

# Hoare type theory, polymorphism and separation<sup>1</sup>

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## Abstract

We consider the problem of reconciling a dependently typed functional language with imperative features such as mutable higher-order state, pointer aliasing, and nontermination. We propose Hoare type theory (HTT), which incorporates Hoare-style specifications into types, making it possible to statically track and enforce correct use of side effects.

The main feature of HTT is the Hoare type  $\{P\}x:A\{Q\}$  specifying computations with precondition  $P$  and postcondition  $Q$  that return a result of type  $A$ . Hoare types can be nested, combined with other types, and abstracted, leading to a smooth integration with higher-order functions and type polymorphism.

We further show that in the presence of type polymorphism, it becomes possible to interpret the Hoare types in the “small footprint” manner, as advocated by separation logic, whereby specifications tightly describe the state required by the computation.

We establish that HTT is sound and compositional, in the sense that separate verifications of individual program components suffice to ensure the correctness of the composite program.

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## 1 Introduction

The static type systems of today’s programming languages, such as Java, ML, and Haskell, provide a degree of lightweight specification and verification that has proven remarkably effective at eliminating a class of coding errors. Equally important, these type systems have scaled to cover and integrate with necessary linguistic features such as higher-order functions, objects, and imperative references and arrays.

Nevertheless, there is a range of errors, such as array-index-out-of-bounds and division-by-zero, which are not caught by today’s type systems. And of course, there are higher-level correctness issues, such as invariants or protocols on mutable data structures, that fall well outside the range where types are effective.

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An alternative approach to address these issues is to utilize a form of *dependent types* to provide precise specifications of these requirements. For example, we can specify that a vector has a certain length, by using the type  $\text{vector}(n)$  that depends on the length  $n$ . Inner product can then be given the signature

$$\text{inner\_prod} : \prod n:\text{nat}. \text{vector}(n) \times \text{vector}(n) \rightarrow \text{nat}$$

signifying that the argument vectors must be of equal length.

Dependent types work well with higher-order features and are convenient for capturing relations on functional data structures, but do not work so well in the presence of side effects, such as state updates and nontermination. An alternative approach is to consider some form of *program logic*, such as Hoare's original logic (Hoare 1969) or the more recent forms of separation logic (O'Hearn et al. 2001; Reynolds 2002; O'Hearn et al. 2004), which are tuned for specifying and reasoning about imperative programs. However, these logics do not integrate *into* the type system. Rather, specifications, such as invariants on data structures or refinements on types, must be separately specified as preconditions and postconditions on expressions that manipulate these data. In turn, this makes it difficult to scale the logics to support the abstraction mechanisms such as higher-order functions, polymorphism, and modules.

In this paper, we propose a new approach that smoothly combines dependent types and a Hoare-style logic for a language with higher-order functions and imperative commands (i.e., core, polymorphic ML). The key mechanism is a distinguished type constructor of Hoare (partial) triples  $\{P\}_x:A\{Q\}$ , which serves to simultaneously isolate and describe the effects of imperative commands. Intuitively, such a type can be ascribed to a stateful computation if when executed in a heap satisfying the precondition  $P$ , the computation diverges or results in a heap satisfying the postcondition  $Q$  and returns a value of type  $A$ . Hoare types can be viewed as a refinement of the concept of *monad* (Moggi 1989; Moggi 1991; Peyton Jones & Wadler 1993; Wadler 1998), which is extensively used in simply typed functional programming to statically track the computational effects. In the dependent setting, monadic isolation is crucial for ensuring soundness, as it prevents the effects from polluting the logical properties of the underlying pure language. At the same time, it makes it possible for encapsulated effectful terms to freely appear in type dependencies and specifications. Furthermore, Hoare types can be nested, combined with other types, and abstracted within terms, types, and predicates alike, thus improving upon the data abstraction and information hiding mechanisms of the original Hoare Logic, and leading to a unified system for programming, specification, and reasoning about programs.

As with any sufficiently rich specification system, checking that Hoare type theory (HTT) programs respect their types is generally undecidable. However, type-checking in HTT is carefully designed to split into two independent phases. The first phase performs a combination of basic type-checking and verification-condition generation, and is completely automatic. The second phase must then show the validity of the generated verification-conditions. The conditions can be fed to an automated theorem prover, discharged by hand, or even ignored, making it possible to use HTT

in various ways, ranging from a simple bug-finding tool that can discover some, but not necessarily all errors, to a full-scale verification and certification framework.

We believe that the HTT approach enjoys many of the benefits and few of the drawbacks of the alternatives mentioned above. In particular, we believe HTT is the right foundational framework for modeling emerging tools, such as ESC (Detlefs *et al.* 1998; Leino *et al.* 2000), SPLint (Evans & Larochelle 2002), Spec# (Barnett *et al.* 2004), Cyclone (Jim *et al.* 2002), and JML (Leavens *et al.* 1999; Burdy *et al.* 2005), that provide support for extended static checking of programs.

The semantics of program heaps in HTT combines the treatment of functional arrays of McCarthy (1962) and Cartwright and Oppen (1978), with ideas from separation logic.

In the functional-array approach, preconditions and postconditions in Hoare triples usually describe the whole heap, rather than just the heap fragment that any particular program requires. Keeping track of whole heaps may be cumbersome as it requires careful tracking of location inequalities (i.e., lack of aliasing). It is much better to simply assert those properties of the beginning and the ending heap that are actually influenced by the computation, and automatically assume that all the unspecified heap portions remain invariant. This is known as the “small footprint” approach to specification, and has been advocated recently by the work on separation logic.

We note that in the presence of type polymorphism, the connectives from separation logic can already be defined using functional arrays, so that the two approaches may be fruitfully combined. We thus endow the Hoare types with the small footprint interpretation, and show that the customary inference rules from separation logic (e.g., the Frame rule) are admissible in HTT.

Moreover, an important example that becomes possible in HTT, but is formally not admitted in separation logic, is naming and explicitly manipulating individual fragments of the heap. We contend that it is useful to be able to do so directly. In particular, it alleviates the need for an additional representation of heaps in assertions as was used in the verification of Cheney’s garbage collection algorithm in separation logic by Birkedal *et al.* (2004). An additional feature admitted by polymorphism is that HTT can support strong updates, whereby a location can point to values of different types in the course of the execution.

We prove that HTT is sound and compositional, in the sense that separate verifications of individual program components suffice to ensure the correctness of the composite program. In other words, the types of HTT are strong enough to serve as adequate interfaces for program components, so that verification does *not* require whole-program reasoning. As customary in type theory, compositionality is expressed via substitution principles. We contrast this to the statement of compositionality for ESC/Modula-3 (Leino & Nelson 2002), which requires a significantly more involved statement and proof.

We start the paper with the overview of the syntactic features of HTT (Section 2), and then discuss the type system (Section 3), and the equational reasoning (Section 4). We establish the basic syntactic properties, like substitution principles (Section 5), then define a call-by-value operational semantics (Section 6), and prove the soundness

of HTT with respect to this operational semantics using denotational methods (Section 7). We close with the discussion of related and future work.

## 2 Syntax and overview

A crucial operation in any type system is comparing types for equality. In the case of dependent types, which we use in HTT to express Hoare-style partial correctness, types can contain terms, so type equality must compare terms as well, which is an undecidable problem in any Turing complete language (in fact, it is not even recursively enumerable). It is therefore crucial for HTT that we select equations on terms that strike the balance between the preciseness and decidability of the equality relation. In this choice, we are guided by the decision to separate type-checking and verification-condition generation, from proving of program specifications. We introduce two different notions: *definitional equality*, which is coarse but decidable, and is employed during type-checking, and *propositional equality*, which is fine but undecidable and is used only in proving. The split into definitional and propositional equalities is a customary way to organize equational reasoning in type theories (Hofmann 1995).

Most of the HTT design is geared toward facilitating a formulation of a decidable definitional equality. Propositional equality can be arbitrarily complex, so it does not require much attention. For example, we split the HTT programs into two fragments: pure and impure—precisely in order to separate the concerns about equality. The pure fragment consists of higher-order functions, and constructs for type polymorphism. It admits the usual term equations of beta reduction and eta expansion. The impure fragment contains the constructs usually found in first-order imperative languages: allocation, lookup, strong update, deallocation of memory, conditionals, and loops (in HTT formulated as recursion). All of these constructs admit reasoning in the style of Hoare Logic by preconditions and postconditions, so we use the Hoare type  $\{P\}x:A\{Q\}$  to classify the impure programs. In this paper, conditionals are considered impure even though they are a customary component of most pure functional languages. This design decision is justified by the well-known complications in the equational reasoning about conditionals (Ghani 1995; Altenkirch et al. 2001), which are related to the laws for eta expansion. We plan to address these issues in the future work, but for now, simply avoid them by placing the conditionals into the impure fragment.

The split between pure and impure fragments is a familiar one in functional programming. For example, the programming language Haskell (Peyton Jones 2003) uses monads to classify impure code. It should therefore not come as a surprise that the Hoare type in HTT is a monad, and that we admit the usual generic monadic laws (Moggi 1991) for reasoning about the impure code.

It may be interesting that HTT monads take a slightly bigger role than to simply serve as type markers for effects. The HTT monadic judgments actually formalize the process of generating the verification conditions for an effectful computation by calculating strongest postconditions. If the verification condition is provable, then the computation matches its specification. The verification conditions are

computed during the first phase of type-checking, in a process that is mutually recursive with normalization and equational reasoning, as formalized in Section 3. However, the conditions can be proved in the second phase, so that the complexity and undecidability of proving does not have any bearing on normalization and equational reasoning.

We next present the syntax of HTT and comment on the various constructors.

<i>Types</i>	$A, B, C ::= \alpha \mid \text{bool} \mid \text{nat} \mid 1 \mid \forall\alpha.A \mid \Pi x:A.B \mid \Psi.X.\{P\}_{x:A}\{Q\}$
<i>Monotypes</i>	$\tau, \sigma ::= \alpha \mid \text{bool} \mid \text{nat} \mid 1 \mid \Pi x:\tau. \sigma \mid \Psi.X.\{P\}_{x:\tau}\{Q\}$
<i>Assertions</i>	$P, Q, R ::= \text{id}_A(M, N) \mid \text{seleq}_\tau(H, M, N) \mid \top \mid \perp \mid P \wedge Q \mid$ $P \vee Q \mid P \supset Q \mid \neg P \mid \forall x:A.P \mid \forall\alpha.P \mid \forall h:\text{heap}.P \mid$ $\exists x:A.P \mid \exists\alpha.P \mid \exists h:\text{heap}.P$
<i>Heaps</i>	$H, G ::= h \mid \text{empty} \mid \text{upd}_\tau(H, M, N)$
<i>Elim terms</i>	$K, L ::= x \mid K M \mid K \tau \mid M : A$
<i>Intro terms</i>	$M, N, O ::= K \mid \text{eta}_\alpha K \mid () \mid \lambda x. M \mid \Lambda\alpha. M \mid \text{do } E \mid \text{true} \mid \text{false} \mid$ $z \mid \text{succ } M \mid M + N \mid M \times N \mid M == N$
<i>Commands</i>	$c ::= \text{alloc}_\tau M \mid !_\tau M \mid M :=_\tau N \mid \text{dealloc } M \mid$ $\text{if}_A M \text{ then } E_1 \text{ else } E_2 \mid \text{case}_A M \text{ of } z. E_1 \text{ or succ } x. E_2 \mid$ $\text{fix } f(y:A):B = \text{do } E \text{ in eval } f M$
<i>Computations</i>	$E, F ::= \text{return } M \mid x \leftarrow K ; E \mid x \Leftarrow c ; E \mid x =_A M ; E$
<i>Variable context</i>	$\Delta, \Psi ::= \cdot \mid \Delta, x:A \mid \Delta, \alpha$
<i>Heap context</i>	$X ::= \cdot \mid X, h$
<i>Assertion context</i>	$\Gamma ::= \cdot \mid \Gamma, P$

*Terms.* Terms form the purely functional fragment of HTT. They are further split into introduction (intro) terms and elimination (elim) terms, according to their standard logical classification. For example,  $\lambda x. M$  is an intro term for the dependent function type, and  $K M$  is the corresponding elim term. Similarly,  $\Lambda\alpha. M$  and  $K \tau$  are the intro and elim terms for polymorphic quantification. The intro term for the unit type is  $()$ , and, as customary, there is no corresponding elimination term. The intro term for Hoare types is  $\text{do } E$ , which encapsulates the effectful computation  $E$  and suspends its evaluation. The name of the constructor is chosen to closely resemble the familiar Haskell-style  $\text{do}$ -notation for writing effectful programs. The corresponding elimination form  $x \leftarrow K ; E$  activates a suspended computation determined by  $K$ . However, this elim form is not a term, but a computation, and is described later in more detail. We also list the intro term  $\text{eta}_\alpha K$ , which is not supposed to be used in programming, but is needed internally during type-checking. The reader can safely ignore this constructor for the time being; its significance will become apparent in Section 3.

The separation into intro and elim terms facilitates bidirectional type-checking (Pierce & Turner 2000), whereby most of the type information can be omitted from the terms, and inferred automatically. When type information must be supplied explicitly, the constructor  $M:A$  can be used. In the typing rules in Section 3,  $M:A$  will indicate direction switch during bidirectional type-checking. More importantly for our purposes, this kind of formulation also facilitates equational reasoning via

hereditary substitutions (Section 4), as it admits a simple syntactic criterion for normality with respect to beta reduction. For example, the reader may notice that an HTT term which does not use the constructor  $M:A$  may not contain beta redexes.

*Computations.* Computations form the effectful fragment of HTT, and can be described intuitively as semicolon-separated lists of commands, terminated with a return value. The commands are executed in the order in which they appear in the list, and usually bind their result to a variable. For example, the computation  $x \leftarrow c$  executes the command  $c$  and binds the result to  $x$ . As already described before,  $x \leftarrow K$  executes the computation encapsulated in  $K$ , thus performing all the side effects that may have been suspended in  $K$ . The command  $x =_A M$  does not perform any side effects, but is simply a syntactic sugar for the usual let-binding of  $M:A$  to  $x$ .

It is important that variables in HTT are *immutable*, as customary in functional programming languages, but in contrast to the usual treatment of variables in Hoare Logic. The scope of the variables bound in computations ( $x$  in the above examples) extends to the right, until the end of the block enclosed by the nearest *do*.

The return value of a computation is specified by the construct *return*  $M$ . When considered by itself, *return*  $M$  creates a trivial computation that does not perform any side effects, but immediately returns the value  $M$ . The computations *return*  $M$  and  $x \leftarrow K; E$  correspond to the monadic operations of *unit* and *bind*, respectively.

We illustrate the commands of HTT with the following example. Consider the simple ML-like function  $f:\text{nat} \rightarrow \text{unit}$ , defined as  $\lambda x:\text{nat}. \text{if } !y = 0 \text{ then } y := x \text{ else } ()$ , where  $y:\text{ref nat}$ . The HTT code that implements function  $f$  and then immediately applies it to one can be written as the following effectful computation:

```

f =nat→unit λx. do (t1 ← !nat y;
                   t2 ← ifunit (t1 == z) then
                       t3 ← (y :=nat x);
                       return t3
                     else return();
                   return t2);
t4 ← f (succ z);
return t4

```

The syntax of computations is very explicit and rather cumbersome, as every intermediate result must be named with a temporary variable ( $t_1$ – $t_4$  above). Shortly, we will introduce a number of syntactic conventions to reduce the clutter. But first, we describe the semantics of the HTT commands.

The commands  $!_\tau M$  and  $M :=_\tau N$  perform a read from and write into the memory location  $M$ , respectively. The index  $\tau$  is the type of the value being read or written. Unlike most statically typed functional languages, HTT supports *strong updates*. That is, if  $x$  is a location storing a *nat*, then it can be updated with a value of any type, not just another *nat*. Correspondingly, memory reads from one and the

same location may return differently typed values. To ensure safety, the type system must ensure statically, as a precondition, that the contents of the location matches the expected index type  $\tau$ . On the other hand, the precondition for an update simply requires that the updated location exists (i.e., is allocated), but does not say anything about the type of the old value. The command  $\text{alloc}_\tau M$  allocates a fresh memory location, initializes it with  $M:\tau$ , and returns the address of the location. Dually,  $\text{dealloc } M$  frees the location at address  $M$ , under the precondition that the location exists. In the conditionals *if* and *case*, the index  $A$  is the type of the branches. The fixpoint command  $\text{fix } f(y:A):B = \text{do } E$  in  $\text{eval } f M$  first obtains the function  $f$  such that  $f(y) = \text{do}(E)$ , and then evaluates the encapsulated computation  $f M$ . Here,  $f$  and  $y$  may appear in the body  $E$ . The types  $A$  and  $B$  are the argument and result type of  $f$ , respectively, and  $B$  may depend on the variable  $y$ . The type  $B$  must be a Hoare type, in order to capture that the recursive definition of  $f$  may lead to divergence.

