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ON THE COHOMOLOGY OF CLASSIFYING SPACES OF GROUPS OF HOMEOMORPHISMS

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Abstract Let M be a closed simply connected $2n$-dimensional manifold. The paper is concerned with the cohomology of classifying spaces of connected groups of homeomorphisms of M.

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1. Introduction and statement of the results

Let M be a closed simply connected $2n$-dimensional manifold. The present paper is concerned with the cohomology of classifying spaces of connected groups of homeomorphisms of M.

1.1. Conventions

We make the following assumptions throughout the paper. The topology on a group of homeomorphisms of a manifold is assumed to be compact-open. If G is a topological group, then G_0 denotes its connected component of the identity. We consider the cohomology with real coefficients unless otherwise specified.

1.2. Generic co-adjoint orbits

Theorem 1.1. *Let $M = G\xi \subset \mathfrak{g}^\vee$ be a generic co-adjoint orbit of a compact connected semi-simple Lie group G. Suppose that the action $G \to \mathrm{Homeo}(M)$ has a finite kernel. The homomorphism*

$$H^*(B\,\mathrm{Homeo}_0(M)) \to H^*(BG)$$

induced by the action is then surjective. It is surjective in degree 4 for every (not necessarily generic) co-adjoint orbit.

The proof of this theorem is given in § 5. *Genericity* means that there exists a non-empty Zariski open subset $Z \subset \mathfrak{g}^\vee$ of the dual of the Lie algebra of G such that the theorem holds for an orbit $G\xi$, where $\xi \in Z$. We discuss the generic orbits in § 6. The examples include complex projective spaces and flag manifolds. However, the subset Z for a given group G is not understood.

Remark 1.2. Theorem 1.1 and most of the results in this paper can be directly generalized to a connected topological monoid of homotopy equivalences (compare with Kędra and McDuff [**8**]).

The following result is an immediate consequence of Theorem 1.1.

Corollary 1.3. *Let $M = G\xi$ be as in Theorem 1.1. Let $\mathcal{H} \subset \mathrm{Homeo}(M)$ be a connected group of homeomorphisms containing G as a subgroup. The induced homomorphism*

$$H^*(B\mathcal{H}) \to H^*(BG)$$

is then surjective for a generic co-adjoint orbit M, and in degree 4 it is surjective for all orbits.

The most important examples of groups $\mathcal{H}$ to which we apply the above result are the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms, $\mathrm{Diff}_0(M)$, the connected component of the identity of the group of diffeomorphisms, and $\mathrm{Homeo}_0(M)$, the connected component of the identity of the group of homeomorphisms.

1.3. Circle actions

A circle action $\mathbb{S}^1 \to \mathcal{H} \subseteq \mathrm{Homeo}(M)$ is called *$\mathcal{H}$-inessential* if it defines a contractible loop in $\mathcal{H}$. For example, if a simply connected Lie group G acts on M, then a circle subgroup of G yields an inessential circle action. The following result generalizes the second part of Theorem 1.1.

Theorem 1.4. *Let (M_i,ω_i), $i = 1,2,\dots,m$, be a closed simply connected symplectic manifold admitting an inessential non-trivial Hamiltonian circle action. Let $M = M_1 \times M_2 \times \cdots \times M_m$ be equipped with a product symplectic form ω. If $\mathcal{H}$ is a connected group containing the product $\mathrm{Ham}(M_1,\omega_1) \times \cdots \times \mathrm{Ham}(M_m,\omega_m)$, then $\dim H^4(B\mathcal{H};\mathbb{R}) \geqslant m$ and $\operatorname{rank} \pi_3(\mathcal{H};\mathbb{R}) \geqslant m$.*

1.4. The fibre integral subalgebra

Let $G \to \mathrm{Homeo}_0(M)$ be an action of a connected topological group. In § 2, we define a certain graded subalgebra $\mathbb{A}^*_G(M) \subset H^*(BG)$ associated with the action. It is called the fibre integral subalgebra and it can be calculated in certain cases. It is our main technical tool and its basic properties are presented in § 3. The following observation is an application of Proposition 3.2 and Lemma 2.3.

Example 1.5. Let $M = \mathbb{CP}^{n_1} \times \cdots \times \mathbb{CP}^{n_l}$ be equipped with a product symplectic form ω invariant under the natural action of the product of special unitary groups. The induced homomorphism

$$H^*(B\,\mathrm{Diff}(M)) \to H^*(B(\mathrm{SU}(n_1+1) \times \cdots \times \mathrm{SU}(n_l+1)))$$

is then surjective and factors through $H^*(B\,\mathrm{Ham}(M,\omega))$. In particular,

$$\dim H^4(B\,\mathrm{Diff}(M,\omega)) \geqslant l.$$

We know from [**8**] that the relevant characteristic classes do not vanish on spheres, and, hence, we also have that

$$\mathrm{rank}(\pi_{2m+1}(\mathrm{Ham}(M,\omega))) \geqslant l \quad \text{for } 1 \leqslant m \leqslant \min\{n_1,\dots,n_l\}$$

and that the rank of the image of the homomorphism induced by the inclusion either in the diffeomorphisms or homeomorphisms is at least l.

