ such that

$$|P(x'')| \geqslant \|P\| - \varepsilon, \quad |Q(y'')| \geqslant \|Q\| - \varepsilon \quad \text{and} \quad |PQ(z'')| \geqslant \|PQ\| - \varepsilon.$$

Let $K = \{x'', y'', z''\}$. Since E'' has the metric approximation property, it follows as in the proof of [**15**, Lemma 3.1] that there exist finite-type polynomials $P_o \in \mathcal{P}_{\mathrm{f}}({}^kE'')$ and $Q_o \in \mathcal{P}_{\mathrm{f}}({}^lE'')$ such that

$$\|P - P_o\|_K < \varepsilon, \quad \|Q - Q_o\|_K < \varepsilon, \quad \|P\| - 2\varepsilon \leqslant \|P_o\| \leqslant \|P\|,$$
$$\|Q\| - 2\varepsilon \leqslant \|Q_o\| \leqslant \|Q\| \quad \text{and} \quad \|PQ\| - 2\varepsilon \leqslant \|P_oQ_o\| \leqslant \|PQ\|.$$

Now applying the principle of local reflexivity and arguing as in the proof of Theorem 3.2 of [**15**], there exist finite-type polynomials $R \in \mathcal{P}_{\mathrm{f}}({}^kE)$ and $S \in \mathcal{P}_{\mathrm{f}}({}^lE)$ such that

$$\|R\| \leqslant \|P_o\| + \varepsilon, \quad \|S\| \leqslant \|Q_o\| + \varepsilon, \quad \|RS\| \leqslant \|P_oQ_o\| + \varepsilon,$$
$$\tilde{R}(x'') = P_o(x'') \quad \text{and} \quad \tilde{S}(y'') = Q_o(y'').$$

Hence $\|P_o\| \leqslant \|R\| + 2\varepsilon$, $\|Q_o\| \leqslant \|S\| + 2\varepsilon$, and we have

$$\begin{aligned}(F_{E''}(k,l) - c)\|RS\| &\leqslant (F_{E''}(k,l) - c)(\|P_oQ_o\| + \varepsilon)\\ &\leqslant (\|P_o\| + 2\varepsilon)(\|Q_o\| + 2\varepsilon) + \varepsilon(F_{E''}(k,l) - c)\\ &\leqslant (\|R\| + 4\varepsilon)(\|S\| + 4\varepsilon) + \varepsilon(F_{E''}(k,l) - c).\end{aligned}$$

Since ε was arbitrary, it follows that $F_{E''}(k,l) \leqslant F_E(k,l)$. □

We note that if F is a $\mathcal{L}_\infty$ space and E is a superspace of F, every homogeneous polynomial on F extends to a polynomial on E with the same norm (see [**3**, Corollary 1.3]). It follows that $F_F(k,l) \leqslant F_E(k,l)$.

3. Estimates using the geometry of Banach spaces

For certain Banach spaces and polynomials on those spaces which have approximate norming points which are far apart it may be possible to improve on the estimates given in (2.2) or (2.6). These estimates depend on the geometry of the underlying Banach spaces and the weighted distance between circles that approximate the norms of the polynomials. We introduce some notation. Given a Banach space E, polynomials P and Q of degree k and l, respectively, $1 \leqslant p \leqslant 2$ and $\varepsilon > 0$, we define $\nu_\varepsilon(P,Q)$ by

$$\nu_\varepsilon(P,Q) = \sup_{x,y} \inf_\theta \{\|\mathrm{e}^{\mathrm{i}\theta}x - y\| : \|x\| = \|y\| = 1,\ \theta \in [0,2\pi],\\ |P(x)| \geqslant \|P\| - \varepsilon,\ |Q(y)| \geqslant \|Q\| - \varepsilon\},$$

and $\eta_\varepsilon(P,Q)$ by

$$\eta_{p,\varepsilon}(P,Q) = \sup_{x,y} \inf_\theta \left\{\left\|\left(\frac{k}{n}\right)^{1/p} \mathrm{e}^{\mathrm{i}\theta}x - \left(\frac{l}{n}\right)^{1/p} y\right\| : \|x\| = \|y\| = 1,\ \theta \in [0,2\pi],\\ |P(x)| \geqslant \|P\| - \varepsilon,\ |Q(y)| \geqslant \|Q\| - \varepsilon\right\}.$$