Returning to our example program, we now introduce the following abbreviations. First, we represent natural numbers in their decimal rather than unary form. Second, we omit the variable  $x$  in  $x \leftarrow (M :=_\tau N)$  and  $x \leftarrow \text{dealloc } M$ , as  $x$  is of unit type. Third, computations of the form  $x \leftarrow c; \text{return}(x)$  are abbreviated simply as  $c$ , to avoid the spurious variable  $x$ . Similarly,  $x \leftarrow K; \text{return}(x)$  is abbreviated as  $\text{eval } K$ . Fourth, we omit the index type in operations, when that type is clear from the context. Under such conventions, the above program may be abbreviated as

$$f = \lambda x. \text{do } (t_1 \leftarrow !y;$$

$$\quad \text{if } (t_1 == 0) \text{ then } y := x \text{ else return}());$$

$$\text{eval}(f 1)$$

*Types.* Types include the primitive types of booleans and natural numbers, unit type 1, dependent functions  $\Pi x:A.B$ , and polymorphic types  $\forall \alpha.A$ . We write  $A \rightarrow B$  to abbreviate  $\Pi x:A. B$  when  $B$  does not depend on  $x$ .

The Hoare type  $\Psi.X.\{P\}x:A\{Q\}$  specifies an effectful computation with a precondition  $P$  and a postcondition  $Q$ , returning a result of type  $A$ . The variable  $x$  names the return value of the computation, and  $Q$  may depend on  $x$ . The contexts  $\Psi$  and  $X$  list the variables and heap variables, respectively, that may appear in both  $P$  and  $Q$ , thus helping relate the properties of the beginning and the ending heap. In the literature on Hoare Logic, these are known under the name of *ghost variables* or *logic variables*, and can appear only in the assertions, but not in the programs. Similarly, in our setting, the type  $A$  cannot contain any variable from  $\Psi$  and  $X$ .

The type  $\forall \alpha.A$  polymorphically quantifies over the *monotype* variable  $\alpha$ . For our purposes, it suffices to define a monotype as any type that does not contain polymorphic quantification, except in the assertions. For example,  $\Psi.X.\{P\}x:\tau\{Q\}$  is a monotype when  $\tau$  is a monotype, even if  $\Psi$ ,  $P$  and  $Q$  contain polymorphic types. This kind of predicative polymorphism (quantification over monotypes) is sufficient for modeling languages such as Standard ML, but not more recent languages such as Haskell. However, extending HTT to support impredicative polymorphism seems



difficult as it significantly complicates the termination argument for normalization (Section 4), which is a crucial component of type equality. Therefore, we leave the treatment of impredicative polymorphism to future work.

Note that allowing polymorphic quantification in the assertions of the Hoare types, or in the types of ghost variables, does not change the predicative nature of HTT. The type system will be formulated so that ghost variables and the assertions do not influence the computational behavior or equational properties of effectful computations.

*Heaps and locations.* In this paper, we model memory locations as natural numbers. One advantage of this approach is that it supports some forms of pointer arithmetic which is needed for languages such as Cyclone. We model heaps as finite functions, mapping a location  $N$  to a pair  $(\tau, M)$  where  $\tau$  is the monotype of  $M$ . In this case we say that  $N$  points to  $M$ , or that  $M$  is the contents of location  $N$ .

We introduce the following syntax for heaps:  $\text{empty}$  denotes the empty heap, and  $\text{upd}_\tau(H, M, N)$  is the heap obtained from  $H$  by updating the location  $M$  so that it points to  $N$  of type  $\tau$ , while retaining all the other assignments of  $H$ .

Heap terms and variables play a prominent role in our encoding of assertions about (propositional) equality and disjointness of heaps. If heaps could hold values of polymorphic type, then encoding these properties would require impredicative quantification. Consequently, we limit heaps to hold only values of monotype. Any monotype is allowed, including higher-order function types, and computation types, so that HTT heaps implement higher-order store.

*Assertions.* Assertions comprise the usual connectives of classical multi-sorted first-order logic. The sorts include all the types of HTT, but also the domain of heaps. We allow polymorphic quantification  $\forall\alpha.P$  and  $\exists\alpha.P$  over monotypes.  $\text{id}_A(M, N)$  denotes propositional equality between  $M$  and  $N$  at type  $A$ , and  $\text{seleq}_\tau(H, M, N)$  states that the heap  $H$  at address  $M$  contains a term  $N:\tau$ .

We now introduce some assertions that will frequently feature in Hoare types.

$$\begin{aligned}
 P \Leftrightarrow Q &= P \supset Q \wedge Q \supset P \\
 \text{hid}(H_1, H_2) &= \forall\alpha.\forall x:\text{nat}.\forall v:\alpha. \text{seleq}_\alpha(H_1, x, v) \Leftrightarrow \text{seleq}_\alpha(H_2, x, v) \\
 M \in H &= \exists\alpha.\exists v:\alpha. \text{seleq}_\alpha(H, M, v) \\
 M \notin H &= \neg(M \in H) \\
 \text{share}(H_1, H_2, M) &= \forall\alpha.\forall v:\alpha. \text{seleq}_\alpha(H_1, M, v) \Leftrightarrow \text{seleq}_\alpha(H_2, M, v) \\
 \text{splits}(H, H_1, H_2) &= \forall x:\text{nat}. (x \notin H_1 \wedge \text{share}(H, H_2, x)) \vee \\
 &\quad (x \notin H_2 \wedge \text{share}(H, H_1, x))
 \end{aligned}$$

In English,  $\Leftrightarrow$  is the logical equivalence (and  $\supset$  is the logical implication),  $\text{hid}$  is the heap equality,  $M \in H$  iff the heap  $H$  assigns to the location  $M$ ,  $\text{share}$  states that  $H_1$  and  $H_2$  agree on the location  $M$ , and  $\text{splits}$  states that  $H$  can be split into disjoint heaps  $H_1$  and  $H_2$ .

We next define the assertions familiar from Separation Logic (O'Hearn et al. 2001; Reynolds 2002; O'Hearn et al. 2004). All of these are relative to the free variable



mem, which denotes the current heap fragment of reference.

$$\begin{aligned}
\text{emp} &= \text{hid}(\text{mem}, \text{empty}) \\
M \mapsto_{\tau} N &= \text{hid}(\text{mem}, \text{upd}_{\tau}(\text{empty}, M, N)) \\
M \mapsto_{\tau} - &= \exists v:\tau. M \mapsto_{\tau} v \\
M \mapsto - &= \exists \alpha. M \mapsto_{\alpha} - \\
M \hookrightarrow_{\tau} N &= \text{seleq}_{\tau}(\text{mem}, M, N) \\
M \hookrightarrow_{\tau} - &= \exists v:\tau. M \hookrightarrow_{\tau} v \\
M \hookrightarrow - &= \exists \alpha. M \hookrightarrow_{\alpha} - \\
P * Q &= \exists h_1:\text{heap}. \exists h_2:\text{heap}. \text{splits}(\text{mem}, h_1, h_2) \wedge \\
&\quad [h_1/\text{mem}]P \wedge [h_2/\text{mem}]Q \\
P \multimap Q &= \forall h_1:\text{heap}. \forall h_2:\text{heap}. \text{splits}(h_2, h_1, \text{mem}) \supset \\
&\quad [h_1/\text{mem}]P \supset [h_2/\text{mem}]Q \\
\text{this}(H) &= \text{hid}(\text{mem}, H)
\end{aligned}$$

Here,  $\text{emp}$  states that the current heap  $\text{mem}$  is empty;  $M \mapsto_{\tau} N$  iff  $\text{mem}$  consists of a *single* location  $M$  which points to the term  $N:\tau$ ;  $M \hookrightarrow_{\tau} N$  iff  $\text{mem}$  contains *at least* the location  $M$  pointing to  $N:\tau$ . *Separating conjunction*  $P * Q$  holds iff  $P$  and  $Q$  hold of disjoint subheaps of  $\text{mem}$ .  $P \multimap Q$  holds of  $\text{mem}$  if any extension by a heap satisfying  $P$  produces a heap satisfying  $Q$ .  $\text{this}(H)$  is true iff  $\text{mem}$  equals  $H$ .

We frequently write  $\forall\Psi.A$  and  $\exists\Psi.A$  for an iterated universal (resp. existential) abstraction over the term and type variables of the context  $\Psi$ . Similarly,  $\forall X.A$  and  $\exists X.A$  stands for iterated quantification over heap variables of the context  $X$ . We also abbreviate  $\diamond A$  instead of  $\{\top\}x:A\{\top\}$ .

## 2.1 Examples

*Small and large footprints.* In this example, we illustrate the “small footprint” semantics of Hoare types by considering several different specifications that can be ascribed to the HTT allocation primitive. The shortest one is the following:

$$\text{alloc} : \forall \alpha. \Pi x:\alpha. \{\text{emp}\}y:\text{nat}\{y \mapsto_{\alpha} x\}$$

It states that allocation does not touch any existing heap locations (precondition  $\text{emp}$ ), and returns an address  $y$  which is initialized with the supplied value  $x:\alpha$  (postcondition  $y \mapsto_{\alpha} x$ ). Implicit in the specification is that  $y$  must be fresh, because the precondition prohibits  $\text{alloc}$  from working with existing locations.

We point out that the definitions of the predicates  $\text{emp}$  and  $\mapsto$  require a free variable  $\text{mem}$  to denote the *current heap*. In other words, the precondition and the postcondition of Hoare types are parametrized with respect to a heap variable  $\text{mem}$ . In the precondition,  $\text{mem}$  denotes the beginning heap of the computation, and in the postcondition,  $\text{mem}$  is the ending heap. The variable  $\text{mem}$  is bound in both the preconditions and the postconditions of the Hoare type. That is, if we make the scope of variables in Hoare types explicit, the syntax of Hoare types would be

$$\{\text{mem}. P\}x:A\{\text{mem}. Q\}$$

However, to reduce clutter, we omit the bindings of  $\text{mem}$ , as they can be assumed implicitly, and simply write  $\{P\}x:A\{Q\}$ .

Returning now to the function `alloc`, the precondition `emp` in its specification does not mean that `alloc` can only work if the global heap is empty. Rather, the small footprint specifications require that the global heap contains a subheap satisfying the precondition. The subheap is then changed by the execution of the effectful computation, so that it satisfies the postcondition. The remaining, unspecified, part of the global heap is guaranteed to remain invariant. In the particular case of `alloc`, the precondition `emp` means that `alloc` can be executed in *any* heap, since any heap trivially contains an empty subheap.

For the sake of illustration, let us consider a slightly less permissive specification for `alloc`, one which allows execution only in heaps with *at least* one boolean location.

$$\text{alloc}' : \forall \alpha. \Pi x:\alpha. z:\text{nat}, v:\text{bool}. \{z \mapsto_{\text{bool}} v\} y:\text{nat} \{y \mapsto_{\alpha} x * z \mapsto_{\text{bool}} v\}$$

The ghost variables  $z$  and  $v$  denote the assumed existing location and its contents, respectively, and the specification insists that the contents of  $z$  is not changed by the execution. Notice the use of separating conjunction in the postcondition to specify that  $z$  and  $y$  belong to disjoint heap portions (i.e., are not aliased), and hence  $y$  is fresh.

Instead of listing the pre-existing locations in the precondition, and then repeating them in the postcondition, as in `alloc'` above, we can simply name the heap encountered before allocation by using a ghost heap variable (say  $h$ ), and then explicitly specify in the postcondition that  $h$  is not changed.

$$\text{alloc}'' : \forall \alpha. \Pi x:\alpha. h:\text{heap}. \{\text{this}(h)\} y:\text{nat} \{(y \mapsto_{\alpha} x) * \text{this}(h)\}$$

Thus, heap variables allow us to express some of the invariance that one may express in higher-order separation logic (Biering *et al.*, 2005).

For comparison, let us now consider the specification of `alloc` in the classical style (Cartwright & Oppen 1978), which, by contrast, we can call *large footprint* specifications. Since `alloc` can run in any initial heap  $h$ , the precondition is trivial. The ending heap is obtained as  $\text{upd}_{\alpha}(h, y, x)$ , but we also know that  $y$  is fresh, that is  $y \notin h$ . Using again ghost variables to name the initial heap, we can write this specification as

$$\text{alloc}''' : \forall \alpha. \Pi x:\alpha. h:\text{heap}. \{\text{this}(h)\} y:\text{nat} \{\text{this}(\text{upd}_{\alpha}(h, y, x)) \wedge y \notin h\}$$

By explicitly naming various heap fragments with ghost variables, HTT can freely switch between the small and large footprint specifications. We believe this to be a very desirable property. For example, it is obvious that the types for `alloc`, `alloc''`, and `alloc'''` are all equivalent, but the small footprint specification as in `alloc` is the most convenient for programming, when we want to express nonaliasing and freshness of locations. On the other hand, large footprint specification as in `alloc'''` may be more parsimonious in cases where aliasing is allowed, as we will illustrate in subsequent examples. Finally, it will turn out that the intermediate form of specification as in `alloc''` is the easiest one to connect to the assertion logic of HTT, and we will adopt this intermediate form to define the semantics of Hoare types. The small footprint

specification can then be viewed as a simple syntactic variant, as we only need to introduce a fresh ghost variable to name the untouched part of the heap.

*Diverging computation.* In HTT, the term language is pure. Nontermination is an effect, and is relegated to the fragment of impure computations. Hence, any recursive program in HTT will have a monadic type, which will prevent recursion from unrolling during type-checking and normalization. We can write a diverging computation of an arbitrary Hoare type  $\{P\}x:A\{Q\}$  as follows:

$$\begin{aligned} \text{diverge} & : \{P\}x:A\{Q\} \\ & = \text{do } (\text{fix } f(y : 1) : \{P\}x:A\{Q\} = \text{do } (\text{eval } (f \ y))) \\ & \quad \text{in eval } f \ () \end{aligned}$$

`diverge` is a suspended computation which, when forced, first sets up a recursive function  $f(y : 1) = \text{do } (\text{eval } (f \ y))$ . The function is applied to `()` to obtain another suspended computation `do (eval f ())`, which is immediately forced by `eval`, to trigger another application to `()`, another suspended computation, another forcing, and so on, ad infinitum.

*Polymorphism and higher-order functions.* In this example we present a polymorphic function `swap` for swapping the contents of two locations. In a simply typed language like ML, with a type of references, `swap` can be given the type  $(\text{ref } \alpha \times \text{ref } \alpha) \rightarrow 1$ . This type is an underspecification, of course, as it does not describe what the function does. In HTT, we can be more precise. Furthermore, we can use strong updates to swap locations pointing to values of different types. One possible definition of `swap` is presented as follows:

$$\begin{aligned} \text{swap} & : \forall \alpha. \forall \beta. \Pi x:\text{nat}. \Pi y:\text{nat}. \\ & \quad m:\alpha, n:\beta. \{x \mapsto_{\alpha} m * y \mapsto_{\beta} n\} r : 1 \{x \mapsto_{\beta} n * y \mapsto_{\alpha} m\} \\ & = \Lambda \alpha. \Lambda \beta. \lambda x. \lambda y. \text{do } (t_1 \leftarrow !_{\alpha} x; t_2 \leftarrow !_{\beta} y; \\ & \quad y :=_{\alpha} t_1; x :=_{\beta} t_2; \\ & \quad \text{return}()) \end{aligned}$$

The function takes two monotypes  $\alpha$  and  $\beta$ , two locations  $x$  and  $y$  and produces a computation which reads both locations, and then writes them back in a reversed order. The precondition of this computation specifies a heap in which  $x$  and  $y$  point to values  $m:\alpha$  and  $n:\beta$ , respectively, for some ghost variables  $m$  and  $n$ . The locations must not be aliased, due to the use of spatial conjunction which forces  $x$  and  $y$  to appear in disjoint portions of the heap. Similar specifications that insists on nonaliasing are possible in several related systems, like Alias Types (Smith *et al.* 2000) and ATS with stateful views (Zhu & Xi 2005). However, in HTT, like in separation logic, we can include the aliasing case as well.

One possible small footprint specification which covers both aliasing and nonaliasing is as follows:

$$\begin{aligned} \text{swap}' & : \forall \alpha. \forall \beta. \Pi x:\text{nat}. \Pi y:\text{nat}. \\ & \quad m:\alpha, n:\beta. \{ (x \mapsto_{\alpha} m * y \mapsto_{\beta} n) \vee (x \mapsto_{\alpha} m \wedge y \mapsto_{\beta} n) \} r : 1 \\ & \quad \{ (x \mapsto_{\beta} n * y \mapsto_{\alpha} m) \vee (x \mapsto_{\beta} m \wedge y \mapsto_{\alpha} n) \} \end{aligned}$$

The second disjuncts in the precondition and postcondition of the above specification, use  $\wedge$  instead of spatial conjunction, and can be true only if the heap contains exactly one location, thus forcing  $x = y$ . This specification is interesting because it precisely describes the smallest heap needed for `swap` as the heap containing only  $x$  and  $y$ .