We recall a result of Seidel from [**14**]. Let $0 \leqslant k < n \leqslant m$ and let

$$\Psi_k \colon \pi_{2k+1}(\mathrm{Ham}(\mathbb{CP}^m \times \mathbb{CP}^n)) \to \pi_{2k+1}(\mathrm{Diff}(\mathbb{CP}^m \times \mathbb{CP}^n))$$

be the homomorphism induced by the inclusion. Seidel proved that

$$\mathrm{rank}(\mathrm{coker}(\Psi_k)) = \begin{cases} 2n-k & \text{if } m-k \geqslant n, \\ m+n-2k & \text{if } m-k < n. \end{cases}$$

Applying the observation from the previous example for $l=2$ and Seidel's theorem, we obtain the following result.

Theorem 1.6. *Let $\mathbb{CP}^m \times \mathbb{CP}^n$ be equipped with a product symplectic form ω such that the symplectic areas of the lines in factors are equal. Suppose that $1 \leqslant k < n \leqslant m$. Then,*

$$\mathrm{rank}(\pi_{2k+1}(\mathrm{Diff}(\mathbb{CP}^m \times \mathbb{CP}^n))) \geqslant \begin{cases} 2n-k+2 & \text{if } m-k \geqslant n, \\ m+n-2k+2 & \text{if } m-k < n. \end{cases}$$

1.5. Relation to the previous work

The obvious strategy to understand the topology of the classifying space $B\mathcal{H}$ of a homeomorphism group $\mathcal{H}$ is to consider a map $f \colon B \to B\mathcal{H}$ defined on a space with understood topology and, for example, examine the induced map on the cohomology. In the present paper we mostly investigate the homomorphism $H^*(B\mathcal{H}) \to H^*(BG)$ for the natural action of a compact Lie group G on a homogeneous space G/H. Conjecturally, the homomorphism

$$H^*(B\,\mathrm{Homeo}_0(G/H)) \to H^*(BG)$$

should be surjective for the real cohomology provided that the action of G is effective.

Apart from the classical results about diffeomorphisms of low-dimensional spheres and surfaces, the first such surjectivity result was obtained by Reznikov [**13**]. He proved that the natural Hamiltonian action of $\mathrm{SU}(n)$ on the complex projective plane $\mathbb{CP}^{n-1}$ induces the surjection $H^*(B\,\mathrm{Ham}(\mathbb{CP}^{n-1})) \to H^*(B\,\mathrm{SU}(n))$. He proved it by using a Hamiltonian version of Chern–Weil theory. He also conjectured that a similar statement should be true for other co-adjoint orbits.

The result of Reznikov was improved and generalized to flag manifolds by Kędra and McDuff [**8**]. We proved that the characteristic classes defined by Reznikov are in fact topological in the sense that they can be defined in the cohomology ring of the topological monoid of homotopy equivalences of a symplectic manifold.

The algebraic independence of the Reznikov classes was proved by Gal *et al*. [**3**] for a generic co-adjoint orbit of a semi-simple Lie group. It was shown by examples that such classes cannot be algebraically independent in general (see Example 6.3).

The results in [**3**, **7**, **8**] are applications of the fibre integration. That is, certain characteristic classes are defined as fibre integrals. This is the main tool here as well. The new ingredient is that we consider distinct symplectic forms on a given manifold at the same time. More precisely, the Reznikov characteristic classes are equal to the fibre integrals of powers of a certain universal cohomology class called the coupling class. This class is induced by a fixed symplectic form. In this paper we consider fibre integrals of products of many coupling classes induced by distinct symplectic forms.

2. Strategy and a few technical results

2.1. Fibre integration

Let $M \to E \xrightarrow{\pi} B$ be an oriented bundle with closed n-dimensional fibre. There exists a homomorphism of $H^*(B)$-modules

$$\pi_!\colon H^{n+k}(E) \to H^k(B).$$

It is defined to be the composition

$$H^{n+k}(E) \to E_\infty^{k,n} \to E_2^{k,n} = H^k(B; H^n(M)) = H^k(B),$$

where the $E_m^{p,q}$ is an mth term of the associated Leray–Serre spectral sequence. The property that the fibre integration is a morphism of $H^*(B)$-modules means that

$$\pi_!(\alpha \cdot \pi^*(\beta)) = \pi_!(\alpha) \cdot \beta,$$

where $\alpha \in H^*(E)$ and $\beta \in H^*(B)$.

Moreover, fibre integration is multiplicative with respect to the cross product. More precisely, let $p_1\colon E_1 \to B_1$ and $p_2\colon E_2 \to B_2$ be oriented bundles with closed fibres. Then

$$(p_1 \times p_2)_!(\alpha \times \beta) = (p_1)_!(\alpha) \times (p_2)_!(\beta),$$

where $\alpha \in H^*(E_1)$ and $\beta \in H^*(E_2)$. This multiplicativity easily follows from the definition and good properties of the spectral sequence.

2.2. A very general view

Let G be a topological group acting on a closed oriented n-manifold M. Consider the associated universal fibration

$$M \xrightarrow{i} M_G \xrightarrow{\pi} BG$$

induced by the action. Given a subalgebra $\mathbb{A} \subset H^*(M_G)$, we consider a subalgebra

$$\langle \pi_!(\mathbb{A}) \rangle \subset H^*(BG)$$

generated by the fibre integrals of elements from $\mathbb{A}$. The strategy is to choose an appropriate subalgebra $\mathbb{A}$ for which one can make computations.