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Since $\eta_{p,\varepsilon}(P,Q)$ and $\nu_\varepsilon(P,Q)$ are decreasing function of ε, we may define $\nu(P,Q)$ and $\eta_p(P,Q)$ by

$$\nu(P,Q) = \inf_{\varepsilon} \nu_\varepsilon(P,Q),$$
$$\eta_p(P,Q) = \inf_{\varepsilon} \eta_{p,\varepsilon}(P,Q).$$

Homogeneous polynomials on complex Banach spaces have not only approximate norming points but approximate norming circles. In this case we can interpret $\nu(P,Q)$ as the maximum distance between the approximate norming circles of P and Q and $\eta_p(P,Q)$ as the maximum weighted distance between approximate norming circles of P and Q. When $k = l$, $\nu(P,Q) = 2^{1/p}\eta_p(P,Q)$.

We introduce the notion of Rademacher type as defined by Beauzamy in [**5**]. We shall use r_k to denote the kth Rademacher function on $[0,1]$. A Banach space has type p if there is $C > 0$ such that for all $n \geqslant 1$ and all $x_1, \dots, x_n$ in E we have that

$$\left(\int_0^1 \left\| \sum_{k=1}^n r_k(t) x_k \right\|^p \mathrm{d}t\right)^{1/p} \leqslant C\left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p}.$$

The smallest C which satisfies the above inequality is denoted by T_p and is called the type constant of E. Although this definition of Rademacher type is equivalent to that which appears in most standard texts, it does give different values for the type constant. These values of T_p have the advantage of being relatively small when E is $L_p(\mu)$, $1 < p < \infty$.

Theorem 3.1. *Let E be a complex Banach space which has type p. For $P \in \mathcal{P}(^kE)$ and $Q \in \mathcal{P}(^lE)$,*

$$\|P\|\,\|Q\| \leqslant \left(\frac{n^n}{k^k l^l}\right)^{1/p} (2T_p^p - \eta_p(P,Q)^p)^{n/p} \|PQ\|. \tag{3.1}$$

Proof. Given ε, $\sigma > 0$, choose x and y with $\|x\| = \|y\| = 1$, $|P(x)| \geqslant \|P\| - \varepsilon$, $|Q(y)| \geqslant \|Q\| - \varepsilon$ so that

$$\inf_{\theta} \left\| \left(\frac{k}{n}\right)^{1/p} \mathrm{e}^{\mathrm{i}\theta} x - \left(\frac{l}{n}\right)^{1/p} y \right\| > \eta_{p,\varepsilon}(P,Q) - \sigma.$$

Since E has type p,

$$\begin{aligned}\left\| \left(\frac{k}{n}\right)^{1/p} \mathrm{e}^{\mathrm{i}\theta} x + \left(\frac{l}{n}\right)^{1/p} y \right\|^p + \left\| \left(\frac{k}{n}\right)^{1/p} \mathrm{e}^{\mathrm{i}\theta} x - \left(\frac{l}{n}\right)^{1/p} y \right\|^p &\leqslant 2T_p^p\left(\frac{k}{n} + \frac{l}{n}\right)^p \\ &= 2T_p^p\end{aligned}$$

for all θ in $[0, 2\pi]$ and so

$$\sup_{\theta} \left\| \left(\frac{k}{n}\right)^{p} \mathrm{e}^{\mathrm{i}\theta} x + \left(\frac{l}{n}\right)^{1/p} y \right\|^n \leqslant (2T_p^p - (\eta_{p,\varepsilon}(P,Q) - \sigma)^p)^{n/p}.$$