Another possibility is to switch to a large footprint style and admit an arbitrary heap in the assertions, but then explicitly state the invariance of the heap fragment not containing  $x$  and  $y$ .

$$\begin{aligned} \text{swap}'' & : \forall \alpha. \forall \beta. \Pi x:\text{nat}. \Pi y:\text{nat}. \\ & \quad m:\alpha, n:\beta, h:\text{heap}. \{(x \hookrightarrow_{\alpha} m) \wedge (y \hookrightarrow_{\beta} n) \wedge \text{this}(h)\} r:1 \\ & \quad \{\text{this}(\text{upd}_{\beta}(\text{upd}_{\alpha}(h, y, m), x, n))\} \end{aligned}$$

As mentioned before, the large footprint style may be less verbose in the case of specifications that admit aliasing. For example, expressing that swapping the same locations twice in a row does not change anything can be achieved as follows:

$$\begin{aligned} \text{identity} & : \forall \alpha. \forall \beta. \Pi x:\text{nat}. \Pi y:\text{nat}. \\ & \quad h:\text{heap}. \{x \hookrightarrow_{\alpha} - \wedge y \hookrightarrow_{\beta} - \wedge \text{this}(h)\} r:1 \{\text{this}(h)\} \\ & = \Lambda \alpha. \Lambda \beta. \lambda x. \lambda y. \text{do } (t_1 \leftarrow \text{swap}'' \alpha \beta x y; \\ & \quad t_2 \leftarrow \text{swap}'' \beta \alpha x y; \\ & \quad \text{return}()) \end{aligned}$$

This function uses `swap''` to generate a computation for swapping  $x$  and  $y$ , and then activates it twice using the monadic bind to the temporary variables  $t_1$  and  $t_2$ . A small footprint specification would have to explicitly repeat in the postcondition that the contents of  $x$  and  $y$  remains unchanged, whereas the large footprint specification can simply postulate the equality with the initial heap.

We further note that `identity` is completely parametric in `swap''`, in the sense that any other function with the same specification as `swap''` could have been used instead. Thus, it is possible to *abstract* the variable `swap''` from `identity`, and obtain a function whose type combines Hoare specifications in a higher-order way.

As a further example of scoping in Hoare types, as well as the use of higher-order abstraction over computations, let us consider the following specification that may be ascribed to a standard looping constructor `until`. Here `until` is parametrized by the loop invariant  $I$ , which is an assertion that depends on the current memory `mem` in the preconditions and postconditions of the Hoare types. `until` takes as argument the loop body which includes the computation of the loop guard as well.

$$\begin{aligned} \text{until}_I & : \{I(\text{mem})\} b:\text{bool}\{I(\text{mem})\} \rightarrow \{I(\text{mem})\} b:\text{bool}\{I(\text{mem}) \wedge b = \text{true}\} \\ & = \lambda e. \text{do}(\text{fix } f(r : 1) : \{I(\text{mem})\} b:\text{bool}\{I(\text{mem}) \wedge b = \text{true}\} = \\ & \quad \text{do}(b \leftarrow e; \text{if } b \text{ then true else eval } f()) \\ & \quad \text{in eval } f()) \end{aligned}$$

Because currently HTT lacks the ability to polymorphically abstract over predicates, like the loop invariant  $I$  given earlier, we cannot create a generic `until` constructor. Instead, a separate copy of `until` must be created for each specific loop invariant. A

similar requirement would also appear in the specification of the usual polymorphic functionals like `map` and `fold` over inductively defined types.

Thus, it is important to extend HTT with abstraction over predicates. Such extension would not only allow for generic programs like `until` given earlier, but would also increase the expressive power of the assertion logic to higher order. We forego a further discussion of this research direction here, but refer to Nanevski *et al.* (2007) for a description of some initial steps toward extending HTT to the full power of the Extended Calculus of Constructions, and how this power can be used to capture the local state of functions and modules in the specifications.

### 3 Type system

Type-checking judgments in HTT (as in any other type theory) require testing if two terms are definitionally equal. In HTT, the tests for definitional equality involve *normalization*. Two terms are reduced by normalization into their respective *canonical forms* and are deemed equal only if the canonical forms are syntactically the same, modulo  $\alpha$ -conversion. The formulation of HTT is somewhat unusual, however, as normalization is undertaken simultaneously with type-checking. That is, each typing judgment not only decides if some expression has a required type, but also computes as output the canonical form of the expression. That way, whenever two well-typed expressions must be checked for definitional equality, their canonical forms are readily available for syntactic comparison.

In the following section, we introduce the equations and operations that are used in the normalization algorithm of HTT; a more thorough discussion of this process will be given in Section 4. We then proceed with the definition of HTT typing rules.

#### 3.1 Equational reasoning

The most important equations in the definitional equality of HTT are the beta and eta equalities associated with each type constructor. We orient the equations to emphasize their use as rewriting rules in the normalization algorithm. As customary, beta equality is used as a reduction, and eta equality as an expansion.

For example, the function type  $\Pi x:A. B$  gives rise to the following familiar beta reduction and eta expansion:

$$\begin{array}{l} (\lambda x. M : \Pi x:A. B) N \longrightarrow_{\beta} [N:A/x]M \\ K \longrightarrow_{\eta} \lambda x. K x \quad \text{choosing } x \notin \text{FV}(K) \end{array}$$

Here, of course, the notation  $[T/x]M$  denotes a capture-avoiding substitution of the expression  $T$  for the variable  $x$  in the expression  $M$ . In the above equations, the participating terms are decorated with type annotations, because otherwise the results may not be well formed with respect to the HTT syntax from Section 2.

Equations associated with the type  $\forall \alpha. A$  are also standard.

$$\begin{array}{l} (\Lambda \alpha. M : \forall \alpha. A) \tau \longrightarrow_{\beta} [\tau/\alpha]M \\ K \longrightarrow_{\eta} \Lambda \alpha. K \alpha \quad \text{choosing } \alpha \notin \text{FTV}(K) \end{array}$$

where  $\text{FTV}(K)$  denotes the free monotype variables of  $K$ .

In the case of the unit type, we do not have any beta reduction (as there are no elimination constructors associated with this type), but only one eta expansion.

$$K \longrightarrow_{\eta} ()$$

The equations for the Hoare type require an auxiliary operation of *monadic substitution*,  $\langle E/x:A \rangle F$ , which sequentially composes the computations  $E$  and  $F$ . The result is a computation which should, intuitively, evaluate as  $E$  followed by  $F$ , where the free variable  $x:A$  in  $F$  is bound to the value of  $E$ . Monadic substitution is defined by induction on the structure of  $E$ , as follows:

$$\begin{aligned} \langle \text{return } M/x:A \rangle F &= [M:A/x]F \\ \langle y \leftarrow K; E/x:A \rangle F &= y \leftarrow K; \langle E/x:A \rangle F \quad \text{choosing } y \notin \text{FV}(F) \\ \langle y \leftarrow c; E/x:A \rangle F &= y \leftarrow c; \langle E/x:A \rangle F \quad \text{choosing } y \notin \text{FV}(F) \\ \langle y =_B M; E/x:A \rangle F &= y =_B M; \langle E/x:A \rangle F \quad \text{choosing } y \notin \text{FV}(F) \end{aligned}$$

Now the equations associated with Hoare types which can be defined as follows:

$$\begin{aligned} x \leftarrow (\text{do } E : \{P\}x:A\{Q\}); F &\longrightarrow_{\beta} \langle E/x:A \rangle F \\ K &\longrightarrow_{\eta} \text{do } (x \leftarrow K; \text{return } x) \end{aligned}$$

The monadic substitution is modeled directly after the operation introduced by Pfenning and Davies (2001), who also show that the beta and eta equations given above are equivalent to the unit and associativity laws for a generic monad (Moggi 1991).

The normalization algorithm of HTT does not use the described eta expansions in an arbitrary way, but relies on a very specific strategy. To capture this strategy, we define an auxiliary function  $\text{expand}_A$ , which iterates over the type  $A$  and expands the given argument (elim or intro term), according to each encountered type constructor.

$$\begin{aligned} \text{expand}_a(K) &= K && \text{if } a \text{ is nat or bool} \\ \text{expand}_{\alpha}(K) &= \text{eta}_{\alpha} K && \text{if } \alpha \text{ is a monotype variable} \\ \text{expand}_1(K) &= () \\ \text{expand}_{\forall \alpha. A}(K) &= \Lambda \alpha. \text{expand}_A(K \alpha) && \text{choosing } \alpha \notin \text{FTV}(K) \\ \text{expand}_{\Pi x:A_1. A_2}(K) &= \lambda x. \text{expand}_{A_2}(K M) && \text{where } M = \text{expand}_{A_1}(x) \\ &&& \text{choosing } x \notin \text{FV}(K) \\ \text{expand}_{\{P\}x:A\{Q\}}(K) &= \text{do } (x \leftarrow K; E) && \text{where } E = \text{return}(\text{expand}_A(x)) \\ \text{expand}_A(N) &= N && \text{if } N \text{ is not elim} \end{aligned}$$

The definition of  $\text{expand}$  exposes the significance of the constructor  $\text{eta}_{\alpha} K$ . This constructor records that, once the type variable  $\alpha$  is instantiated with some concrete type, then  $K$  should be expanded, on-the-fly, with respect to this type.

Beta reductions too are used in the normalization algorithm according to a very specific strategy, implemented via a set of auxiliary functions called *hereditary substitutions* (Watkins et al. 2004) that operate on canonical terms only. For example, in places where an ordinary capture-avoiding substitution creates a redex like  $(\lambda x. M) N$ , a hereditary substitution continues by immediately substituting  $N$  for  $x$  in  $M$ . This may produce another redex, which is immediately reduced by initiating another hereditary substitution, and so on.

The definition and properties of hereditary substitutions are discussed in more detail in Section 4. Here, we only note that hereditary substitutions have the form  $[M/x]_A^*(-)$ , and they substitute the canonical form  $M$  for a variable  $x$  into a given argument. The superscript  $*$  ranges over  $\{k, m, e, a, p, h\}$  and determines the syntactic domains of the argument (elim terms, intro terms, computations, types, assertions, and heaps, respectively). The subscript  $A$  is a putative type of  $M$ , and is used as termination metric for the substitution. We also need a monadic hereditary substitution  $\langle E/x \rangle_A(-)$ , and a monotype hereditary substitution  $[\tau/\alpha]^*(-)$ . The latter performs an on-the-fly eta expansion with respect to  $\tau$  of any subterms of the form  $\text{eta}_x K$  in the argument.

We mention two more capture-avoiding substitutions,  $[H/h]G$  and  $[H/h]P$ , which substitute the heap  $H$  for the variable  $h$  in the *heap*  $G$  and *assertion*  $P$ , respectively. We will never substitute heaps into other kinds of expressions, so we do not define similar operations for other syntactic domains. The substitutions simply commute with all the constructors, leaving intact the subexpressions which are not heaps or assertions, as illustrated in the definitional clause given below (which is typical).

$$[H/h](\text{upd}_\tau(G, M, N)) = \text{upd}_\tau([H/h]G, M, N)$$

Not substituting into  $\tau$ ,  $M$ , and  $N$  is justified because HTT types, terms, and computations are *not* allowed to contain free heap variables.

There are several more reductions that factor into definitional equality. For example,  $(\text{succ } M) + N$  is not canonical, as it can be simplified into  $\text{succ } (M + N)$ . The latter is simpler, because it makes more of the structure apparent (e.g., just by looking at the head constructor of  $\text{succ } (M + N)$ , we know that the expression must be nonzero). Similarly, addition  $z + N$  reduces to  $N$ , multiplication  $(\text{succ } M) \times N$  reduces to  $N + (M \times N)$ , and multiplication  $z \times N$  reduces to  $z$ . These reductions are required in order for the normalization to agree with the evaluation of `nat` expressions, so that the actual numerals are the only closed canonical forms of type `nat`.

We close this section with the comment that the rules described here certainly do not constitute a reasoning system that is complete in any sense. For example, algebraic laws like commutativity or associativity of addition, or reasoning principles like induction, cannot be inferred. We will make such reasoning principles available in the *propositional equality* defined in the following section.

### 3.2 Typing rules

The type system of HTT consists of the following judgments:

$$\begin{array}{ll} \Delta \vdash K \Rightarrow A [N'] & \Delta \vdash \Psi \Leftarrow \text{ctx } [\Psi'] \\ \Delta \vdash N \Leftarrow A [N'] & \Delta; X \vdash \Gamma \text{ pctx} \\ \Delta; P \vdash E \Rightarrow x:A. Q [E'] & \Delta; X \vdash P \Leftarrow \text{prop } [P'] \\ \Delta; P \vdash E \Leftarrow x:A. Q [E'] & \Delta \vdash A \Leftarrow \text{type } [A'] \\ \Delta; X; \Gamma_1 \Longrightarrow \Gamma_2 & \Delta \vdash \tau \Leftarrow \text{mono } [\tau'] \\ & \Delta; X \vdash H \Leftarrow \text{heap } [H'] \end{array}$$



The judgments on the right deal with formation and canonicity of variable contexts, assertion contexts, assertions, types, monotypes, and heaps. In these judgments, the output is always the canonical version of the main input ( $\Psi'$  is canonical for  $\Psi$ ,  $P'$  is canonical for  $P$ , etc.). When checking assertion contexts ( $\Gamma$  pctx),  $\Gamma$  is required to be canonical, so there is no need to return the output.

The judgments on the left are the main ones, and are explicitly oriented to symbolize whether the type or the assertion are given as input or are synthesized as output. This is a characteristic feature of bidirectional type-checking (Pierce & Turner 2000), which we here employ for both terms and computations.

The judgment  $\Delta \vdash K \Rightarrow A [N']$  takes an elim form  $K$  and outputs the type  $A$  of  $K$  and the canonical form  $N'$ . For example, we can derive  $f : \text{nat} \rightarrow \text{nat} \vdash f \Rightarrow \text{nat} \rightarrow \text{nat} [\lambda x. f x]$ , as the type of  $f$  can be inferred from the context. The canonical form of  $f$  is its eta expanded form, as described previously.

On the other hand, intro forms must be supplied a type against which to check. The judgment  $\Delta \vdash N \Leftarrow A [N']$  takes an intro form  $N$  and outputs the canonical form  $N'$  if  $N$  matches  $A$ . We supply the checking type, as otherwise  $N$  may type-check against many different types producing potentially different canonical forms. For example,  $\vdash \lambda x. x \Leftarrow \text{nat} \rightarrow \text{nat} [\lambda x. x]$ , but, in contrast,  $\vdash \lambda x. x \Leftarrow (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{nat}) [\lambda f. \lambda y. f y]$ .

The judgment  $\Delta; P \vdash E \Rightarrow x:A. Q [E']$  takes a computation  $E$ , input assertion  $P$ , and input type  $A$ , and outputs the strongest postcondition  $Q$  for  $E$  with respect to the precondition  $P$ , and the canonical form  $E'$  of  $E$ . Symmetrically,  $\Delta; P \vdash E \Leftarrow x:A. Q [E']$  takes computation  $E$  and outputs the canonical form  $E'$ , if  $Q$  is a postcondition (not necessarily the strongest) for  $E$  with respect to  $P$ , and  $A$  is the type of the return value of  $E$ .

The judgment  $\Delta; X; \Gamma_1 \Longrightarrow \Gamma_2$  formalizes the sequent calculus for the assertion logic, which is a classical multisorted logic with polymorphism. Here,  $\Delta$  is a variable context,  $X$  is a heap context, and  $\Gamma_1, \Gamma_2$  are sets of assertions. As usual, the sequent is valid if for every instantiation of the variables in  $\Delta$  and  $X$  such that the conjunction of assertions in  $\Gamma_1$  holds, the disjunction of assertions in  $\Gamma_2$  holds as well. The input and output contexts and types in all the earlier judgments are assumed canonical.