2.3. The fibre integral subalgebra

We assume that M is simply connected and G is connected and let $\mathbb{A}^* = \langle H^2(M_G) \rangle \subset H^*(M_G)$ be the subalgebra generated by the classes of degree 2. Define the *fibre integral subalgebra*

$$\mathbb{A}^*_G(M) := \langle \pi_!(\mathbb{A}^*) \rangle$$

associated with the action of G on M to be the graded subalgebra of $H^*(BG)$ generated by the fibre integrals of the products of cohomology classes of degree 2. In particular, an element of $\mathbb{A}^{2k}_G(M)$ is a linear combination of classes of the form $\pi_!(a_1 \cdots a_{n+k})$, where $a_i \in H^2(M_G)$ and $\dim M = 2n$. We say that the fibre integral subalgebra is *full* if it is equal to the whole of $H^*(BG)$.

2.4. The surjectivity lemma

Let $\mathcal{H} \subset \mathrm{Homeo}(M)$ be a connected group of homeomorphisms of a simply connected manifold M. Let

$$M \xrightarrow{i} M_{\mathcal{H}} \xrightarrow{\pi} B\mathcal{H}$$

be the universal bundle associated with the action of $\mathcal{H}$ on M.

Lemma 2.1. *The homomorphism $i^* \colon H^2(M_{\mathcal{H}}) \to H^2(M)$ induced by the inclusion of the fibre is surjective.*

Proof. Consider the associated Leray–Serre spectral sequence $E^{p,q}_m$. Since M and $B\mathcal{H}$ are simply connected, both the first row and the first column of the sequence are trivial. That is,

$$E^{p,q}_2 = H^p(B\mathcal{H}) \otimes H^q(M) = 0$$

if $p = 1$ or $q = 1$.

Let $a \in H^2(M) = E^{0,2}_2$ be a non-zero cohomology class. Since $E^{2,1}_2 = 0$, we have $d_2(a) = 0$. Thus, to finish the proof we need to show that $d_3(a) = 0$.

Since M is finite dimensional, there exists a number $k \in \mathbb{N}$ such that $a^k \neq 0$ and $a^{k+1} = 0$. Observe that

$$E^{3,2k}_3 \subset E^{3,2k}_2 = H^3(B\mathcal{H}) \otimes H^{2k}(M)$$

because the differential $d_2\colon E_2^{1,2k+1} \to E_2^{3,2k}$ is trivially 0 for $E_2^{1,2k+1} = 0$. Thus, the computation

$$0 = d_3(a^{k+1}) = (k+1)d_3(a) \otimes a^k$$

implies that $d_3(a) = 0$ is required. □

Remark 2.2. In fact, the above argument proves the following. Suppose that $F \to E \to B$ is a fibration over a simply connected base. Let $a \in H^2(F)$ be a cohomology class of finite cup length. If $d_2(a) = 0$, then $d_3(a) = 0$ (see [**8**, proof of Proposition 3.1]).

2.5. A dimension inequality

Lemma 2.3. *Let M be a simply connected closed manifold and let $\mathcal{H} \subseteq \mathrm{Homeo}(M)$ be a connected group of homeomorphisms. Let $G \subset \mathcal{H}$ be a connected subgroup with finite $\pi_1(G)$. Then,*

$$\dim \mathbb{A}_G^{2k}(M) \leqslant \dim \mathbb{A}_{\mathcal{H}}^{2k}(M) \leqslant \dim H^{2k}(B\mathcal{H}).$$

In particular, if $\mathbb{A}_G^*$ is full, then the homomorphism $H^*(B\mathcal{H}) \to H^*(BG)$ induced by the action is surjective.

Proof. Consider the following commutative diagram of fibrations:

$$\begin{tikzcd} M \arrow[r, "="] \arrow[d, "j"'] & M \arrow[d, "i"] \\ M_G \arrow[r, "F"] \arrow[d, "p"'] & M_{\mathcal{H}} \arrow[d, "\pi"] \\ BG \arrow[r, "f"] & B\mathcal{H} \end{tikzcd}$$

Since $\pi_1(G)$ is finite, $H^2(BG) = 0$ and the inclusion of the fibre $j\colon M \to M_G$ induces an isomorphism $j^*\colon H^2(M_G) \to H^2(M)$. It follows from Lemma 2.1 that the homomorphism $F^*\colon H^2(M_{\mathcal{H}}) \to H^2(M_G)$ is surjective and, hence, we have that

$$p_!(a_1 \cdots a_{n+k}) = p_!(F^*(\tilde{a}_1 \cdots \tilde{a}_{n+k})) = f^*(\pi_!(\tilde{a}_1 \cdots \tilde{a}^{n+k})),$$

which completes the proof. □

2.6. Cohomologically symplectic manifolds and coupling classes

A closed $2n$-manifold M is called *cohomologically symplectic*, or *c-symplectic* for short, if there exists a class $\alpha \in H^2(M)$ such that $\alpha^n \neq 0$. Such a class α is called a *symplectic class*.

Assume that M is simply connected. Let a topological group G act on M preserving a symplectic class α. Let

$$M \xrightarrow{i} E \xrightarrow{\pi} B$$

be a fibration with the structure group G. There exists a unique cohomology class $\Omega_E \in H^2(E)$ such that $i^*\Omega_E = \alpha$ and $\pi_!(\Omega_E^{n+1}) = 0$. The class Ω_E is called the *coupling class*. It is natural in the sense that the coupling class of a pullback bundle is the pullback of the coupling class. The symplectic class $\alpha \in H^2(M)$ is said to satisfy the *hard Lefschetz condition* if the multiplication by its kth power defines an isomorphism $H^{n-k}(M) \to H^{n+k}(M)$ for $k = 0, 1, \ldots, n$.