Thus, by (2.1), we have

$$\begin{aligned}(\|P\|-\varepsilon)(\|Q\|-\varepsilon) &\leqslant \left(\frac{n^n}{k^k l^l}\right)^{1/p}\left|P\left(\left(\frac{k}{n}\right)^{1/p}x\right)\right|\left|Q\left(\left(\frac{l}{n}\right)^{1/p}y\right)\right| \\ &\leqslant \|PQ\|\left(\frac{n^n}{k^k l^l}\right)^{1/p}\sup_{\theta}\left\|\left(\frac{k}{n}\right)^{1/p}\mathrm{e}^{\mathrm{i}\theta}x+\left(\frac{l}{n}\right)^{1/p}y\right\|^n \\ &\leqslant \|PQ\|\left(\frac{n^n}{k^k l^l}\right)^{1/p}(2T_p^p-(\eta_{p,\varepsilon}(P,Q)-\sigma)^p)^{n/p}.\end{aligned}$$

Letting ε and σ tend to 0 gives the result. □

If we assume that E is a complex Banach space with type p such that $T_p = 1$ (this happens when E is $L_p(\mu)$, $1 \leqslant p \leqslant 2$) and P and Q are two homogeneous polynomials of degree n on E such that $\eta_p(P,Q) \geqslant \sqrt{2}/2^{1/p}$, then our estimate from Theorem 3.1 is that

$$\|P\|\,\|Q\| \leqslant (4-2^{p/2})^{2n/p}\|PQ\|,$$

while our estimate from Theorem 2.1 is that

$$\|P\|\,\|Q\| \leqslant (2)^{2n}\|PQ\|.$$

The value in (3.1) is an improvement on (2.2) when

$$2^p + 2^{p/2} - 4 \geqslant 0.$$

This happens when

$$\log(\tfrac{1}{2}(9-\sqrt{17}))/\log 2 < p \leqslant 2,$$

which is the interval $(1.2859\dots, 2]$. In particular, when E is a complex Hilbert space and P and Q are two n-homogeneous polynomials on E such that $\eta_p(P,Q) = 1$, the estimate we get with (3.1) is

$$\|P\|\,\|Q\| \leqslant 2^n\|PQ\|,$$

compared with

$$\|P\|\,\|Q\| \leqslant 4^n\|PQ\|$$

using (2.2).

A Banach space E is said to be uniformly convex if, for all $r > 0$, there is $\gamma(r) > 0$ such that if x, y are unit vectors in E that satisfy $\|x-y\| \geqslant r$, then $\|x+y\| \leqslant 2(1-\gamma(r))$. Using (2.1) we obtain the following result.

Theorem 3.2. *Let E be a complex uniformly convex Banach space. For $P \in \mathcal{P}({}^kE)$ and $Q \in \mathcal{P}({}^lE)$ and $n = k + l$,*

$$\|P\|\,\|Q\| \leqslant 2^n(1-\gamma(\nu(P,Q)))^n\|PQ\|.$$

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When $P \in \mathcal{P}({}^k L_p(\mu))$, $Q \in \mathcal{P}({}^l L_p(\mu))$, for $p \geqslant 2$ we obtain two estimates for $\|P\|\,\|Q\|$. Using the fact that $L_p(\mu)$ has type 2, arguing as in Theorem 3.1, we obtain

$$\|P\|\,\|Q\| \leqslant (4B_p^2 - \nu(P,Q)^2)^{n/2}\|PQ\|,$$

where B_p is the constant on the right-hand side of Khinchin's inequality. While using uniform convexity and the fact that the modulus of convexity of $L^p(\mu)$ with $p \geqslant 2$ satisfies

$$\gamma(r) \geqslant 1 - \tfrac{1}{2}(2^p - r^p)^{1/p},$$

we obtain

$$\|P\|\,\|Q\| \leqslant (2^p - \nu(P,Q)^p)^{n/p}\|PQ\|.$$

This last inequality can also be obtained from Clarkson's inequality, which says that for x, y in $L^p(\mu)$ we have