*Terms.* We need two auxiliary functions to compute canonical forms of application and type specialization. The functions  $\text{apply}_A(M, N)$  and  $\text{spec}(M, \tau)$  reduce the applications  $M N$  and  $M \tau$ , if  $M$  is a function or type abstraction, respectively. Here,  $M, N$ , and  $\tau$  are assumed to be canonical.

$$\begin{array}{lll}
 \text{apply}_A(K, M) & = & K M & \text{if } K \text{ is an elim term} \\
 \text{apply}_A(\lambda x. N, M) & = & N' & \text{where } N' = [M/x]_A^m(N) \\
 \text{apply}_A(N, M) & & \text{fails} & \text{otherwise} \\
 \\ 
 \text{spec}(K, \tau) & = & K \tau & \text{if } K \text{ is an elim term} \\
 \text{spec}(\Lambda \alpha. M, \tau) & = & [\tau/\alpha]^m(M) & \\
 \text{spec}(N, \tau) & & \text{fails} & \text{otherwise}
 \end{array}$$

The typing rules now formalize the intuition that intro terms can be checked against a supplied type, and elim terms can synthesize their type. The latter holds because

elim terms are generally of the form  $x T_1 T_2 \dots T_n$ , applying a variable  $x$  to a sequence of intro terms or types  $T_i$ . Since the type of  $x$  can be read from the context, the type of the whole application can always be inferred by instantiating the type of  $x$  with  $T_i$ .

$$\begin{array}{c}
 \frac{}{\Delta, x:A, \Delta_1 \vdash x \Rightarrow A[x]} \text{var} \qquad \frac{}{\Delta \vdash () \Leftarrow 1 [()]} \text{unit} \\
 \\
 \frac{\Delta, x:A \vdash M \Leftarrow B[M']}{\Delta \vdash \lambda x. M \Leftarrow \Pi x:A. B[\lambda x. M']} \Pi^x \\
 \\
 \frac{\Delta \vdash K \Rightarrow \Pi x:A. B[N'] \quad \Delta \vdash M \Leftarrow A[M']}{\Delta \vdash K M \Rightarrow [M'/x]_A^g(B) [\text{apply}_A(N', M')]} \Pi E \\
 \\
 \frac{\Delta, \alpha \vdash M \Leftarrow A[M']}{\Delta \vdash \Lambda \alpha. M \Leftarrow \forall \alpha. A[\Lambda \alpha. M']} \forall I^x \\
 \\
 \frac{\Delta \vdash K \Rightarrow \forall \alpha. B[N'] \quad \Delta \vdash \tau \Leftarrow \text{mono}[\tau']}{\Delta \vdash K \tau \Rightarrow [\tau'/\alpha]^a(B) [\text{spec}(N', \tau')]} \forall E \\
 \\
 \frac{\Delta \vdash A \Leftarrow \text{type}[A'] \quad \Delta \vdash M \Leftarrow A'[M']}{\Delta \vdash M : A \Rightarrow A'[M']} \Leftarrow\Rightarrow \\
 \\
 \frac{\Delta \vdash K \Rightarrow A[N'] \quad A = B}{\Delta \vdash K \Leftarrow B[\text{expand}_B(N')]} \Rightarrow\Leftarrow \qquad \frac{\Delta \vdash K \Rightarrow \alpha[K]}{\Delta \vdash \text{eta}_\alpha K \Leftarrow \alpha[\text{eta}_\alpha K]} \text{eta}
 \end{array}$$

For example,  $\Pi I$  checks that term  $\lambda x. M$  has the given function type, and if so, returns the canonical form  $\lambda x. M'$ . In  $\Pi E$  we first synthesize the canonical type  $\Pi x:A. B$  and the canonical form  $N'$  of the function part of the application. Then the synthesized type is used in checking the argument part of the application. The result type of the whole application is synthesized using hereditary substitutions in order to remove the dependency of the type  $B$  on the variable  $x$ . Finally, we compute the canonical form of the whole application, using the auxiliary function `apply` to reduce the term  $N' M'$  should  $N'$  actually be a lambda abstraction. A similar description applies to the rules for polymorphic quantification.

In the rule  $\Leftarrow\Rightarrow$ , we need to synthesize the canonical type for the ascription  $M:A$ . This type should clearly be the canonical version of  $A$ , under the condition that  $M$  actually has this type. Thus, we first test that  $A$  is well formed and compute its canonical form  $A'$ , and then proceed to check  $M$  against  $A'$ . If  $M$  and  $A'$  match, we obtained the canonical version  $M'$  of  $M$ . Then  $M'$  and  $A'$  are returned as the output of the judgment.

In the rule  $\Rightarrow\Leftarrow$ , we are checking an elim term  $K$  against a canonical type  $B$ . But  $K$  can already synthesize its canonical type  $A$ , so we simply need to check that  $A$  and  $B$  are actually syntactically equal. The canonical form synthesized from  $K$  in the premise, may be an elim term (because it is generated by a judgment for elim

terms), but we need to use it in the conclusion as an intro term. The switch from an elim term to the equivalent intro term is achieved by eta expansion with respect to the supplied type  $B$ . For example, if  $x:\text{nat}\rightarrow\text{nat}$  is a variable in context, then its canonical form is  $\lambda y. x y$ , and we could use the rule  $\Rightarrow\Leftarrow$  to derive the judgment  $x:\text{nat}\rightarrow\text{nat} \vdash x \Leftarrow \text{nat}\rightarrow\text{nat} [\lambda y. x y]$ .

When the types  $A$  and  $B$  in the rule  $\Rightarrow\Leftarrow$  are equal to some type variable  $\alpha$ , we cannot eta expand the canonical forms, so we use the constructor `eta` to remember that expansion must be done as soon as  $\alpha$  is instantiated with a concrete monotype. The constructor `eta` is applicable only to canonical forms; notice how its typing rule insists that the argument  $K$  equals its own canonical form. Thus, `eta` is never used in the source programs, but is only required internally, for equational reasoning.

*Computations.* The judgment  $\Delta; P \vdash E \Rightarrow x:A. Q [E']$  generates the strongest postcondition  $Q$  for the program  $E$  with respect to the precondition  $P$  (although approaches based on other kinds of predicate transformers, like the weakest preconditions, are possible). However, unlike in the Hoare type, where preconditions and postconditions are unary relations over the heap `mem`, here the assertions  $P$  and  $Q$  are *binary relations on heaps*, and the typing rule for introduction of Hoare types will mediate the switch from unary to binary relations. Since  $P$  and  $Q$  are binary relations, we make them depend on two free heap variables: `init`, which denotes a specific heap in the past of the computation, and `mem`, which denotes the current heap. The computation  $E$  describes how the heap `mem` from the precondition is modified into the heap `mem` in the postcondition, one effectful step at a time, while `init` is a fixed heap of reference, shared by both  $P$  and  $Q$ . In other words,  $P$  describes how `mem` relates to `init` before the execution of  $E$ , and  $Q$  describes how `mem` relates to `init` after the execution of  $E$ .

Alternatively, the computation  $E$  may be viewed semantically as a relation between the input and output heaps. Then the typing judgment simply serves to translate the computational syntax of  $E$  into the corresponding binary relation (here, the postcondition  $Q$ ). Intuitively,  $P$  is a relation that the translation starts with, and  $Q$  is the relation that the translation ends with, thus capturing the semantics of  $E$ . In addition,  $P$  has to be strong enough to guarantee that the execution of  $E$  will never get stuck. Translating programs into relations on heaps is a well-known approach to formulating Hoare Logic (Greif & Meyer 1979), and we adapt it here to a type-based system.

In order to define the small footprint semantics of the Hoare types, we first need two new connectives. The *relational composition*

$$P \circ Q = \exists h:\text{heap}. [h/\text{mem}]P \wedge [h/\text{init}]Q$$

expresses temporal sequencing of heaps. The informal reading of  $P \circ Q$  is that  $Q$  holds of the current heap, which is itself obtained from another past heap of which  $P$  holds.

The *difference operator* on assertions is defined as

$$R_1 \multimap R_2 = \forall h:\text{heap}. ([\text{init}/\text{mem}](R_1 * \text{this}(h))) \supset R_2 * \text{this}(h)$$

where  $R_1$  is assumed to have a free variable  $\text{mem}$ , but not  $\text{init}$ . The informal reading of  $R_1 \multimap R_2$  is that the heap  $\text{mem}$  is obtained from the initial heap  $\text{init}$  by replacing a fragment satisfying  $R_1$  with a new fragment which satisfies  $R_2$ . The rest of the heaps  $\text{init}$  and  $\text{mem}$  agrees. It is not specified, however, which particular fragment of  $\text{init}$  is replaced. If there are several fragments satisfying  $R_1$ , then each of them could be replaced, but the replacement is always such that the result satisfies  $R_2$ . The operator  $\multimap$  is used in the typing judgments to describe a difference between two successive heaps of the computation. For example,  $R_1 \multimap R_2$  captures the transformation incurred on the heap by the execution of an unspecified computation with type  $\{R_1\}x:A\{R_2\}$ . Notice how the definition of  $\multimap$  relies on naming the heap  $h$  by means of universal quantification in order to state its invariance. This corresponds to the equivalence of types  $\{R_1\}x:A\{R_2\}$  and  $h:\text{heap}.\{R_1 * \text{this}(h)\}x:A\{R_2 * \text{this}(h)\}$  that we commented on in Section 2.1.

Now consider a suspended computation  $\text{do } E : \Psi.X.\{R_1\}x:A\{R_2\}$ . Intuitively, the computation and the type should correspond if the following three requirements are satisfied: (1) Assuming that the initial heap can be split into two disjoint parts  $h_1$  and  $h_2$  such that  $R_1$  holds of  $h_1$ , then  $E$  does not get stuck if executed in this initial heap. Moreover,  $E$  never touches  $h_2$  (not even for a lookup); in other words,  $h_2$  is not in the footprint of  $E$ . (2) Upon termination of  $E$ , the fragment  $h_1$  is replaced with a new fragment which satisfies  $R_2$ , while  $h_2$  remains unchanged. (3) The split into  $h_1$  and  $h_2$  is not decided upon before  $E$  executes, and need not be unique. We only know that if a split is possible, then the execution of  $E$  defines one such split, but which split is chosen may depend on the run-time conditions. Whichever values  $h_1$  and  $h_2$  end up taking, however, we know that (2) holds.

The above requirements define what it means for the specification in the form of Hoare type  $\Psi.X.\{R_1\}x:A\{R_2\}$  to possess the small footprint property. We argue next that the requirements are satisfied by  $E$  if we can establish that  $\Delta; P \vdash E \Leftarrow x:A. Q$ , where  $P = \text{this}(\text{init}) \wedge \exists \Psi.X.(R_1 * \top)$  and  $Q = \forall \Psi.X.R_1 \multimap R_2$ .

The assertion  $P$  is related to the requirements (1) and (3). Indeed,  $P$  states that the initial heap can be split into  $h_1$  and  $h_2$  so that  $h_1$  satisfies  $R_1$  and  $h_2$  satisfies  $\top$ , as required. In order to ensure progress, the typing judgment will allow  $E$  to touch only locations whose existence can be proved. Because there is no information available about  $h_2$  and its locations (knowing  $\top$  amounts to knowing nothing),  $E$  will be restricted to working with  $h_1$  only. The split into  $h_1$  and  $h_2$  is arbitrary, satisfying an aspect of (3).

The assertion  $Q$  is related to the requirements (2) and (3). After unraveling the definition of the  $\multimap$  operator,  $Q$  essentially states that any split into  $h_1$  and  $h_2$  that  $E$  may have induced on  $\text{init}$  results in a final heap where  $h_1$  is replaced with a fragment satisfying  $R_2$ , while  $h_2$  remains unchanged. The invariance of  $h_2$  is precisely what (2) requires, and the parametricity of  $R_2$  with respect to the split is the remaining aspect of (3).

Before we can state the inference rules of the computation judgments, we need an auxiliary function  $\text{reduce}_A(M, x, E)$  which reduces the term  $x \leftarrow M; E$ , if  $M$  is a do-suspended computation. Here,  $A$ ,  $M$ , and  $E$  are assumed canonical.

$$\begin{aligned} \text{reduce}_A(K, x, E) &= x \leftarrow K; E && \text{if } K \text{ is an elim term} \\ \text{reduce}_A(\text{do } F, x, E) &= E' && \text{where } E' = \langle F/x \rangle_A(E) \\ \text{reduce}_A(N, x, E) &\text{ fails} && \text{otherwise} \end{aligned}$$

We start with the rules for the general monadic operations, and then proceed with the individual effectful commands.

$$\frac{\Delta; P \vdash E \Rightarrow x:A. R[E'] \quad \Delta, x:A; \text{init, mem}; R \Longrightarrow Q}{\Delta; P \vdash E \leftarrow x:A. Q[E']} \text{ consq}$$

$$\frac{\Delta \vdash M \leftarrow A[M']}{\Delta; P \vdash \text{return } M \Rightarrow x:A. P \wedge \text{id}_A(\text{expand}_A(x), M') [\text{return } M']} \text{ return}$$

$$\frac{\Delta; \text{this}(\text{init}) \wedge \exists \Psi.X.(R_1 * \top) \vdash E \leftarrow x:A. \forall \Psi.X.R_1 \multimap R_2[E']}{\Delta \vdash \text{do } E \leftarrow \Psi.X.\{R_1\}_x:A\{R_2\} [\text{do } E']} \{ \}$$

$$\frac{\Delta \vdash K \Rightarrow \Psi.X.\{R_1\}_x:A\{R_2\} [N'] \quad \Delta; \text{init, mem}; P \Longrightarrow \exists \Psi.X.(R_1 * \top) \quad \Delta, x:A; P \circ (\forall \Psi.X.R_1 \multimap R_2) \vdash E \Rightarrow y:B. Q[E']}{\Delta; P \vdash x \leftarrow K; E \Rightarrow y:B. (\exists x:A. Q) [\text{reduce}_A(N', x, E')]} \{ \}E$$

The rule `consq` allows the weakening of the strongest postcondition  $R$  into an arbitrary postcondition  $Q$ , assuming that  $R$  implies  $Q$ . The rule `return` types the trivial computation that immediately returns the result  $x = M$  and performs no changes to the heap. The precondition is simply propagated into the postcondition, but the postcondition must also assert the equality between  $M$  and (the canonical form of)  $x$ . The rule `{ }` defines the small footprint semantics of Hoare types. This is achieved with the premise  $\Delta; P \vdash E \leftarrow x:A. Q$ , for  $P$  and  $Q$  as discussed earlier.

The rule `{ }E` describes how a suspended computation  $K \Rightarrow \Psi.X.\{R_1\}_x:A\{R_2\}$  can be sequentially composed with another computation  $E$ . The composition is meaningful if the following are satisfied. First, the assertion logic must establish that  $P$  ensures that the current heap contains a fragment satisfying  $R_1$ , as required by  $K$ . In other words, we need to show that  $P \Longrightarrow \exists \Psi.X.(R_1 * \top)$ . Second, the computation  $E$  needs to check against the postcondition obtained after executing  $K$ . The latter is taken to be  $P \circ \forall \Psi.X.R_1 \multimap R_2$ , expressing that the execution of  $K$  changed the heap  $P$  by replacing a fragment satisfying  $R_1$  with a new fragment satisfying  $R_2$ . The normal form of the whole computation is obtained by invoking the auxiliary function `reduce`. We emphasize that the type  $B$  in the conclusion of the `{ }E` rule is an *input* of the typing judgments, and is by assumption well formed in the context  $\Delta$ . In particular,  $x$  does not appear in  $B$ , so no special considerations are needed passing from the premise of the rule to the conclusion. No such assumptions are made about the postcondition  $Q$ , which is an output of the judgment, so we need to existentially abstract  $x$  in the postcondition of the conclusions, to avoid dangling variables. A similar remark applies to the rules for the specific effectful constructs

for allocation, lookup, strong update, and deallocation which we will present next.

$$\frac{\Delta \vdash \tau \Leftarrow \text{mono} [\tau'] \quad \Delta \vdash M \Leftarrow \tau' [M'] \quad \Delta, x:\text{nat}; P * (x \mapsto_{\tau'} M') \vdash E \Rightarrow y:B. Q [E']}{\Delta; P \vdash x = \text{alloc}_{\tau}(M); E \Rightarrow y:B. (\exists x:\text{nat}. Q) [x = \text{alloc}_{\tau}(M'); E']}$$

$$\frac{\Delta \vdash M \Leftarrow \text{nat} [M'] \quad \Delta \vdash \tau \Leftarrow \text{mono} [\tau'] \quad \Delta; \text{init, mem}; P \Longrightarrow M' \hookrightarrow_{\tau'} - \quad \Delta, x:\tau'; P \wedge (M' \hookrightarrow_{\tau'} \text{expand}_{\tau}(x)) \vdash E \Rightarrow y:B. Q [E']}{\Delta; P \vdash x = !_\tau M; E \Rightarrow y:B. (\exists x:\tau'. Q) [x = !_\tau M'; E']}$$

$$\frac{\Delta \vdash M \Leftarrow \text{nat} [M'] \quad \Delta \vdash \tau \Leftarrow \text{mono} [\tau'] \quad \Delta \vdash N \Leftarrow \tau' [N'] \quad \Delta; \text{init, mem}; P \Longrightarrow M' \hookrightarrow - \quad \Delta; P \circ ((M' \mapsto -) \multimap (M' \mapsto_{\tau'} N')) \vdash E \Rightarrow y:B. Q [E']}{\Delta; P \vdash M :=_{\tau} N; E \Rightarrow y:B. Q [M' :=_{\tau'} N'; E']}$$

$$\frac{\Delta \vdash M \Leftarrow \text{nat} [M'] \quad \Delta; \text{init, mem}; P \Longrightarrow M' \hookrightarrow - \quad \Delta; P \circ ((M' \mapsto -) \multimap \text{emp}) \vdash E \Rightarrow y:B. Q [E']}{\Delta; P \vdash \text{dealloc}(M); E \Rightarrow y:B. Q [\text{dealloc}(M'); E']}$$

$$\frac{\Delta \vdash A \Leftarrow \text{type} [A'] \quad \Delta \vdash M \Leftarrow A' [M'] \quad \Delta; P \vdash [M:A/x]E \Rightarrow y:B. Q [E']}{\Delta; P \vdash x =_A M; E \Rightarrow y:B. Q [E']}$$

In the case of allocation,  $E$  is checked against the assertion  $P * (x \mapsto_{\tau'} M')$ , which describes the state after the allocation, and is the strongest postcondition for allocation with respect to  $P$ . The assertion simply states that the newly allocated memory whose address is stored in  $x$  is disjoint from any already allocated memory described in  $P$ .