Example 2.4. All Kähler manifolds (e.g. co-adjoint orbits) satisfy the hard Lefschetz condition [**4**].

2.7. Consequences of the hard Lefschetz condition

The following lemma was first proved for complex algebraic manifolds by Blanchard [**1**]. The proof of the following topological version of the lemma can be found in [**10**].

Lemma 2.5 (Blanchard [1]). *Let M be a closed simply connected c-symplectic $2n$-manifold satisfying the hard Lefschetz condition. If $M \to E \to B$ is a bundle with a connected structure group $\mathcal{H} \subset \mathrm{Homeo}(M)$, then the homomorphism $i^*\colon H^*(E) \to H^*(M)$ induced by the inclusion of the fibre is surjective.*

Note that the surjectivity of the homomorphism i^* in the above lemma implies, due to the Leray–Hirsch theorem [**6**, Theorem 4D.1], that $H^*(E)$ is isomorphic as an $H^*(B)$-module to the tensor product $H^*(B) \otimes H^*(M)$. In particular, the homomorphism $p^*\colon H^*(B) \to H^*(E)$ induced by the projection is injective.

The next proposition is motivated by the fact that if a cohomology class of a space X evaluates non-trivially on a sphere, then it is indecomposable. That is, it cannot be expressed as the sum of products of classes of positive degree. Hence, one can think of such a class as a generator of the cohomology ring of X.

Proposition 2.6. *Let M be a closed simply connected c-symplectic $2n$-manifold satisfying the hard Lefschetz conditions. Let*

$$M \xrightarrow{j} E \xrightarrow{p} S^{2k}$$

be a bundle over a sphere of positive dimension and with a connected structure group $\mathcal{H} \subset \mathrm{Homeo}(M)$. Let $\sigma \in H^{2k}(S^{2k})$ denote a generator. If $p^*(\sigma) = \sum a \cdot b$, where all $a, b \in H^*(E)$ are of positive degree, then the homomorphism

$$f^*\colon H^{2k}(B\mathcal{H}) \to H^{2k}(S^{2k})$$

induced by the classifying map is surjective.

Proof. Since M satisfies the hard Lefschetz condition, $p^*(\sigma) \neq 0$. Next, observe that the base sphere has to be of dimension higher than 2. Indeed, if $k = 1$, then $p^*(\sigma) = \sum a \cdot b$ for $a, b \in H^1(E)$. Since E is simply connected, it implies that $p^*(\sigma) = 0$, which cannot happen. Consequently, we have $k > 1$. Note that the homomorphism $j^*\colon H^m(E) \to H^m(M)$ induced by the inclusion of the fibre is an isomorphism for $m < 2k$.

Let $M \xrightarrow{i} M_{\mathcal{H}} \xrightarrow{\pi} B\mathcal{H}$ be the universal fibration and let $\Omega \in H^2(M_{\mathcal{H}})$ be the coupling class associated with the symplectic class α. Let $\hat{a}, \hat{b} \in H^*(M_{\mathcal{H}})$ be such that $i^*\hat{a} = j^*a$ and $i^*\hat{b} = j^*b$.

In the following calculation, $\Omega_E = F^*(\Omega)$ denotes the coupling class. Also, since j^* is an isomorphism in degrees smaller than $2k$, we have that $F^*(\hat{a}) = a$ and $F^*(\hat{b}) = b$. This implies the second equality:

$$\begin{aligned} f^*\pi_!\Big(\Omega^n \sum \hat{a}\hat{b}\Big) &= p_!\Big(F^*\Big(\Omega^n \sum \hat{a}\hat{b}\Big)\Big) \\ &= p_!\Big(\Omega_E^n \sum ab\Big) \\ &= p_!(\Omega_E^n p^*\sigma) = \sigma\,\mathrm{volume}(M). \end{aligned}$$

$\square$

3. Basic properties of the fibre integral subalgebra

Throughout this section, M and N are assumed to be closed connected and simply connected manifolds.

Proposition 3.1. *Let $H \to G \to \mathrm{Homeo}_0(M)$ be a sequence of actions of connected topological groups on a manifold M. Let $f\colon BH \to BG$ denote the induced map. If $f^*\colon H^2(BG) \to H^2(BH)$ is surjective, then*

$$\mathbb{A}_H^*(M) \subset f^*(\mathbb{A}_G^*(M)).$$

Proof. Consider the diagram of universal fibrations:

$$\begin{tikzcd} M \arrow[r, "="] \arrow[d, "j"'] & M \arrow[d, "i"] \\ M_H \arrow[r, "F"] \arrow[d, "p"'] & M_G \arrow[d, "\pi"] \\ BH \arrow[r, "f"] & BG \end{tikzcd}$$