$$2(\|x\|^p + \|y\|^p) \leqslant \|x+y\|^p + \|x-y\|^p.$$

4. Estimates for general polynomials

For $\theta \in [0, 2\pi]$, $f \in \mathcal{H}(E)$, we define $f_\theta \in \mathcal{H}(E)$ by $f_\theta(x) = f(\mathrm{e}^{-\mathrm{i}\theta}x)$. We have the following generalization of the identity (2.1).

Proposition 4.1. *Let f and g be entire holomorphic functions on a complex locally convex space E. Then for every x, y in E we have*

$$f(x)\overline{g(y)} = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} f_\theta(\mathrm{e}^{\mathrm{i}\theta}x + \mathrm{e}^{\mathrm{i}\phi}y)\overline{g_\phi(\mathrm{e}^{\mathrm{i}\theta}x + \mathrm{e}^{\mathrm{i}\phi}y)}\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

Proof. Given polynomials P and Q on E we can write

$$P_\theta(\mathrm{e}^{\mathrm{i}\theta}x + \mathrm{e}^{\mathrm{i}\phi}y) \quad \text{and} \quad \overline{Q_\phi(\mathrm{e}^{\mathrm{i}\theta}x + \mathrm{e}^{\mathrm{i}\phi}y)}$$

as polynomials in $\mathrm{e}^{\mathrm{i}\theta}$, $\mathrm{e}^{\mathrm{i}\phi}$, $\mathrm{e}^{-\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\phi}$, expand and obtain the identity

$$P(x)\overline{Q(y)} = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} P_\theta(\mathrm{e}^{\mathrm{i}\theta}x + \mathrm{e}^{\mathrm{i}\phi}y)\overline{Q_\phi(\mathrm{e}^{\mathrm{i}\theta}x + \mathrm{e}^{\mathrm{i}\phi}y)}\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{4.1}$$

Let $(P)_n$ (respectively, $(Q)_n$) denote the Taylor polynomial for f (respectively, g) of degree n. Since $(P_\theta)_n\overline{(Q_\phi)_n}$ converges uniformly to $f\bar{g}$ on the compact set

$$\{\mathrm{e}^{\mathrm{i}\theta}x + \mathrm{e}^{\mathrm{i}\phi}y\}_{\theta,\phi\in[0,2\pi]},$$

the result follows from (4.1). □

We shall use this identity to give another estimate for the norm of the product of general polynomials. Let us recall that δ is the Mahler measure of $F(x,y) = 1 + x + y - xy$ and is equal to $1.7916\ldots$.

The support of a function f in $L^p(\mu)$ is defined as $\overline{\{x : f(x) \neq 0\}}$.

Theorem 4.2. *Consider $1 \leqslant p \leqslant \infty$ and let P and Q be polynomials on $L_p(\mu)$ of degree k and l, respectively, which achieve their norms at points with disjoint support. Let $n = k + l$. Then*

$$\|P\|\,\|Q\| \leqslant \left(\frac{n^n}{k^k l^l}\right)^{1/p} \delta^n \|PQ\|. \tag{4.2}$$

Proof. Choose norming points x and y for P and Q, respectively, which have disjoint support. By Proposition 4.1,

$$\begin{aligned}\left|P\left(\left(\frac{k}{n}\right)^{1/p} x\right)\right| \left|Q\left(\left(\frac{l}{n}\right)^{1/p} y\right)\right| &\leqslant \sup_{\theta\phi} \|P_\theta Q_\phi\| \left\|\left(\frac{k}{n}\right)^{1/p} \mathrm{e}^{\mathrm{i}\theta} x + \left(\frac{l}{n}\right)^{1/p} \mathrm{e}^{\mathrm{i}\phi} y\right\|^n \\ &\leqslant \sup_{\theta\phi} \|P_\theta Q_\phi\|.\end{aligned}$$