In the case of lookup, the strongest postcondition states that the heap is not changed (i.e.,  $P$  still holds) but we have the additional knowledge that the variable  $x$  stores the looked up value. The variable  $x$  is expanded because we only consider assertions in canonical form. In order to ensure progress, we must prove the sequent  $P \Longrightarrow M' \hookrightarrow_{\tau'} -$  showing that the location  $M'$  actually exists in the current heap, and points to a value of an appropriate type. It is important to notice that proving the sequent  $P \Longrightarrow M' \hookrightarrow_{\tau'} -$  may be postponed, as it does not influence the other premises. The sequent can be seen as part of the verification condition which is generated during type-checking.

The strongest postcondition for update states that the heap has changed by replacing some assignment  $M' \mapsto -$  with an assignment  $M' \mapsto_{\tau'} N'$ . A prerequisite is to prove the sequent  $P \Longrightarrow M' \hookrightarrow -$ , thus showing that  $M'$  was allocated with an arbitrary type (hence the update is strong).

The strongest postcondition for deallocation states that the heap is changed by replacing the assignment  $M' \mapsto -$  with  $\text{emp}$ . The side condition is the sequent  $P \Longrightarrow M' \hookrightarrow -$  showing that  $M'$  was allocated.

Type-checking the command  $x =_A M; E$  reduces to type-checking the substitution  $[M:A/x]E$ , reflecting that assignment is just a syntactic sugar for substitution.





Definitional equality is embedded into propositional equality by postulating the following rules:

$$\frac{\Delta; X; \Gamma_1 \Longrightarrow \text{id}_A(M, M), \Gamma_2}{\Delta, x:A; X \vdash P \Leftarrow \text{prop } [P] \quad \Delta; X; \Gamma_1, \text{id}_A(M, N) \Longrightarrow [M/x]_A^P(P), [N/x]_A^P(P), \Gamma_2} \\ \Delta; X; \Gamma_1, \text{id}_A(M, N) \Longrightarrow [M/x]_A^P(P), \Gamma_2$$

The first rule postulates that propositional equality is reflexive. In combination with the requirement that all terms appearing in sequents are *in canonical form*, this rule in effect embeds definitional equality as a subrelation of propositional equality. The second rule axiomatizes that equal terms can be substituted in an arbitrary context  $P$ .

The last equality rule does not admit extensionality of functions. The terms  $M$  and  $N$  must depend only on the variables in  $\Delta$  and  $X$ , while functions require an additional variable. Hence, we introduce a separate rule for function extensionality, and similarly, a separate rule for type abstraction.

$$\frac{\Delta, x:A; X; \Gamma_1 \Longrightarrow \text{id}_B(M, N), \Gamma_2}{\Delta; X; \Gamma_1 \Longrightarrow \text{id}_{\Pi x:A. B}(\lambda x. M, \lambda x. N), \Gamma_2} \\ \frac{\Delta, \alpha; X; \Gamma_1 \Longrightarrow \text{id}_B(M, N), \Gamma_2}{\Delta; X; \Gamma_1 \Longrightarrow \text{id}_{\forall \alpha. B}(\Lambda \alpha. M, \Lambda \alpha. N), \Gamma_2}$$

In the above rules, it is assumed that the bound variables  $x$  and  $\alpha$  do not appear free in the involved contexts.

Heaps are axiomatized as partial functions from nats to values as follows:

$$\frac{}{\Delta; X; \Gamma_1, \text{seleq}_\tau(\text{empty}, M, N) \Longrightarrow \Gamma_2} \\ \frac{}{\Delta; X; \Gamma_1 \Longrightarrow \text{seleq}_\tau(\text{upd}_i(H, M, N), M, N), \Gamma_2} \\ \frac{}{\Delta; X; \Gamma_1, \text{seleq}_\tau(\text{upd}_o(H, M_1, N_1), M_2, N_2) \Longrightarrow \text{id}_{\text{nat}}(M_1, M_2), \text{seleq}_\tau(H, M_2, N_2), \Gamma_2} \\ \frac{}{\Delta; X; \Gamma_1, \text{seleq}_\tau(H, M, N_1), \text{seleq}_\tau(H, M, N_2) \Longrightarrow \text{id}_\tau(N_1, N_2), \Gamma_2}$$

The first rule states that an empty heap does not contain any assignments. The second rule and the third rule implement the McCarthy axioms for functional arrays (McCarthy 1962), relating the `seleq` and `upd` functions. The fourth axiom asserts a version of heap functionality: a heap may assign at most one value to a location, for each given type.

We would prefer a slightly stronger fourth axiom here, which would state that a heap assigns at most one type and value to a location, instead of at most one value for each type. As an illustration, in our previous example we used the assertion  $P = x \mapsto_\alpha m \wedge y \mapsto_\beta n$  to specify a heap which contains exactly one location thus forcing  $x$  and  $y$  to be aliases. While  $x = y$  could be derived from  $P$ , we cannot derive that  $\alpha = \beta$  and  $m = n$  with our weak fourth axiom. Stating the full functionality of heaps requires a new assertion for *heterogeneous equality* (a.k.a. John Major

equality, as introduced by McBride (1999)), which allows equating terms at different underlying types. We leave this extension for future work.

Finally, the rules for the primitive types include the standard Peano axioms for natural numbers, including the induction principle, and a similar set of rules for booleans.

$$\frac{}{\Delta; X; \Gamma_1, \text{id}_{\text{nat}}(\text{succ } M, z) \Longrightarrow \Gamma_2}$$

$$\frac{}{\Delta; X; \Gamma_1, \text{id}_{\text{nat}}(\text{succ } M, \text{succ } N) \Longrightarrow \text{id}_{\text{nat}}(M, N), \Gamma_2}$$

$$\frac{\Delta, x:\text{nat}; X \vdash P \Leftarrow \text{prop } [P] \quad \Delta \vdash M \Leftarrow \text{nat } [M] \quad \Delta, x:\text{nat}; X; \Gamma_1, P \Longrightarrow [\text{succ } x/x]_{\text{nat}}^p(P), \Gamma_2}{\Delta; X; \Gamma_1, [z/x]_{\text{nat}}^p(P) \Longrightarrow [M/x]_{\text{nat}}^p(P), \Gamma_2}$$

$$\frac{}{\Delta; X; \Gamma_1, \text{id}_{\text{bool}}(\text{true}, \text{false}) \Longrightarrow \Gamma_2}$$

$$\frac{\Delta, x:\text{bool}; X \vdash P \Leftarrow \text{prop } [P] \quad \Delta \vdash M \Leftarrow \text{bool } [M]}{\Delta; X; \Gamma_1, [\text{true}/x]_{\text{bool}}^p(P), [\text{false}/x]_{\text{bool}}^p(P) \Longrightarrow [M/x]_{\text{bool}}^p(P), \Gamma_2}$$

Peano axioms usually include equations on primitive operations like  $+$  and  $\times$ . Such equations are not required here, as they are already incorporated into definitional equality. Similarly, we do not postulate any specific equations over effectful computations. Currently, HTT only supports the generic monadic unit and associativity laws (Moggi 1991), and these too are already a part of definitional equality.

*Example.* Here we consider the function `sumfunc` that takes an argument  $n$  and computes the sum  $1 + \dots + n$ . The function first allocates  $a$  which will store the partial sums, then increments the contents of  $a$  with successive nats in a loop, until  $n$  is reached. Then  $a$  is deallocated before its contents is returned as the final result.

We present the code for `sumfunc` in Figure 1, and annotate it with assertions (preceded by “—”) that are generated during type-checking at the various control points. In the code, we assume given the ordering  $\leq$ , and introduce the following abbreviations: (1)  $\text{sum}(r, n) = \text{id}_{\text{nat}}(2 \times r, n \times (n + 1))$  denotes that  $r = 1 + \dots + n$ ; (2)  $I = i \leq n \wedge \exists t:\text{nat}. a \mapsto_{\text{nat}} t \wedge \text{sum}(t, i)$  denotes the loop invariant during the summation; (3)  $Q = a \mapsto_{\text{nat}} - \wedge \text{sum}(x, n)$  asserts what holds upon the exit from the loop.

The specification for `sumfunc` states that the function starts and ends with an empty heap. The most interesting part of the code is the recursive loop. It introduces the fixpoint variable  $f$ , whose type we take to be  $f:\Pi i:\text{nat}. \{I\}x:\text{nat}\{Q\}$ , giving the loop invariant in the precondition. The variable  $i$  is the counter which drives the loop. The initial value for  $i$  is 0, as specified in `eval f 0`, and the loop terminates when  $i$  reaches  $n$ .

The following sequents are generated during type-checking, and they constitute the verification conditions that should be discharged in order to validate the program: (1)  $P_2 \Longrightarrow a \hookrightarrow_{\text{nat}} -$ , so that  $a$  can be looked up; (2)  $P_5 \Longrightarrow a \hookrightarrow -$  so that  $a$

---

```

sumfunc  :  Πn:nat. {emp}r:nat{emp ∧ sum(r, n)}
=  λn. do (— P0 : {this(init)})
      a = allocnat(0);
      — P1 : {this(init) * (a ↦nat 0)}
      x = fix f(i : nat) : {I}x:nat{Q} =
          do (— P2 : {this(init) ∧ (I * ⊤)})
            s = !nat a;
            — P3 : {P2 ∧ a ↦nat s}
            t = ifnat(i == n) then
                — P4 : {P3 ∧ idnat(i, n)}
                return(s)
            else
                — P5 : {P3 ∧ ¬idnat(i, n)}
                a :=nat s + i + 1;
                — P6 : {P5 ◦ (a ↦nat — ◯ a ↦nat s + i + 1)}
                x ← f (i + 1);
                — P7 : {P6 ◦ ([i + 1/i]I ◯ Q)}
                return(x)
          — P8 : {(P4 ∧ idnat(t, s)) ∨ (∃x:nat.P7 ∧ idnat(t, x))}
          return(t)
      in eval f 0;
      — P9 : {P1 ◦ ([0/i]I ◯ Q)}
      dealloc(a);
      — P10 : {P9 ◦ (a ↦nat — ◯ emp)}
      return(x)

```

---

Fig. 1. Annotated code for summation. Lines preceded by “—” describe what is true at the respective program points.

can be updated; (3)  $P_6 \implies [i + 1/i]I * \top$ , so that the computation obtained from  $f(i + 1)$  can be executed; (4)  $P_8 \wedge \text{id}_{\text{nat}}(x, t) \implies I \multimap Q$ , so that the fixpoint satisfies the prescribed postcondition; (5)  $P_9 \implies a \multimap -$  so that  $a$  can be deallocated; and (6)  $P_{10} \wedge \text{id}_{\text{nat}}(r, x) \implies \text{emp} \multimap (\text{emp} \wedge \text{sum}(r, n))$ , so that `sumfunc` has the required postcondition. It is not too hard to see that all these sequents are valid.

#### 4 Hereditary substitutions

In Section 3.1, we briefly sketched how equational reasoning is organized in the typing judgments of HTT: to compare two terms for equality, we first normalize each of them to a canonical form, and then check the canonical forms for  $\alpha$ -equivalence. In this section, we describe the details of the normalization operation.

An HTT term is in canonical form if it is beta-normal (i.e. it contains no beta redexes), and eta-long (i.e., all of its intro subterms are eta expanded). For example, if  $f : (\text{nat} \rightarrow \text{nat}) \rightarrow (\text{nat} \rightarrow \text{nat}) \rightarrow \text{nat}$  and  $g : \text{nat} \rightarrow \text{nat}$ , then the canonical version of the term  $f g$  is  $\lambda h. f (\lambda y. g y) (\lambda x. h x)$ . This definition of canonicity accounts for both beta and eta equations.

The main insight, due to Watkins et al. (2004), is that conversion to canonical forms (i.e., normalization) can be defined on possibly ill-typed terms, and can be shown to terminate using a simple syntactic argument. The current section is a

self-contained presentation of these ideas, augmented with a treatment of predicative polymorphism (which is not a difficult extension).

At the center of the development are *hereditary substitutions*, which are capture-avoiding substitutions defined only on canonical terms. Hereditary substitutions must avoid producing results with beta redexes, so whenever an ordinary substitution would create a beta redex, hereditary substitution must immediately reduce it. This reduction may create another redex that must be reduced as well, and so on. Hereditary substitutions may therefore be viewed as implementing a very specific normalization strategy, which lends itself to a simple proof of termination.

In the current paper, the definitional equality of HTT terms does not depend on the full HTT type, but only on its dependency-free version. For example, two terms that are equal at some Hoare type are equal at any other Hoare type that they belong to. Thus, when computing with canonical forms, we can ignore the assertions from the Hoare types. With this in mind, given an HTT type  $A$ , we define the *shape*  $A^-$  to be the simple type obtained by erasing the dependencies.

$$\begin{aligned}
 (\alpha)^- &= \alpha \\
 (\text{nat})^- &= \text{nat} \\
 (\text{bool})^- &= \text{bool} \\
 (1)^- &= 1 \\
 (\forall\alpha. A) &= \forall\alpha. A^- \\
 (\Pi x:A. B)^- &= A^- \rightarrow B^- \\
 (\Psi.X.\{P\}x:A\{Q\})^- &= \diamond(A^-)
 \end{aligned}$$

We impose an ordering on shapes and write  $S_1 \leq S_2$  and  $S_1 < S_2$ , if  $S_1$  is a subexpression of  $S_2$  (proper subexpression in the second case). Here, we consider that proper subexpressions of a quantified type also include all the substitution instances obtained by replacing the bound type variable with a *simple* monotype. This is clearly a well-founded ordering, as instantiating a type quantifier with a simple monotype decreases the overall number of quantifiers.

We define the following hereditary substitutions: (1)  $[M/x]_S^*(-)$  substitute the canonical term  $M$  for  $x$  in the argument. Here  $* \in \{k, m, e, a, p, h\}$  ranges over elim terms, intro terms, computations, types, assertions, and heaps, respectively, and determines the domain of the argument of the substitution. The index  $S$  is the putative shape of the type of  $M$ , and will serve as the termination metric for the substitution; (2)  $\langle E/x \rangle_S(F)$  is the hereditary version of the *monadic substitution*; and (3)  $[\tau/\alpha]^*(-)$  is the hereditary *type* substitution. To reduce clutter, we will frequently write  $[M/x]_A^*(-)$  and  $\langle E/x \rangle_A(F)$ , instead of  $[M/x]_{A^-}^*(-)$  and  $\langle E/x \rangle_{A^-}(F)$ , correspondingly.

The substitutions are defined by nested induction, first on the structure of  $S$ , and then on the structure of the term being substituted into (in case of the monadic substitution, we use the substituted computation instead). In other words, we either go to a smaller shape, in which case the expressions may become larger, or the shape remains the same, but the expressions decrease.

We note that the hereditary substitutions are partial functions. If the involved expressions are not well typed, the substitution, while terminating, may fail to return

a meaningful result. As conventional when working with expressions that may fail to be defined, whenever we state an equality  $T_1 = T_2$ , we imply that  $T_1$  and  $T_2$  are also defined.