Let $p_!(\Omega_1 \cdots \Omega_k) \in \mathbb{A}_H^*(M)$, where $j^*(\Omega_i) = a_i$. According to Lemma 2.1, there exist classes $\hat{\Omega}_i \in H^2(M_G)$ such that $i^*(\hat{\Omega}_i) = a_i$. Since H and G are connected, their classifying spaces are simply connected and we have that

$$\Omega_i - F^*(\hat{\Omega}_i) = p^*(\alpha_i)$$

for some $\alpha_i \in H^2(BH)$. It follows from the hypothesis that $\alpha_i = f^*(\beta_i)$, and we get that

$$\Omega_i = F^*(\hat{\Omega}_i - \pi^*(\beta_i)).$$

We finally have that

$$p_!(\Omega_1 \cdots \Omega_k) = f^*\pi_!((\hat{\Omega}_1 - \pi^*\beta_1) \cdots (\hat{\Omega}_k - \pi^*\beta_k)),$$

which completes the proof. $\square$

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Proposition 3.2. *Let G and H be connected groups acting on manifolds M and N, respectively. Then, $G \times H$ acts on $M \times N$ and*

$$\mathbb{A}^*_{G\times H}(M \times N) \cong \mathbb{A}^*_G(M) \otimes \mathbb{A}^*_H(N).$$

In particular, if both $\mathbb{A}^*_G(M)$ and $\mathbb{A}^*_H(N)$ are full, then $\mathbb{A}^*_{G\times H}(M \times N)$ is also full.

Proof. The statement is true due to the multiplicativity property of the fibre integration with respect to the cross product and the isomorphism $H^*(BH \times BG) = H^*(BH) \otimes H^*(BG)$. $\square$

Proposition 3.3. *Let a connected group G act on M and N. The cup product in $H^*(BG)$ induces a map*

$$\mathbb{A}^*_G(M) \otimes \mathbb{A}^*_G(N) \to H^*(BG)$$

with the image equal to $\mathbb{A}^*_G(M\times N)$. In particular, if G acts on N trivially, then $\mathbb{A}^*_G(M) = \mathbb{A}^*_G(M \times N)$.

Proof. The map in the statement is the composition

$$\mathbb{A}^*_G(M) \otimes \mathbb{A}^*_G(N) \xrightarrow{\cong} \mathbb{A}^*_{G\times G}(M \times N) \xrightarrow{\Delta^*} H^*(BG),$$

where the first isomorphism is due to Proposition 3.2 and the second map is induced by the diagonal $\Delta\colon BG \to BG \times BG$.

It follows from Proposition 3.1 that $\mathbb{A}^*_G(M \times N)$ is contained in the image of the above map. Thus, we need to show that the converse inclusion holds:

$$\Delta^*(\mathbb{A}^*_{G\times G}(M \times N)) \subset \mathbb{A}^*_G(M \times N).$$

According to simple connectivity we have that

$$H^2(M_G \times N_G) = H^2(M_G) \oplus H^2(N_G)$$

and, hence, an element in the subalgebra of $H^*(M_G\times N_G)$ generated by degree 2 classes is a sum of products of the form $(\alpha_1 \cdots \alpha_k)\times(\beta_1 \cdots \beta_l)$ for $\alpha_i \in H^2(M_G)$ and $\beta_i \in H^2(N_G)$. Since this is itself a product of degree 2 classes, we get that its pullback via the map $\hat{\Delta}\colon (M \times N)_G \to M_G \times N_G$ is a product of degree 2 classes. This, according to the functoriality of the fibre integration, completes the proof. $\square$

Lemma 3.4. *Suppose that G is a connected group with finite $\pi_1(G)$ acting on a closed simply connected $2n$-manifold M. The fibre integral subalgebra $\mathbb{A}^*_G(M)$ is then generated by the fibre integrals of powers, i.e. by the classes of the form $\pi_!(a^m)$.*

Proof. Let $A \in H_{2k}(BG)$ be a homology class. A certain non-zero multiple of A is represented by a map $f\colon B \to BG$ defined on a closed oriented connected $2k$-manifold B. Suppose that

$$f^*(\pi_!(a_1 \cdots a_{n+k})) \neq 0,$$

where $a_i \in H^2(M_G)$. The map f induces a bundle $M \to E \to B$, and the above inequality is equivalent to

$$0 \neq F^*(a_1 \cdots a_{n+k}) \in H^{2(n+k)}(E) = \mathbb{R},$$

where $F\colon E \to M_G$ is the induced map of total spaces. It follows that the product map

$$F^*(H^2(M_G)) \otimes \cdots \otimes F^*(H^2(M_G)) \to H^{2(n+k)}(E) = \mathbb{R}$$

is non-trivial. Since it is a polynomial map, due to the polarization formula, the power map $F^*(a) \mapsto F^*(a)^{n+k}$ is also non-trivial.

This shows that $f^*(\pi_!(a^{n+k})) \neq 0$ for some class $a \in H^2(M_G)$.

We have shown that, for every homology class in $A \in H_{2k}(BG)$ that evaluates non-trivially on the fibre integral subalgebra, there exists a class $a \in H^2(M_G)$ such that the fibre integral of its $(n+k)$th power evaluates non-trivially on A. This proves that such fibre integrals generate $\mathbb{A}^*_G(M)$. □

Remark 3.5. Since the power map $F^*(H^2(M_G)) \to H^{2(n+k)}(E)$ in the above proof is polynomial and non-trivial, there exists a non-empty open and dense subset $U \subset F^*(H^2(M_G))$ such that $a^{n+k} \neq 0$ for $a \in U$.

4. Calculations for co-adjoint orbits

4.1. Symplectic preliminaries

Let G be a compact connected semi-simple Lie group and let $\xi \in \mathfrak{g}^\vee$ be a covector. The co-adjoint orbit $G \cdot \xi$ admits a G-invariant symplectic form. The Killing form provides an equivariant isomorphism between the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^\vee$, and, hence, also a bijective correspondence between adjoint and co-adjoint orbits.