Fix an arbitrary z in the closed unit ball of E and consider the polynomial, $\tilde{P}$, of degree k on $\boldsymbol{C}$ defined by

$$\tilde{P}(\lambda) = \lambda^k P\left(\frac{1}{\lambda} z\right).$$

It follows from the maximum modulus theorem that $\tilde{P}(1) \leqslant \|\tilde{P}\|_{(k/n)^{1/p} B_{\boldsymbol{C}}}$. This gives us that

$$P(z) \leqslant \left(\frac{n}{k}\right)^{k/p} \sup_{|\lambda|=1} \left|P\left(\left(\frac{k}{n}\right)^{1/p} \lambda z\right)\right| \leqslant \left(\frac{n}{k}\right)^{k/p} \|P\|_{(n/k)^{1/p} B_E}.$$

Taking the supremum over all z in the unit ball of E we get that

$$\|P\| \leqslant \left(\frac{n}{k}\right)^{k/p} \|P\|_{(k/n)^{1/p} B_E}.$$

Similarly, we obtain

$$\|Q\| \leqslant \left(\frac{n}{l}\right)^{l/p} \|Q\|_{(l/n)^{1/p} B_E}.$$

Thus we have

$$\|P\|\,\|Q\| \leqslant \left(\frac{n^n}{k^k l^l}\right)^{1/p} \sup_{\theta\phi} \|P_\theta Q_\phi\|.$$

Fix z in E and for $f \in \mathcal{H}(E)$ let $f^z \in \mathcal{H}(\boldsymbol{C})$ be defined by $f^z(\lambda) = f(\lambda z)$. Then, by Theorem 2 of [**8**], we have

$$\sup_{\theta\phi} \|(P_\theta Q_\phi)^z\| = \sup_{\theta\phi} \|P_\theta^z Q_\phi^z\| = \|P^z\|\,\|Q^z\| \leqslant \delta^n \|P^z Q^z\| = \delta^n \|(PQ)^z\|.$$

Taking the supremum over all z with $\|z\| \leqslant 1$ we obtain

$$\sup_{\theta\phi} \|P_\theta Q_\phi\| \leqslant \delta^n \|PQ\|,$$

which gives (4.2). □

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If P and Q are two norm-attaining polynomials of degree n on $L_p(\mu)$ that attain their norms at points with disjoint support, then Theorem 4.2 tells us that

$$\|P\|\,\|Q\| \leqslant 2^{2n/p}\delta^{2n}\|PQ\|.$$

For

$$p > \frac{1}{1-\log_2\delta} = 6.24\ldots$$

this is better than the estimate in [**7**] of 2^{2n}.

Theorem 3 of [**7**] uses the isometry between the space of all polynomials of degree at most n on a Banach space E and the space of n-homogeneous polynomials on the Banach space $E\bigoplus_\infty \boldsymbol{C}$ to obtain the best constant for general polynomials. This isometry can be used in conjunction with Theorem 2 of [**8**] to give the following result.

Proposition 4.3. *Given* $P \in \mathcal{P}({}^k\ell_\infty^2)$ *and* $Q \in \mathcal{P}({}^l\ell_\infty^2)$, *let* $n = k + l$. *Then*

$$\|P\|\,\|Q\| \leqslant \delta^n\|PQ\|$$

and this estimate is asymptotically sharp as $n \to \infty$.

For any infinite set I, $\ell_\infty(I)$ is isometrically isomorphic to $\ell_\infty(I)\bigoplus_\infty \boldsymbol{C}$. Therefore, we see that if P and Q are polynomials of degree k and l, respectively, on $\ell_\infty(I)$, then

$$\|P\|\,\|Q\| \leqslant F_{\ell_\infty(I)}(k,l)\|PQ\|.$$

Thus, the best constant for the product of homogeneous polynomials is also the best constant for general polynomials.

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