We next present the characteristic cases of the definitions of hereditary substitutions. The substitution into elim terms may return either another elim term  $K$ , or an intro term  $N$ . In the latter case,  $N$  is annotated with the shape  $S$  of its putative type.

$$\begin{array}{ll}
[M/x]_S^k(x) & = M :: S \\
[M/x]_S^k(y) & = y & \text{if } y \neq x \\
[M/x]_S^k(K N) & = K' N' & \text{if } [M/x]_S^k(K) = K' \text{ and } [M/x]_S^k(N) = N' \\
[M/x]_S^k(K N) & = O' :: S_2 & \text{if } [M/x]_S^k(K) = \lambda y. M' :: S_1 \rightarrow S_2, \text{ where} \\
& & [M/x]_S^k(N) = N' \text{ and } O' = [N'/y]_{S_1}^m(M') \\
[M/x]_S^k(K \tau) & = K' \tau' & \text{if } [M/x]_S^k(K) = K' \text{ and } [M/x]_S^k(\tau) = \tau' \\
[M/x]_S^k(K \tau) & = N' :: [\tau^-/\alpha]S_2 & \text{if } [M/x]_S^k(K) = \Lambda\alpha. M' :: \forall\alpha. S_2, \text{ where} \\
& & [M/x]_S^k(\tau) = \tau' \text{ and } N' = [\tau'/\alpha]^m(M') \\
[M/x]_S^k(K') & \text{fails} & \text{otherwise}
\end{array}$$

Notice that the substitution into  $K N$  and  $K \tau$  may fail to be defined depending on what is returned as a result of substituting into  $K$ . For example, a failure will appear if  $[M/x]_S^k(K)$  returns an intro term which is not a lambda abstraction, or if the returned shape is not a function type. We also note that the definition does not require a case for substitution into elim terms  $N:A$ , because this kind of terms cannot appear in a canonical form.

Next we present several cases from the hereditary substitution into intro terms.

$$\begin{array}{ll}
[M/x]_S^m(K) & = K' & \text{if } [M/x]_S^k(K) = K' \\
[M/x]_S^m(K) & = N' & \text{if } [M/x]_S^k(K) = N' :: S' \\
[M/x]_S^m(\text{eta}_\alpha K) & = \text{eta}_\alpha K' & \text{if } [M/x]_S^k(K) = K' \\
[M/x]_S^m(\text{eta}_\alpha K) & = \text{eta}_\alpha K' & \text{if } [M/x]_S^k(K) = \text{eta}_\alpha K' :: \alpha \\
[M/x]_S^m(()) & = () \\
[M/x]_S^m(\lambda y. N) & = \lambda y. N' & \text{where } [M/x]_S^m(N) = N' \\
& & \text{choosing } y \notin \text{FV}(M) \text{ and } y \neq x \\
[M/x]_S^m(\Lambda\alpha. N) & = \Lambda\alpha. N' & \text{where } [M/x]_S^m(N) = N' \\
& & \text{choosing } \alpha \notin \text{FTV}(M) \text{ and } \alpha \neq x \\
[M/x]_S^m(\text{do } E) & = \text{do } E' & \text{if } [M/x]_S^e(E) = E'
\end{array}$$

We omit the clauses for substitution into primitive operations, as they are not particularly interesting. In fact, the only interesting case here is substituting into  $\text{eta}_\alpha K$ , which fails if substituting into the subterm  $K$  does not return  $\text{eta}_\alpha K'$  indexed by the same type variable  $\alpha$ . Any other intro term would restrict which types could be substituted for  $\alpha$ , thus violating the parametricity of type abstraction.

The characteristic cases of hereditary substitution into computations follow:

$$\begin{aligned}
 [M/x]_S^e(\text{return } N) &= \text{return } N' && \text{if } [M/x]_S^m(N) = N' \\
 [M/x]_S^e(y \leftarrow K; E) &= y \leftarrow K'; E' && \text{if } [M/x]_S^k(K) = K' \text{ and } [M/x]_S^e(E) = E' \\
 &&& \text{choosing } y \notin \text{FV}(M) \text{ and } y \neq x \\
 [M/x]_S^e(y \leftarrow K; E) &= F' && \text{if } [M/x]_S^k(K) = \text{do } F :: \diamond S_1 \\
 &&& \text{and } [M/x]_S^e(E) = E' \text{ and } F' = \langle F/y \rangle_{S_1}(E') \\
 &&& \text{choosing } y \notin \text{FV}(M) \text{ and } y \neq x
 \end{aligned}$$

We omit the clauses for primitive computations as they are straightforward. Also, we do not require a clause for substitution into  $y =_A N; E$ , because this computation is not canonical (indeed it reduces to  $[N/y]_A^e(E)$ ). The only interesting case arises when the substitution into the branch  $K$  of  $y \leftarrow K; E$  returns a *do*-suspended computation. This creates a redex which is immediately reduced by invoking a monadic hereditary substitution.

Hereditary monadic substitution is a simple adaptation of the ordinary monadic substitution defined in Section 3.1.

$$\begin{aligned}
 \langle \text{return } M/x \rangle_A(F) &= F' && \text{if } [M/x]_A^e(F) = F' \\
 \langle y \leftarrow K; E/x \rangle_A(F) &= y \leftarrow K; F' && \text{if } \langle E/x \rangle_A(F) = F', \text{ choosing } y \notin \text{FV}(F) \\
 \langle y \leftarrow c; E/x \rangle_A(F) &= y \leftarrow c; F' && \text{if } \langle E/x \rangle_A(F) = F', \text{ choosing } y \notin \text{FV}(F) \\
 \langle E/x \rangle_A(F) & \text{ fails} && \text{otherwise}
 \end{aligned}$$

Hereditary (mono)type substitutions  $[\tau/\alpha]^*(-)$  in most cases simply commute with the constructors, and the only interesting case is

$$[\tau/\alpha]^m(\text{eta}_x K) = \text{expand}_t(K') \text{ if } [\tau/\alpha]^k(K) = K'$$

which expands out the argument expression with respect to  $\tau$ . Notice that the result must itself be a canonical intro term, as expansion cannot create new redexes (please see the definition of *expand* in Section 3.1).

We omit the hereditary substitutions into types, propositions, and heaps, as they simply commute with all the connectives.

*Theorem 1 (Termination of hereditary substitutions)*

1. If  $[M/x]_S^k(K) = N' :: S_1$ , then  $S_1 \leq S$ .
2.  $[M/x]_S^*(-)$ ,  $\langle E/x \rangle_S(-)$ , and  $[\tau/\alpha]^*(-)$  terminate, either by returning a result, or failing in a finite number of steps.

*Proof*

The first part is proven by induction on  $K$ . The second part is trivial in the case of type substitutions, as type substitutions cannot create new redexes, and are hence terminating. In the case of  $[M/x]_S^*$  and  $\langle E/x \rangle_S$ , we proceed by nested induction, first on the index shape  $S$  (under the ordering  $\leq$ ), and second on the structure of the argument we apply the substitution to. In each case of the definition, we either decrease  $S$ , or failing that, we apply the function to strict subexpressions of the input. Since  $\leq$  is well founded, this process must terminate.

Note that without the restriction to predicative polymorphism, types could actually grow after a substitution into the elim form  $K \tau$ , and hereditary substitutions may

fail to terminate if applied to ill-typed terms. Hence our restriction to predicative polymorphism insists that  $\tau$  must be a monotype.

In the future work, we plan to extend HTT with impredicative polymorphism. The extension will require a significantly more involved proof of termination, probably based on logical relations, unlike the present case where a simple syntactic argument suffices.  $\square$

## 5 Properties

In this section, we present the most characteristic properties of HTT, leading up to the substitution principles.

*Theorem 2 (Relative decidability of type-checking)*

Given an oracle for deciding the validity of assertion logic sequents, all the typing judgments of the HTT are decidable.

*Proof*

The typing judgments of HTT are syntax directed; their premises always involve type-checking smaller expressions, or deciding syntactic equality of types, or computing hereditary substitutions, or deciding sequents of the assertion logic. Checking syntactic equality is obviously a terminating algorithm, and as shown in Theorem 1, hereditary substitutions are terminating as well. Thus, if the validity of each assertion logic sequent can be decided, so too can the typing judgments.  $\square$

The proof of Theorem 2 essentially shows that type-checking of HTT can be divided into the two phases described in the beginning of Section 2. Whenever a sequent is encountered during type-checking, we can use the oracle to decide its validity, or simply view the sequent as a verification condition, and postpone calling the oracle (that is, proving the sequent) until later. The proof of Theorem 2 guarantees that if deciding sequents is postponed, then type-checking terminates. Thus, we can divide type-checking into two phases. The first phase checks the underlying simple types, and generates the verification conditions. Since HTT is a higher-order language, this process is nontrivial, as it involves *normalization*. However, by Theorem 2, it is terminating. In the second phase, the verification conditions are discharged.

It should be possible to remove the assumption about the oracle in Theorem 2, by extending HTT with certificates for the sequents, in the style of Proof-Carrying Code (Necula 1997). With this extension, a computation judgment of HTT will contain all the information needed to establish its own derivation, as the derivation is completely guided by the syntax of the computation. In the terminology of Martin-Löf (1996), the judgments become *analytic*, or self-evident. Alternatively, we can say that an HTT computation can be viewed as a proof of its own specification, and thus the effectful fragment of HTT establishes a correspondence in the style of Curry–Howard isomorphism (Howard 1980) between computations and specification proofs.

The next lemma connects the elim term  $K$  with its eta expansion.



*Lemma 3 (Identity principle)*

If  $\Delta \vdash K \Rightarrow A [K]$ , then  $\Delta \vdash \text{expand}_A(K) \Leftarrow A [\text{expand}_A(K)]$ .

*Proof*

By induction on the structure of  $A$ , using an auxiliary lemma about expansion of variables (omitted here), which roughly states that if  $x:A$  is a free variable in a well-typed expression  $N$ , then hereditarily substituting  $\text{expand}_A(x)$  for  $x$  does not change  $N$ . Intuitively, the latter holds because typing ensures that  $x$  is correctly used in  $N$ , so that when  $\text{expand}_A(x)$  is substituted, the created redexes are immediately hereditarily reduced, preserving the syntactic form of  $N$ .  $\square$

We proceed to restate in the context of HTT the usual properties of Hoare Logic, such as weakening of the consequent and strengthening of the precedent. Also included is the property of relational composition, which states that a computation does not depend on how the heap in which it executes has been obtained; what matters is only the current state of the heap. In other words, if a computation has a precondition  $P$  and postcondition  $Q$  (which, in the typing judgments are binary relations on heaps), these can be composed with an arbitrary relation  $R$  into a new precondition  $R \circ P$  and a new postcondition  $R \circ Q$ . This property is important for the metatheory of HTT, because the typing rules compute strongest postcondition by composing binary relations.

*Lemma 4 (Properties of computations)*

Suppose that  $\Delta; P \vdash E \Leftarrow x:A. Q [E']$ . Then

1. *Weakening consequent.* If  $\Delta, x:A; \text{init, mem}; Q \Longrightarrow R$ , then  $\Delta; P \vdash E \Leftarrow x:A. R [E']$ .
2. *Strengthening precedent.* If  $\Delta; \text{init, mem}; R \Longrightarrow P$ , then  $\Delta; R \vdash E \Leftarrow x:A. Q [E']$ .
3. *Relational composition.* If  $\Delta; \text{init, mem} \vdash R \Leftarrow \text{prop} [R]$ , then  $\Delta; R \circ P \vdash E \Leftarrow x:A. (R \circ Q) [E']$ .

*Proof*

Weakening of consequent is straightforward. From  $\Delta; P \vdash E \Leftarrow x:A. Q [E']$  we know that there exists an assertion  $S$ , such that  $\Delta; P \vdash E \Rightarrow x:A. S$  where  $\Delta, x:A; \text{init, mem}; S \Longrightarrow Q$ . Applying the rule of cut, we get  $\Delta, x:A; \text{init, mem}; S \Longrightarrow R$ , and thus  $\Delta; P \vdash E \Leftarrow x:A. R [E']$ .

Strengthening precedent and relational composition are proved by induction on the structure of  $E$ , using the identity principles.  $\square$

We next use the properties of computations to prove that the Frame Rule from separation logic is admissible. The Frame Rule captures the essence of small footprints, and guarantees that a computation cannot touch the portion of the heap that is not described by its precondition.

*Lemma 5 (Frame)*

If  $\Delta \vdash \text{do } E \Leftarrow \Psi.X.\{P\}x:A\{Q\} [E']$ , and  $\Delta, \Psi; X, \text{mem} \vdash R \Leftarrow \text{prop} [R]$ , then  $\Delta \vdash \text{do } E \Leftarrow \Psi.X.\{P * R\}x:A\{Q * R\} [E']$ .

*Proof*

From the assumption on the typing of  $\text{do } E$ , we obtain  $\Delta; \text{this}(\text{init}) \wedge \exists \Psi.X.(P * \top) \vdash E \Leftarrow x:A. \forall \Psi.X.(P \multimap Q)$ .

Notice that that following sequents are derivable:

1.  $\text{this}(\text{init}) \wedge \exists \Psi.X.(P * R * \top) \Longrightarrow \text{this}(\text{init}) \wedge \exists \Psi.X.(P * \top)$ , and
2.  $\forall \Psi.X.(P \multimap Q) \Longrightarrow \forall \Psi.X.(P * R \multimap Q * R)$ .

Both are proved easily, in the second case after expanding the definition of  $\multimap$ . Now the result follows from the typing of  $E$ , by strengthening the precedent using (1) and weakening the consequent using (2).  $\square$

We close this section with the statement of the substitution principles, which formalize the interplay between the ordinary substitutions on general terms, and hereditary substitutions on the canonical versions of the same terms. Intuitively, *substitution commutes with normalization*. That is, ordinary substitution followed by normalization, produces the same result as computing the canonical forms first, and then employing hereditary substitutions.

Thus, unlike in most other works on dependent types, where proving soundness usually requires proving strong normalization, here we establish a somewhat weaker, but sufficient, result. Instead of showing that *every* reduction order yields the same canonical form, we only need to know that the particular reduction order implemented by the type-checker is stable under substitution.

*Lemma 6 (Type substitution principles)*

Suppose that  $\Delta \vdash \tau \Leftarrow \text{mono}[\tau']$ , and denote by  $\bar{T}$  the operation of hereditary type substitution  $[\tau'/\alpha]^*(T)$ , for any kind of expression  $T$ . Then the following hold:

1. If  $\Delta, \alpha, \Delta_1 \vdash K \Rightarrow B [O]$ , then  $\Delta, \bar{\Delta}_1 \vdash [\tau/\alpha]K \Rightarrow \bar{B} [\bar{O}]$ .
2. If  $\Delta, \alpha, \Delta_1 \vdash N \Leftarrow B [O]$ , then  $\Delta, \bar{\Delta}_1 \vdash [\tau/\alpha]N \Leftarrow \bar{B} [\bar{O}]$ .
3. If  $\Delta, \alpha, \Delta_1; P \vdash E \Leftarrow y:B. Q [F]$ , and  $y \notin \text{FV}(M)$ , then  $\Delta, \bar{\Delta}_1; \bar{P} \vdash [\tau/\alpha]E \Leftarrow y:\bar{B}. \bar{Q} [\bar{F}]$ .
4. If  $\Delta, \alpha, \Delta_1; X; \Gamma_1 \Longrightarrow \Gamma_2$ , then  $\Delta, \bar{\Delta}_1; X; \bar{\Gamma}_1 \Longrightarrow \bar{\Gamma}_2$ .

*Lemma 7 (Term substitution principles)*

Suppose that  $\Delta \vdash A \Leftarrow \text{type}[A']$  and  $\Delta \vdash M \Leftarrow A' [M']$ , and denote by  $\bar{T}$  the operation of hereditary substitution  $[M'/x]_{A'}^*(T)$ , for any kind of expression  $T$ . Then the following hold:

1. If  $\Delta, x:A', \Delta_1 \vdash K \Rightarrow B [O]$ , then  $\Delta, \bar{\Delta}_1 \vdash [M:A/x]K \Rightarrow \bar{B} [\bar{O}]$ .
2. If  $\Delta, x:A', \Delta_1 \vdash N \Leftarrow B [O]$ , then  $\Delta, \bar{\Delta}_1 \vdash [M:A/x]N \Leftarrow \bar{B} [\bar{O}]$ .
3. If  $\Delta, x:A', \Delta_1; P \vdash E \Leftarrow y:B. Q [F]$ , and  $y \notin \text{FV}(M)$ , then  $\Delta, \bar{\Delta}_1; \bar{P} \vdash [M:A/x]E \Leftarrow y:\bar{B}. \bar{Q} [\bar{F}]$ .
4. If  $\Delta, x:A, \Delta_1; X; \Gamma_1 \Longrightarrow \Gamma_2$ , then  $\Delta, \bar{\Delta}_1; \Psi; \bar{\Gamma}_1 \Longrightarrow \bar{\Gamma}_2$ .
5. If  $\Delta; P \vdash E \Leftarrow x:A'. Q [E']$  and  $\Delta, x:A'; Q \vdash F \Leftarrow y:B. R [F']$ , where  $x \notin \text{FV}(B, R)$ , then  $\Delta; P \vdash \langle E/x:A \rangle F \Leftarrow y:B. R [\langle E'/x \rangle_A (F')]$ .