Let $T \subset G$ be a maximal torus and denote by $\mathfrak{t}$ its Lie algebra. Every adjoint orbit has a representative in $\mathfrak{t}$. The Lie algebra $\mathfrak{t}$ is decomposed into the Weyl chambers. Let $C \subset \mathfrak{t}$ denote the closure of a Weyl chamber. It is a polyhedral cone. If two elements $\xi, \eta \in \mathfrak{t}$ belong to the interior of a face of C, then the corresponding adjoint orbits $G \cdot \xi$ and $G \cdot \eta$ are diffeomorphic. In this case the isotropy groups G_ξ and G_η are conjugate in G. The conjugation by an element of G provides a G-equivariant diffeomorphism between the orbits $G \cdot \xi$ and $G \cdot \eta$.

Thus, we can fix one orbit M and consider it as a smooth manifold equipped with various G-invariant symplectic forms. Since the conjugation induces a map of BG homotopic to the identity, the universal fibration $M \to M_G \to BG$ is Hamiltonian with respect to these symplectic forms. In such a case we have the coupling class $\Omega_\xi \in H^2(M_G)$ corresponding to the symplectic form on the orbit $G \cdot \xi \cong M$.

4.2. Flag manifolds

Let $\dim G/T = 2n$ and consider the universal bundle $G/T \to BT \xrightarrow{\pi} BG$ associated with the action.

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Lemma 4.1. *Let G be a connected group with finite $\pi_1(G)$. The fibre integral subalgebra $\mathbb{A}^*_G(G/T)$ is full. Moreover, it is generated by the fibre integrals of powers of coupling classes.*

Proof. Let $\dim G/T = 2n$ and let $G/T \to BT \xrightarrow{\pi} BG$ be the universal bundle associated with the action. Consider the composition

$$H^2(BT) \otimes \cdots \otimes H^2(BT) \to H^{2(n+k)}(BT) \xrightarrow{\pi_!} H^{2k}(BG),$$

where the first map is the product and the second is the fibre integration. Observe that the first map is polynomial and surjective, since $H^*(BT)$ is the polynomial algebra generated by $H^2(BT)$. The second map is surjective, which follows from the injectivity of $\pi^*\colon H^*(BG) \to H^*(BT)$. Indeed, if $b \in H^{2k}(BG)$, then $\pi_!(\pi^*(b) \cdot \Omega^n) = b$ for a coupling class $\Omega \in H^2(BT)$. This proves that the fibre integral subalgebra is full.

The second statement follows from Lemma 3.4 and Remark 3.5. Indeed, there exists an open and dense subset of $H^2(BT)$ consisting of coupling classes. In other words, a generic cohomology class in $H^2(BT)$ pulls back to a class represented by an invariant symplectic form on the flag manifold G/T. □

4.3. Fibre integral of a power of the coupling class as an invariant polynomial

The cohomology of the classifying space of a compact Lie group is isomorphic to the algebra of invariant polynomials on the Lie algebra

$$H^{2k}(BG) \cong S^k(\mathfrak{g}^\vee)^G.$$

The right-hand side is isomorphic to $S^k(\mathfrak{t})^{W_G}$, the polynomials on the Lie algebra of the maximal torus invariant under the Weyl group of G. The next lemma follows from [**8**, Lemmas 3.6 and 3.9].

Lemma 4.2. *Let $M = G \cdot \xi$ be a $2n$-dimensional co-adjoint orbit of a semi-simple Lie group G. The fibre integral of the $(n+k)$th power of the coupling class $\Omega_\xi \in H^2(M_G)$ corresponds to the following invariant polynomial:*

$$P_k(\xi, X) := (-1)^k \binom{n+k}{k} \cdot \int_G \langle X, \mathrm{Ad}_g^\vee(\xi)\rangle^k \,\mathrm{vol}_G\,.$$

Since the polynomials $P_k(\xi, X)$ depend continuously (with respect to the Zariski topology) on ξ, and since the algebraic independence is an open condition, we obtain the following result.

Proposition 4.3. *Let $\xi \in \mathfrak{g}^\vee$. There exists a Zariski open neighbourhood $Z \subset \mathfrak{g}^\vee$ of ξ such that*

$$\mathbb{A}^*_G(G\xi) \subset \mathbb{A}^*_G(G\eta)$$

for every $\eta \in Z$.

Corollary 4.4. *If $M = G\cdot\xi \subset \mathfrak{g}^\vee$ is a generic co-adjoint orbit of a compact semi-simple Lie group G, then the fibre integral subalgebra $\mathbb{A}^*_G(M)$ is full.*

Proof. It follows from Proposition 4.3 that if there is an orbit with the full fibre integral subalgebra, then a generic co-adjoint orbit has full fibre integral subalgebra. The statement follows from Lemma 4.1. □

5. Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.1. According to Corollary 4.4 we have that the fibre integral subalgebra $\mathbb{A}^*_G(M)$ is full for a generic co-adjoint orbit M. Since G is semi-simple and its fundamental group is finite, the first statement then follows from Lemma 2.3.

To prove the second statement, first observe that a compact semi-simple group G is finitely covered by a product $G_1 \times \cdots \times G_m$ of simple groups. Thus, there is a splitting of the Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$.

