# A remark about canonical forms

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A comparison of the rational and classical canonical forms of a square matrix reveals that for a nilpotent matrix the two are identical. In this note I describe how we may utilise this fact in solving the problem of reducing a given matrix to classical canonical form. I believe that the point which I try to make in what follows is one which is not always explicitly remarked upon in the literature, and it has therefore seemed to me to be worth while to stress it here.

Suppose $A$ is an $n \times n$ matrix with $p$ distinct characteristic roots $a_1, \ldots, a_p$, and with minimum function $f(\lambda) = (\lambda - a_1)^{r_1} \ldots (\lambda - a_p)^{r_p}$. We interpret column matrices as vectors of the $n$-dimensional vector space $V_n$, and $A$ as the matrix of a linear transformation in $V_n$. Denote by $E_1, \ldots, E_p$ the null-manifolds of $(a_1 I - A)^{r_1}, \ldots, (a_p I - A)^{r_p}$ respectively. It may be shown that $E_1, \ldots, E_p$ span $V_n$ and that $E_i$ meets the join of $E_1, \ldots, E_{i-1}, E_{i+1}, \ldots, E_p$ only in the zero vector. Write dimension $E_i = d_i$. If $P = (P_1 \ldots P_p)$, where $P_i$ is an $n \times d_i$ matrix of rank $d_i$ whose columns are linearly independent vectors spanning $E_i$, then

$$P^{-1}AP = \begin{pmatrix} \Gamma_1 & 0 & \ldots & 0 \\ 0 & \Gamma_2 & \ldots & 0 \\ . & . & \ldots & . \\ 0 & 0 & \ldots & \Gamma_p \end{pmatrix}, \tag{1}$$

where $\Gamma_i$ is a $d_i \times d_i$ matrix with the single characteristic root $a_i$. When non-singular matrices $Q_i$ (say) have been found reducing $\Gamma_i$ to classical canonical form then the problem of reducing $A$ to classical canonical form is solved.

In practice, when $A$ is of small order and its minimum function is known in factorised form, it is not difficult to find a matrix $P$ reducing $A$ to the form (1) above. The method of reduction of a matrix to rational canonical form (by the construction of a complete principal sequence) is well known and is practicable when the matrix is small.[1] If this method is applied to the nilpotent matrix $a_i I - \Gamma_i$, it evidently leads to the construction of a matrix $Q_i$ reducing $a_i I - \Gamma_i$ to classical canonical form and therefore reducing $\Gamma_i$ to classical canonical form.

## REFERENCE.

[1] Todd, J. A., *Projective and Analytical Geometry* (London, 1947), pp. 163-4.

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