Most of the substitution principles are simple adaptations of the fairly standard principles encountered in any dependent type theory. However, it may be interesting

to point out here that the monadic substitution principle (Lemma 7.5) actually states that the *sequential composition rule* usually encountered in Hoare Logics is *admissible* in HTT. Indeed, Lemma 7.5 states that a computation  $E$  with a postcondition  $Q$  can be sequentially composed with another computation  $F$ , if  $F$  has a precondition  $Q$ .

Both type and term substitution principles are proved by simultaneous induction of the main derivations, after generalizing to include the formation judgments of HTT. The proofs makes essential use of auxiliary *hereditary substitution principles* (omitted here), which have similar statement as the lemmas above, but involve only canonical forms.

## 6 Operational semantics

In this section, we define the call-by-value, left-to-right operational semantics for HTT and prove that the type system is sound with respect to the operational semantics. In particular, we argue that if  $\Delta; P \vdash E \Leftarrow x:A. Q$  is derivable in the type system, then it is indeed the case that evaluating  $E$  in a heap in which  $P$  holds produces a heap in which  $Q$  holds (if  $E$  terminates).

The operational semantics is only defined for well-typed terms. Since HTT types correspond to specifications, our approach is different from the traditional approach of Hoare Logic but it is similar to the approach in Birkedal *et al.* (2005), which also only gives semantics to well-specified programs.

*Syntax.* The syntactic domains required by the operational semantics are as follows.

<i>Values</i>	$v, l$	$::=$	$() \mid \lambda x. M \mid \Lambda \alpha. M \mid \text{do } E \mid \text{true} \mid \text{false} \mid z \mid \text{succ } v$
<i>Value heaps</i>	$\chi$	$::=$	$\cdot \mid \chi, l \mapsto_{\tau} v$
<i>Continuations</i>	$\kappa$	$::=$	$\cdot \mid x:A. E; \kappa$
<i>Control expressions</i>	$\rho$	$::=$	$\kappa \triangleright E$
<i>Abstract machines</i>	$\mu$	$::=$	$\chi, \kappa \triangleright E$

The definition of values is standard from mostly functional programming languages. We use  $l$  to range over nats when they are used as pointers. Value heaps are assignments from nats to values, where each assignment is indexed by a type. Value heaps are a run-time concept, and are used in the evaluation judgments to describe the state in which programs execute. This is in contrast to heaps from Section 2 which are used for reasoning in the assertion logic. That the two notions correspond to each other is expressed by our definition of heap soundness that will be given later in this section. We will need to convert a value heap into a heap canonical form, so we introduce the following conversion function:

$$\begin{aligned} \llbracket \cdot \rrbracket &= \text{empty} \\ \llbracket \chi, l \mapsto_{\tau} v \rrbracket &= \text{upd}_{\tau}(\llbracket \chi \rrbracket, l, M), \quad \text{where } \cdot \vdash v \Leftarrow \tau[M] \end{aligned}$$

A continuation is a sequence of computations of the form  $x:A.E$ , where  $E$  may depend on the bound variable  $x:A$ . The continuation is executed by passing a value to the variable  $x$  in the first computation  $E$ . If that computation terminates, its return value is passed to the second computation, and so on.

A control expression  $\kappa \triangleright E$  pairs up a computation  $E$  and a continuation  $\kappa$ , so that  $E$  provides the initial value with which the execution of  $\kappa$  can start. Thus, a control expression is in a sense a self-contained computation. Control expressions are introduced because they make the call-by-value semantics of the computation  $x \leftarrow \text{do } E; F$  explicit. Evaluation of this computation is carried out by creating the control expression  $x. F \triangleright E$ ; or in other words, first push  $x. F$  onto the continuation, and proceed to evaluate  $E$ .

An abstract machine  $\mu$  is a pair of a value heap  $\chi$  and a control expression  $\kappa \triangleright E$ . The control expression is evaluated against the heap, to eventually produce a result and possibly change the heap.

Our theorems require a typing judgment for abstract machines, in order to specify the type of the return value and the properties of the heap in which the abstract machine terminates (if it does). Given  $\mu = \chi, \kappa \triangleright E$ , we write  $\vdash \mu \Leftarrow x:A. Q$  if we can prove that  $Q$  is a postcondition for  $\kappa \triangleright E$  with respect to the assertion  $\llbracket \chi \rrbracket$  generated from  $\chi$ .

*Evaluation.* There are three evaluation judgments in HTT; one for elimination terms  $K \hookrightarrow_k K'$ , one for introduction terms  $M \hookrightarrow_m M'$ , and one for abstract machines  $\chi, \kappa \triangleright E \hookrightarrow_e \chi', \kappa' \triangleright E'$ . Each judgment relates an expression with its one-step reduct. We present selected rules in Figure 2.

For example, in the evaluation of intro terms, if the intro term is obtained by coercion from an elim term, we invoke the judgment for elim terms. If the returned result is of the form  $v:A$ , we remove the type annotation. This prevents accumulation of type annotations, as in  $v:A_1 : \dots : A_n$ . In the evaluation of the abstract machine  $\chi, \kappa \triangleright E$ , we first reduce  $E$  to a value, which is plugged into the continuation  $\kappa$  to proceed. Occasionally we must check that the types given at the input abstract machine are well formed, so that the output abstract machine is also well formed. The outcome of the evaluation, however, does not depend on type information, if we assume that the input to evaluation is well formed.

*Soundness.* Perhaps somewhat surprisingly for a program logic like HTT, we formulate soundness via Preservation and Progress theorems as often used for simpler type systems. This is a consequence of our decision to formulate HTT as a type theory, rather than as an ordinary Hoare Logic. Of course, our Preservation and Progress theorems are significantly stronger (and also harder to prove) than corresponding theorems for simpler type systems since our types are much more expressive.

*Theorem 8 (Preservation)*

1. if  $K_0 \hookrightarrow_k K_1$  and  $\cdot \vdash K_0 \Rightarrow A [N']$ , then  $\cdot \vdash K_1 \Rightarrow A [N']$ .
2. if  $M_0 \hookrightarrow_m M_1$  and  $\cdot \vdash M_0 \Leftarrow A [M']$ , then  $\cdot \vdash M_1 \Leftarrow A [M']$ .
3. if  $\mu_0 \hookrightarrow_e \mu_1$  and  $\vdash \mu_0 \Leftarrow x:A. Q$ , then  $\vdash \mu_1 \Leftarrow x:A. Q$ .

*Proof*

The first two statements are proved by simultaneous induction on the evaluation judgment, using inversion on the typing derivation, and substitution principles. The third statement is proved by case analysis on the evaluation judgment, using the first

## Evaluation of elim terms

$$\frac{K \hookrightarrow_k K'}{K N \hookrightarrow_k K' N} \quad \frac{N \hookrightarrow_m N'}{(v : A) N \hookrightarrow_k (v : A) N'} \quad \frac{K \hookrightarrow_k K'}{K \tau \hookrightarrow_k K' \tau}$$

$$\frac{}{(\lambda x. M : \Pi x:A_1. A_2) v \hookrightarrow_k [v : A_1/x]M : [v : A_1/x]A_2}$$

$$\frac{}{(\lambda \alpha. M : \forall \alpha. A) \tau \hookrightarrow_k [\tau/\alpha]M : [\tau/\alpha]A} \quad \frac{M \hookrightarrow_m M'}{M : A \hookrightarrow_k M' : A}$$

## Evaluation of intro terms

$$\frac{K \hookrightarrow_k K' \quad K' \neq v : A}{K \hookrightarrow_m K'} \quad \frac{K \hookrightarrow_k v : A}{K \hookrightarrow_m v}$$

## Evaluation of abstract machines

$$\frac{}{\chi, x:A. E; \kappa \triangleright v \hookrightarrow_e \chi, \kappa \triangleright [v : A/x]E}$$

$$\frac{}{\chi, \kappa \triangleright x \leftarrow (\text{do } F) : \Psi.X.\{P\}x:A\{Q\}; E \hookrightarrow_e \chi, (x:A. E; \kappa) \triangleright F}$$

$$\frac{\cdot \vdash \tau \Leftarrow \text{mono} [\tau'] \quad l \notin \text{dom}(\chi)}{\chi, \kappa \triangleright x = \text{alloc}_\tau(v); E \hookrightarrow_e (\chi, l \mapsto_{\tau'} v), \kappa \triangleright [l:\text{nat}/x]E}$$

$$\frac{\cdot \vdash \tau \Leftarrow \text{mono} [\tau'] \quad l \mapsto_{\tau'} v \in \chi}{\chi, \kappa \triangleright x = !_\tau l; E \hookrightarrow_e \chi, \kappa \triangleright [v : \tau/x]E}$$

$$\frac{\cdot \vdash \tau \Leftarrow \text{mono} [\tau']}{(\chi_1, l \mapsto_\sigma v', \chi_2), \kappa \triangleright l :=_\tau v; E \hookrightarrow_e (\chi_1, l \mapsto_{\tau'} v, \chi_2), \kappa \triangleright E}$$

$$\frac{}{(\chi_1, l \mapsto_\sigma v, \chi_2), \kappa \triangleright \text{dealloc}(l); E \hookrightarrow_e (\chi_1, \chi_2), \kappa \triangleright E}$$

$$\frac{}{\chi, \kappa \triangleright x =_A v; E \hookrightarrow_e \chi, \kappa \triangleright [v:A/x]E}$$

$$\frac{}{\chi, \kappa \triangleright x = \text{if}_A(\text{true}) \text{ then } E_1 \text{ else } E_2; E \hookrightarrow_e \chi, x:A. E; \kappa \triangleright E_1}$$

$$\frac{}{\chi, \kappa \triangleright x = \text{if}_A(\text{false}) \text{ then } E_1 \text{ else } E_2; E \hookrightarrow_e \chi, x:A. E; \kappa \triangleright E_2}$$

$$\frac{}{\chi, \kappa \triangleright x = \text{case}_A(z) \text{ of } z.E_1 \text{ or } s y.E_2; E \hookrightarrow_e \chi, x:A. E; \kappa \triangleright E_1}$$

$$\frac{}{\chi, \kappa \triangleright x = \text{case}_A(sv) \text{ of } z.E_1 \text{ or } s y.E_2; E \hookrightarrow_e \chi, x:A. E; \kappa \triangleright [v:\text{nat}/y]E_2}$$

$$\frac{N = \lambda z. \text{do } (y = \text{fix } f(x:A):B = \text{do } E \text{ in eval } f z; y) \quad B = \Psi.X.\{R_1\}y:C\{R_2\}}{\chi, \kappa \triangleright y = \text{fix } f(x:A):B = \text{do } E \text{ in eval } f v; F \hookrightarrow_e \chi, (y:[v:A/x]C. F; \kappa) \triangleright [v:A/x, N:\Pi x:A.B/f]E}$$

Fig. 2. Selected evaluation rules (omitted are primitive arithmetic operations and nonredex cases of abstract machines).

two statements. We also tacitly use an auxiliary replacement lemma (omitted here), which roughly states that replacing the computation  $E$  in the control expression  $\kappa \triangleright E$  with another computation  $F$  preserves the typing of the control expression, as long as  $E$  and  $F$  have the same postconditions, and thus both provide the same precondition for the execution of  $\kappa$ .  $\square$

The preservation theorem states that the evaluation step on a well-specified term/abstract machine does not change the specification of the result. In the case of abstract machines, after taking the step, the evaluation is still on its way to produce a value of type  $A$ , and to terminate in a heap satisfying  $Q$ . In the case of pure terms, there is an additional claim that evaluation preserves the canonical form—and thus the equational properties—of the evaluated term. In other words, normalization is adequate for the operational semantics.

Before we can state the progress theorem, we need to define a property of the assertion logic which we call *heap soundness*.

*Definition 9 (Heap soundness)*

The assertion logic of HTT is heap sound iff for every value heap  $\chi$ ,

1. if  $\cdot; \text{mem}; \text{this}(\llbracket \chi \rrbracket) \Longrightarrow l \hookrightarrow_{\tau} -$ , then  $l \mapsto_{\tau} v \in \chi$ , for some value  $v$ , and
2. if  $\cdot; \text{mem}; \text{this}(\llbracket \chi \rrbracket) \Longrightarrow l \hookrightarrow -$ , then  $l \mapsto_{\tau} v \in \chi$  for some monotype  $\tau$  and a value  $v$ .

Heap soundness essentially shows that the assertion logic correctly reasons about value heaps, so that facts established in the assertion logic will be true during evaluation. The clauses of the definition of heap soundness correspond to the side conditions that need to be derived in the typing rules for the primitive commands of lookup, update, and deallocation. If the assertion logic proves that  $l \hookrightarrow_{\tau} -$ , then the evaluation will be able to associate a value  $v$  with this location, and carry out the lookup. If the assertion logic proves that  $l \hookrightarrow -$ , then the evaluation will be able to associate a monotype  $\tau$  and a value  $v:\tau$  with this location, and carry out the update or deallocation.

Now we can state the Progress theorem, under the assumption of heap soundness; in the following section we prove that the assertion logic of HTT is indeed heap sound.

*Theorem 10 (Progress)*

Suppose that the assertion logic of HTT is heap sound. Then the following hold:

1. If  $\cdot \vdash K_0 \Rightarrow A [N']$ , then either  $K_0 = v : A$  or  $K_0 \hookrightarrow_k K_1$ , for some  $K_1$ .
2. If  $\cdot \vdash M_0 \Leftarrow A [M']$ , then either  $M_0 = v$  or  $M_0 \hookrightarrow_m M_1$ , for some  $M_1$ .
3. If  $\vdash \chi_0, \kappa_0 \triangleright E_0 \Leftarrow x:A. Q$ , then either  $E_0 = v$  and  $\kappa_0 = \cdot$ , or  $\chi_0, \kappa_0 \triangleright E_0 \hookrightarrow_e \chi_1, \kappa_1 \triangleright E_1$ , for some  $\chi_1, \kappa_1, E_1$ .

*Proof*

By straightforward case analysis on the involved expressions. In addition, the proof of the third statement requires heap soundness in order to discharge the cases involving the primitive commands for allocation, lookup, and update.  $\square$

*Example.* From Progress and Preservation theorems it is now clear that `sumfunc 10` produces a computation that, if it terminates when executed in an empty heap, returns the value 55 and an empty heap.

## 7 Heap soundness

In this section, we sketch a proof that the assertion logic is heap sound. For the most part, the assertion logic is a standard first-order logic with polymorphism. However, it also admits reasoning about heaps and monadic computations. We need to show that the presence of heaps and computations does not cause unsoundness, and we do so by means of a simple, and somewhat crude set-theoretic model of HTT.

Our model depends on the observation that the assertion logic does not include axioms for computations; reasoning about computations is formalized via the typing rules and soundness of those is proved earlier via progress and preservation assuming soundness of the assertion logic (heap soundness).

Thus in our set-theoretic model, we chose to simply interpret the computation types  $\Psi.X.\{P\}_x:A\{Q\}$  and  $\Psi.X.\{P\}_x:\tau\{Q\}$  as one-element sets, emphasizing that the assertion logic cannot distinguish between different computations. Given this basic decision we are really just left with interpreting a type theory similar to the Extended Calculus of Constructions (ECC) with a (assertion) logic on top. The type theory has two universes (mono and other types) and is similar to the ECC, except that the mono universe is not impredicative. Hence, we can use a simplified version of Luo's model of ECC (Luo 1990) for the types. Thus, our model is really fairly standard and hence we only include a sketch of it here.

As in Luo (1990) our model takes place in ZFC set theory with infinite inaccessible cardinals  $\kappa_0, \kappa_1, \dots$ . The universe `mono` is the set of all sets of cardinality smaller than  $\kappa_0$ . The type `nat` is interpreted as the set of natural numbers, `bool` as the set of booleans,  $\Pi x:A. B$  and  $\Pi x:\tau. \sigma$  as dependent product in sets,  $\Sigma x:A. B$  and  $\Sigma x:\tau. \sigma$  as dependent sum in sets. Heaps are interpreted as finite partial functions from the set of natural numbers to  $\Sigma \alpha:\text{mono}. \alpha$ . Predicates  $P$  on a type are interpreted as subsets in the classical way.

Thus, we clearly get a sound model of classical higher-order logic and the assertion logic is heap sound.