The composition $G_1 \times \cdots \times G_m \to G \to \mathrm{Aut}(\mathfrak{g}^\vee)$ of the covering projection and the co-adjoint action is the co-adjoint action of the product, and, hence, it is the product of co-adjoint actions $\mathrm{Ad}^\vee : G_i \to \mathrm{Aut}(\mathfrak{g}_i^\vee)$. Thus, M is diffeomorphic to a product $M_1 \times \cdots \times M_m$ of the corresponding co-adjoint orbits of simple groups.

We next show that each of the above orbits M_i is of positive dimension. This follows from the assumption on the kernel of the action. Indeed, since the kernel of the action $\mathrm{Ad}^\vee : G \to \mathrm{Aut}(\mathfrak{g}^\vee)$ is finite, so is the kernel of the co-adjoint action of the product. But the latter is isomorphic to the product of the kernels of $\mathrm{Ad}^\vee : G_i \to \mathrm{Aut}(\mathfrak{g}^\vee)$ and, hence, each M_i is a non-trivial orbit.

The statement now follows from Theorem 1.4, proof of which we present next. □

Proof of Theorem 1.4. Let $\dim M = 2n$, $\dim M_i = 2n_i$ and let $\omega_a = \sum a_i\omega_i$, where $a := (a_1, \ldots, a_m)$ is an m-tuple of positive real numbers. Let $\Omega_a \in H^2(M_{\mathcal{H}})$ be the coupling class associated with the symplectic form ω_a:

$$\begin{tikzcd}
M_1 \times \cdots \times M_m \arrow[r, "="] \arrow[d] & M \arrow[d] \\
(M_1)_{\mathrm{Ham}} \times \cdots \times (M_m)_{\mathrm{Ham}} \arrow[r, "F"] \arrow[d, "p"] & M_{\mathcal{H}} \arrow[d, "\pi"] \\
B\mathrm{Ham}(M_1, \omega_1) \times \cdots \times B\mathrm{Ham}(M_m, \omega_m) \arrow[r, "f"] & B\mathcal{H}
\end{tikzcd}$$

The pullback of the coupling class $F^*(\Omega_a)$ is equal to the sum of coupling classes $\sum a_i\Omega_i$. In the following computation C_k is a positive constant depending on a tuple $k = (k_1, \ldots, k_m)$ of non-negative integers, and the fourth equality follows from the multiplicativity of the fibre integration with respect to the cross product:

$$\begin{aligned} f^*(\pi_!(\Omega_a)^{n+2}) &= p_!(F^*(\Omega_a)^{n+2}) \\ &= p_!\Big(\Big(\sum a_i\Omega_i\Big)^{n+2}\Big) \\ &= \sum C_k p_!((a_1\Omega_1)^{k_1} \times \cdots \times (a_m\Omega_m)^{k_m}) \\ &= \sum C_k (p_1)_!((a_1\Omega_1)^{k_1}) \times \cdots \times (p_m)_!((a_m\Omega_m)^{k_m}). \end{aligned}$$

Note that the fibre integral $(p_i)_!(\Omega_i^k) = 0$ if $k < n_i$. Thus, the above sum is, up to a positive constant, equal to the sum of terms of the form

$$(p_1)_!(a_1\Omega_1)^{n_1} \times \cdots \times (p_i)_!(a_i\Omega_i)^{n_i+2} \times \cdots \times (p_m)_!(a_m\Omega_m)^{n_m}.$$

It follows from [**8**, Theorem 1.1] that an inessential non-trivial circle action induces an element of $\sigma_i \in \pi_4(B\operatorname{Ham}(M_i))$ on which the class $(p_i)_!(\Omega_i^{n+2})$ evaluates non-trivially. By varying the m-tuple $a = (a_1, \dots, a_m)$ we obtain that the image of the homomorphism

$$f^*\colon H^4(B\mathcal{H}) \to H^4(B\operatorname{Ham}(M_1,\omega_1) \times \cdots \times B\operatorname{Ham}(M_m,\omega_m))$$

is at least m dimensional.

Choosing the parameters a appropriately, the classes $\pi_!(\Omega_a^{n+2})$ define m linearly independent functionals on $\pi_4(B\mathcal{H}) \otimes \mathbb{R}$. Evaluating them on the images of the classes σ_i, we obtain that the rank $\pi_4(B\mathcal{H})$ is at least m. This rank is equal to the rank of $\pi_3(\mathcal{H})$. $\square$

6. Examples

Let G be a compact connected semi-simple Lie group with a maximal torus T. Let $\mathfrak{g}$ and $\mathfrak{t}$ denote the corresponding Lie algebras. The closed positive Weyl chamber $C \subset \mathfrak{t}$ is a simplicial cone. Let F be a face of C. If ξ and η belong to the interior of F, then they are diffeomorphic. Moreover, if $\xi \in \operatorname{interior}(F)$, then

$$\dim H^2(G \cdot \xi) = \dim F.$$

Conversely, a co-adjoint orbit $G \cdot \xi$ has a representative that belongs to the interior of a face of dimension equal to $\dim H^2(G \cdot \xi)$.

Remark 6.1. The above observations can be deduced from Bott's results in [**2**] (see also [**11**, § 2] and [**5**, § 2.3]).