### *Theorem 11 (Heap soundness)*

The assertion logic of HTT is heap sound.

### *Proof*

Let  $\chi$  be a value heap. Here we only sketch the argument for case (1) of heap soundness (Definition 9); case (2) is proved analogously.

By assumption  $\cdot; \text{mem}; \text{this}(\llbracket \chi \rrbracket) \Longrightarrow l \hookrightarrow_{\tau} -$  is derivable. By soundness of the assertion logic we have that  $\cdot; \text{mem}; \text{this}(\llbracket \chi \rrbracket) \Longrightarrow l \hookrightarrow_{\tau} -$  is true. By definition of the interpretation of  $\hookrightarrow_{\tau}$  this means that  $\exists v \in \llbracket \tau \rrbracket. \llbracket \llbracket \chi \rrbracket \rrbracket(l) = v$ . By the definition of  $\llbracket \chi \rrbracket$  and the semantics of heaps, we have that  $l \mapsto_A v_0 \in \chi$ , for some value  $v_0$ , as required (and  $\llbracket v_0 \rrbracket$  is the  $v$  that exists).  $\square$



Note that the denotational model presented earlier does not model predicates as *admissible*<sup>2</sup> subsets, but rather as all subsets. One might have expected admissibility to show up since HTT contains a rule for fixed points (see Section 3) but because the denotational model is so crude and since it is only used to show heap soundness, while operational methods are used to show soundness of the typing rule for fixed points, we do not need to restrict attention to admissible predicates in the denotational model. We are not aware of similar combinations of models and proof methods for models of higher-order store in the literature.

## 8 Related work

We divide the related work into three groups: (1) Hoare Logics for higher-order effectful programs, (2) dependent type systems for pure and impure functional programming, and (3) languages and tools for extended static checking of effectful programs.

*Hoare Logics for higher-order programs.* Berger *et al.* (2005) and Honda *et al.* (2005) present several Hoare Logics for total correctness of PCF with references, where specifications in the form of Hoare triples are taken as propositions. Krishnaswami (2006) proposes a version of separation logic for a core monomorphic ML-like language. Similar to HTT, Krishnaswami bases his logic on a monadic presentation of the underlying programming language. Both proposals do not support polymorphism, strong updates, deallocation, or pointer arithmetic. Both are Hoare-like logics, rather than type theories, and do not integrate expressive specifications into types.

*Dependent types for programming.* The work on dependently typed systems with stateful features has mostly focused on how to appropriately restrict the effects from appearing in the language of types. If types only depend on pure terms, it becomes possible to use logical reasoning about them. Such systems have mostly employed singleton types to enforce purity. Examples include dependent ML by Xi and Pfenning (1998; 1999), applied type systems by Chen and Xi (2005), and Zhu and Xi (2005), and a type system for certified binaries by Shao *et al.* (2005). HTT differs from these approaches, because types are allowed to depend on monadically encapsulated effectful computations. We also mention the theory of type refinements by Mandelbaum *et al.* (2003), which reasons about programs with effects, by employing a restricted fragment of linear logic. The restriction on the logic limits the class of properties that can be described, but is undertaken to preserve decidability of type-checking.

There are also several recent proposals for purely functional languages with dependent types. Examples include Cayenne (Augustsson 1998), Epigram (McBride & McKinna, 2005), Omega (Sheard 2004) and Sage (Flanagan 2006).

Cayenne is an extension of Haskell that allows arbitrary values to occur in types. Like Haskell, Cayenne considers diverging computations to be pure, and admits

<sup>2</sup> A subset of a pointed cpo is admissible if it is pointed and closed under sups of chains.

them into type dependencies. As commented in (Augustsson 1998), this makes it possible to construct a proof for every proposition that appears in a Cayenne type, by simply building a diverging program whose type equals the required proposition. Hence, Cayenne, as a logic, is unsound. From the computational perspective, this is not necessarily bad, as long as unsoundness is manifested as divergence, rather than a run-time type error. But, it would seem to imply that all program transformations must preserve diverging behavior of the underlying code, which may complicate some optimizations (Gill *et al.* 1993). Thus, we believe that it is much better if divergence is considered to be an effect, and is appropriately isolated by the type system, as done in HTT.

Epigram is based closely on Luo's UTT (Luo 1994), and supports a number of practical programming features, like inference of implicit arguments, nonuniform families of types, sophisticated forms of pattern matching, and the use of explicit proofs within programs. Epigram does not currently support imperative features.

Omega (Sheard 2004) employs singleton types, in a way similar to DML and ATS, to refine the type system of the underlying language. It allows the use of explicit proofs within programs, in order to discharge the various conditions that arise during type-checking.

Sage (Flanagan 2006) makes type-checking of expressive specifications manageable by combining proving and run-time checks. When the Sage type-checker cannot automatically discharge some verification condition, it inserts an explicit test in the code. If the test fails at run-time, the program is forced into an error state, and the failing test instance is remembered as a falsehood for future use in theorem proving. We believe that a similar functionality can easily be built into HTT, at least for those properties that can actually be computationally verified (this excludes any nontrivial use of quantifiers).

In addition to the purely functional languages described above, we list RSP1 (Westbrook *et al.* 2005) and Deputy (Condit *et al.* 2007), as two proposals for using dependent types for specification and reasoning about state in first-order and low-level languages, respectively. HTT differs from these approaches, as we support fully higher-order, as well as polymorphic, imperative programs and stores.

*Extended static checking.* There are also a number of languages and tools that integrate Hoare-style specifications into the type system and provide a mode of extended static checking of imperative programs. Examples include ESC/Modula-3 (Detlefs *et al.* 1998), ESC/Java (Leino *et al.* 2000), JML (Leavens *et al.* 1999; Burdy *et al.* 2005), Spec# (Barnett *et al.* 2004), SPLint (Evans & Larochelle 2002), and SPARK/Ada (Barnes 2003). All these systems use the annotations to statically generate the appropriate verification conditions, which are then discharged by an automated theorem prover, or by inserting a run-time check.

For example, SPARK/Ada syntactically limits the programming language, so that the assertions are always decidable. This makes it inapplicable in many situations, but provides a desired guarantee that well-typed programs do not go wrong. Spec# inserts a run-time check whenever it cannot prove a certain condition (with similar restrictions as described above for Sage). In contrast, ESC/Java is designed simply

to warn the programmer of a potential error, but does not offer any soundness guarantees.

Finally, we mention two theoretical frameworks for programming with expressive specifications. Abadi and Leino (2004) describe a logic for object-oriented programs where specifications, like in HTT, are treated as types. One of the problems that the authors describe concerns the scoping of variables; certain specifications cannot be proved because the inference rule for  $\text{let val } x = E \text{ in } F$  does not allow sufficient interaction between the specifications of  $E$  and  $F$ . Such problems do not appear in HTT. Birkedal *et al.* (2005) describe a dependent type system for well-specified programs in idealized Algol extended with heaps. The type system includes a wide collection of higher-order frame rules, which are shown sound by a denotational model. A serious limitation of the type system compared to HTT is that the heap in *loc. cit.* can only contain simple integer values.

## 9 Future work

We briefly sketch several ideas that we plan to carry out in the future: scaling HTT to include quantification over propositions, capturing locality and ownership of state in the type system, investigation of reasoning principles for effectful computations, addition of inductive and recursive types, and stateful types like references and arrays, and other applications of type systems with Hoare types.

*Higher-order quantification and local state.* The type system presented in this paper does not provide any constructs for quantification over assertions. Yet, this clearly is an important operation. For example, abstracting a proposition at the level of terms and types makes it possible to hide certain details about a function or a module, so that different implementations may share the same signature. This is crucial for modular programming and code reuse. Abstracting a proposition at the level of assertions increases the power of the assertion logic to higher order, thus providing internal means for defining new predicates (including inductive and coinductive ones) and for reasoning about a large class of types (Church 1940; Paulson 1990; SRI International & DSTO 1991).

In Nanevski *et al.* (2007), we have already carried out preliminary investigations into the higher-order extensions, and scaled HTT to include a significant fragment of the ECC (Luo 1990). It turns out that when combined with Hoare types, abstraction over predicates can represent the local state of modules within the type system. Local state is shared by several functions of the module, but can be hidden from the clients. The precision of the abstracted predicate controls how much information about local state is revealed. By judiciously choosing this predicate, modules may completely protect their local state, or grant a partial or total access to, or even ownership of portions of local state to the clients.

We also believe that higher-order assertion logic is the appropriate framework for studying type and annotation inference for HTT, as higher-order logic should be strong enough to represent any precondition and postcondition that may appear

during program verification. This is directly related to the property of Cook completeness (Cook 1978), which we would also like to prove for HTT.

*Reasoning about effectful code.* One of the main obstacles in the design of languages for integrated programming and reasoning, as for example outlined in Burdy et al. (2005), is that effectful code usually cannot appear in specifications. This is problematic from the software engineering point of view, as it leads to code duplication. Almost every functionality must be implemented once purely, to be used in specification and reasoning, and once impurely, for efficient execution.

In HTT, we admit effectful code in specifications, but that is only the first step toward solving the above problem, as the degree to which such code can be reasoned about is determined by the available equational laws. Currently, HTT admits only the generic monadic laws, which do not capture all the semantic properties of computations. In the future, we plan to investigate which additional equations can be soundly added. This might require developing more refined denotational models than the one described in this paper, and formalizing them in the assertion logic of HTT, so that the equalities between computations can be certified when required.

*Incorporating other notions of effects.* In this paper, we applied HTT to the problem of reasoning about state with aliasing. But other computational effects and applications seem possible too. For example, we would like HTT to support control effects like exceptions and continuations. Some related work in this direction may be Tan and Appel (2006), which extends Hoare Logic with compositional reasoning about programs with labels and goto commands. We also plan for HTT to support concurrency. Separation logic has been used recently to reason about concurrent programs (O'Hearn et al. 2004; Feng et al. 2007) and we hope that these ideas may be embedded into HTT as well. On the other hand, an approach to concurrency based on software transactional memory, as recently implemented in Haskell (Harris et al. 2005), may be particularly well suited to the monadic nature of Hoare types. Another interesting extension may involve reasoning through Hoare types about information flow and security (Amtoft et al. 2006).

*Stateful types.* The types of HTT are currently all pure, in the sense that we abstract away from their actual layout in memory. It is possible to describe the layout of the memory and reason about it *using assertions*, but it is not possible to capture the memory into a type, such as the type of references or arrays, which are inherently stateful.

Yet, such stateful types are important because they can lead to a higher degree of automation of the reasoning principles, as well as to a significant reduction in the size and number of verification conditions. Indeed, in the current paper, we must prove before every location access that the location at the given address exists. On the other hand, if locations were a separate type, rather than natural numbers, we could have a more fine-grained control over them. For example, we could make locations

persistent, as in ML and Haskell. Then simple type-checking would guarantee access safety, with no need for additional verification conditions.

One of the challenges in adding stateful types is deciding which axioms they should satisfy. For example, we can always consider references simply to be natural numbers (as currently in HTT), but references are often expected to exhibit a certain recursive behavior, and thus may need to be defined as some sort of recursive type (see later).

The extension with stateful types may be related to the recent substructural type systems for state, such as the work of Ahmed *et al.* (2005), Fluet *et al.* (2006), Morrisett *et al.* (2005), and Collinson and Pym (2006), and the logic of Reus and Schwinghammer (2006).

*Recursive types.* Inductive types have been extensively studied in dependent type theories (Dybjer 1994; Dybjer & Setzer 2006), including pattern-matching against them (McBride & McKinna, 2005). However, general recursive types have not received much attention, possibly because recursive types introduce divergence into the term language, and thus destroy the logical properties of the theory.

In HTT, it should be possible to add recursive types safely, as long as divergence is monadically encapsulated within the Hoare types. This may be achieved by pushing the elimination construct for the recursive type into the effectful fragment. Typing rules for the introduction and elimination of recursive types may roughly be as follows:

$$\frac{\Delta \vdash M \Leftarrow [\mu\alpha.A/\alpha]^a(A)}{\Delta \vdash \text{fold}_{\mu\alpha.A} M \Rightarrow \mu\alpha.A}$$

$$\frac{\Delta \vdash K \Rightarrow \mu\alpha.A \quad \Delta, x:[\mu\alpha.A/\alpha]^a(A); P \wedge \text{id}_{\mu\alpha.A}(\text{fold } x, K) \vdash E \Rightarrow y:B. Q}{\Delta; P \vdash x = \text{unfold } K; E \Rightarrow y:B. (\exists x:[\mu\alpha.A/\alpha]^a(A).Q)}$$

We believe that recursive types are also the correct way of controlling the recursive behavior that arises in the presence of higher-order store and the type of references (known in the literature as “tying the Landin’s knot”). The store in HTT is higher order, as locations may point to arbitrary higher-order functions and stateful computations, but HTT does not admit this kind of recursion through the store. Our typing rules require that each computation specifies all the locations that it may touch. Thus, such locations must be created *before* the computation and its type. It is not possible to create a location of a Hoare type that depends on the very same location, thus potentially creating a loop. If a loop in the store is required, it should be specified either using assertions in the preconditions and postconditions of computations (already available in HTT), or using recursive types.

We consider this to be an advantage of HTT over other stateful languages. It seems to us as a good design to delegate all the concerns about recursion to explicit fixpoint constructs, rather than allow implicit recursive behavior to arise through interaction of otherwise unrelated programming features.

## 10 Conclusions

We have presented HTT, which is a novel framework for extending a dependently typed functional language with imperative programming features like state and nontermination. The supported operations on state involve allocation, deallocation, lookup, and strong update, so that a location may be updated with values of varying types. The store in HTT is higher-order, in the sense that locations may point to arbitrary higher-order functions and effectful computations. We also admit pointer arithmetic.

The main idea that allows for safe combination of dependent types and effects follows the familiar “specifications-as-types” principle of type theory (Howard 1980). We introduce the distinguished Hoare type  $\{P\}x:A\{Q\}$ , which classifies effectful programs with a precondition  $P$ , a postcondition  $Q$ , and a return value  $x:A$ , thus internalizing into the type system the well-known specification methodology from Hoare Logic (Hoare 1969). The addition of Hoare types makes it possible to statically track the use of stateful operations, and make sure they are performed only if the required preconditions are satisfied. Hoare types also generalize the notion of monads, used extensively in modern simply typed functional programming (Peyton Jones 2003).

Embedding specifications into types has significant advantages over the original Hoare Logic. In particular, if specifications are treated as types, then it becomes possible to combine the specifications with the source code, allowing for nesting, combination with other types, abstraction and packaging of a specification with the data whose properties it describes. All of these operations are essential from the point of view of modularity, information hiding, and data abstraction, but are not available in Hoare Logic. For example, we are able to seamlessly incorporate Hoare types with other higher-order features like higher-order functions and polymorphism, which were traditionally a difficult extension in Hoare Logic (Cartwright & Oppen 1978; O’Donnell 1982).

We also show that in the presence of polymorphism, we can define the propositional connectives from separation logic, which has been recently proposed as an extension to Hoare Logic, particularly suited for reasoning about pointer aliasing. In addition, HTT specifications can quantify over heap variables to name the particular heap fragments of interest. This ability is not directly available in separation logic, but in HTT we can use it to freely combine the separation logic specifications in the so-called “small footprint” style, with the classical “large footprint” approach. We believe that the former is preferable when specifying the lack of aliasing or freshness of newly generated locations, but the latter may be more compact in the cases when aliasing is allowed. Also, explicit naming comes handy when specifying the invariance of a subheap which the computation may read from during execution. Such a subheap is in the footprint of the computation, so that its invariance cannot be inferred using the Frame Rule of Separation Logic, but must instead be explicitly stated in the specification.

We further establish that HTT is sound, in the usual sense of Preservation and Progress theorems. That is, evaluation of HTT terms preserves the type

information, and well-typed terms never get stuck. Furthermore, we establish that HTT satisfies the customary substitution principles of dependent type theories, and also show that the usual Hoare-style rule for sequential composition is admissible. It may be interesting that the latter admissible rule also takes the form of a substitution principle, associated with the operation of monadic substitution of effectful programs.

The substitution principles formally establish the modular nature of HTT, and guarantee that separate verifications of program modules suffice to verify the composite program. Crucial in proving the substitution principles was the *syntax-directed* nature of HTT type-checking, ensuring that the verification conditions for any given computation depend on the computation alone, rather than on an unpredictable global context in which the computation may execute. This property is achieved by organizing the typing rules so as to compute the strongest postconditions for effectful code, following the general idea of *predicate transformers* (Dijkstra 1975). Thus, in addition to the “specifications-as-types” principle which we adopted by introducing Hoare types, we believe that HTT also provides a form of “proofs-as-programs” principle for effectful code (Howard 1980).

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