Proposition 4.3 states that $\mathbb{A}_G^*(G \cdot \xi) \subset \mathbb{A}_G^*(G \cdot \eta)$ for any η in an open and dense neighbourhood $U \subset C$ of ξ. Thus, most interesting are the orbits corresponding to the edges (i.e. one-dimensional faces) of C. This is because near an edge there are points corresponding to many topologically distinct orbits. The orbits corresponding to edges are characterized by their second Betti number being equal to 1. They are called *minimal* in the terminology of Guillemin *et al.* [**5**].

Example 6.2. The minimal co-adjoint orbits of $\mathrm{SU}(n)$ are the complex Grassmannians $\mathrm{G}(k, n)$.

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6.1. Special unitary group $\mathrm{SU}(n)$

It is known that the fibre integral subalgebra associated with the natural projective action of $\mathrm{SU}(n)$ on the complex projective space $\mathbb{CP}^{n-1}$ is full [**8**, Proposition 1.7]. Thus, it follows from Proposition 4.3 that every co-adjoint orbit M of $\mathrm{SU}(n)$ close to the projective space has a full fibre integral subalgebra $\mathbb{A}^*_{\mathrm{SU}(n)}(M)$. Such orbits are of the form

$$\mathrm{SU}(n)/\mathrm{S}(\mathrm{U}(n_1) \oplus \cdots \oplus \mathrm{U}(n_k) \oplus \mathrm{U}(1)),$$

where $n_1 + \cdots + n_k + 1 = n$.

Example 6.3 (Gal *et al.* [3, Proposition 3.6]). The fibre integral subalgebra of the complex Grassmannian $\mathrm{G}(n, 2n)$ with respect to the natural action of $\mathrm{SU}(2n)$ is not full. More precisely, $H^6(B\,\mathrm{SU}(2n))$ is not contained in the fibre integral subalgebra. This is due to the fact that the relevant invariant polynomial has odd degree and its zero represents the Grassmannian (see [**3**, § 3] for more examples and details).

Remark 6.4. It is shown in [**8**, Proposition 4.8] that the natural action of $\mathrm{SU}(n)$ on a generalized flag manifold M induces a surjective homomorphism $H^*(B\,\mathrm{Homeo}_0(M)) \to H^*(B\,\mathrm{SU}(n))$. The proof is also an application of fibre integrals. It is, however, specialized to this particular case.

6.2. Special orthogonal group $\mathrm{SO}(2m)$, $m > 2$

The cohomology ring of the classifying space of the special orthogonal group is generated by the Pontryagin classes $p_1, \ldots, p_m$ and the Euler class e. They have degrees $\deg(p_k) = 4k$ and $\deg(e) = 2m$, respectively.

Consider a minimal co-adjoint orbit of the form $M := \mathrm{SO}(2m)/\mathrm{U}(m)$. The classes $p_1, e, p_{m+1}, \ldots, p_{2m}$ belong to the fibre integral subalgebra $\mathbb{A}^*_{\mathrm{SO}(2m)}(M)$. For the first Pontryagin class this follows from the second part of Theorem 1.1.

To see that the higher Pontryagin classes p_k for $k > m$ belong to the fibre integral subalgebra, consider the map $\pi \colon B\,\mathrm{U}(m) \to B\,\mathrm{SO}(2m)$ induced by the inclusion. The pullback of a Pontryagin class is expressed in terms of Chern classes by the well-known formula

$$\pi^*(p_k) = c_k^2 - 2c_{k-1}c_{k+1} + \cdots \pm 2c_{2k}.$$

Thus, if $k > m$, then $\pi^*(p_k)$ is a sum of products of classes of positive degrees, and Proposition 2.6 applies.

Finally, a result of Reznikov [**9**, **13**] implies that the Euler class belongs to the fibre integral subalgebra.

Since $H^2(M) = \mathbb{R}$, there are, in general, orbits close to M that are topologically different from M. Their fibre integral subalgebras contain $\mathbb{A}^*_{\mathrm{SO}(2m)}(M)$.

Remark 6.5. We excluded the case of $\mathrm{SO}(4)$ because in that case the action of the group $\mathrm{SO}(4)$ on the orbit $\mathrm{SO}(4)/\mathrm{U}(2) = \mathbb{CP}^1$ is not effective.

7. A remark on stability

If G acts on a manifold M, then it also acts on the product $M \times N$ (acting trivially on the second factor). We think of the composition of actions

$$f \colon G \to \operatorname{Homeo}(M) \to \operatorname{Homeo}(M \times N)$$

as a kind of stabilization. If G is connected, then the second part of Proposition 3.3 states that $\mathbb{A}^*_G(M) = \mathbb{A}^*_G(M \times N)$ and, hence,

$$\mathbb{A}^*_G(M) \subset f^*(B \operatorname{Homeo}_0(M \times N)).$$

This means that the part of the topology of the classifying space of the group of homeomorphisms of M captured by the fibre integral subalgebra persists when we stabilize M.

Example 7.1. Let M be a co-adjoint orbit of a semi-simple compact Lie group G. Let N be a closed simply connected symplectic manifold. The fibre integral subalgebra $\mathbb{A}^*_G(M)$ is then contained in the image of the homomorphism

$$H^*(B \operatorname{Ham}(M \times N)) \to H^*(BG)$$

induced by the action of G on the product. In particular, if the fibre integral subalgebra is full, then the above homomorphism is surjective.

Remark 7.2. The fundamental group of the group of Hamiltonian diffeomorphisms of a product symplectic manifold has been recently investigated by Pedroza in [**12**].

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