NOTES ON SPLITTING EXTENSIONS OF GROUPS

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In [1] Gaschütz has shown that a finite group G splits over an abelian normal subgroup N if its Frattini subgroup $\phi(G)$ intersects N trivially. When N is a non-abelian nilpotent normal subgroup of G the condition $\phi(G) \cap N = 1$ cannot be satisfied: for if N is non-abelian then the commutator subgroup C(N) of N is non-trivial. Now N is nilpotent, whence $1 \neq C(N) \subset \phi(N)$. Since G is a finite group, therefore, by (3, theorem 7.3.17) $\phi(N) \subset \phi(G)$. It follows that $\phi(G) \cap N \neq 1$. Thus the condition $\phi(G) \cap N = 1$ must be modified. In §1 we shall derive some similar type of conditions for G to split over N when the restriction of N being an abelian normal subgroup is removed. In §2 we shall give a characterization of splitting extensions of N in which every subgroup splits over its intersection with N.

1. In this section G will always mean finite group.

LEMMA 1.1. *Let N be a nilpotent normal subgroup of G and L any subgroup of G. If $\phi(L) \cap N = 1$ then $L \cap N$ is abelian.*

Proof. Let $A = L \cap N$. Since N is a nilpotent normal subgroup of G, therefore, A is a nilpotent normal subgroup of L. Thus $C(A) \subset \phi(A) \subset \phi(L)$, where C(A) is the commutator subgroup of A. Hence $C(A) \neq 1$ will imply $\phi(L) \cap N \neq 1$, whence A must be abelian.

THEOREM 1.2. *Let N be a nilpotent normal subgroup of G. G splits over N if and only if G contains a subgroup L such that $G = LN$ and $\phi(L) \cap N = 1$.*

Proof. We need only prove the second half of the theorem. Let $A = L \cap N$. Now $\phi(L) \cap N = 1$. Therefore, by lemma 1.1, A is abelian. Moreover by Gaschütz's theorem L splits over A. Let C be a complement of A in L. It is clear that C is also a complement of N in G.

THEOREM 1.3. *Let N be a solvable normal subgroup of G. G splits over N if and only if G contains a subgroup L, minimal with respect to $G = LN$, such that $\phi(L) \cap N = 1$.*

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Proof. Again we shall only prove the second half of the theorem. Let $A = L \cap N$. Since N is a solvable normal subgroup of G, A is a solvable normal subgroup of L. Let A be a solvable group of length n. This means that the (n-1)th derived subgroup $A^{(n-1)}$ is abelian. Since $A^{(n-1)}$ is fully invariant in A it follows that $A^{(n-1)}$ is an abelian normal subgroup of L. Moreover $\phi(L) \cap A^{(n-1)} = 1$. Therefore L splits over $A^{(n-1)}$. But this will violate the hypothesis that L is minimal with respect to $G = LN$ unless $A = 1$. Hence L is a complement of N in G.

Applying a similar argument, the following statement can be proved.

THEOREM 1.4. _If_ N _is a normal subgroup of_ G _such that there exists a solvable subgroup_ L _of_ G, _minimal with respect to_ $G = LN$, _and_ $\phi(L) \cap N = 1$, _then_ G _splits over_ N.

THEOREM 1.5. G _splits over a normal subgroup_ N _if and only if_ G _contains a subgroup_ L, _minimal with respect to_ $G = LN$, _such that for each prime_ p _there is a Sylow p-subgroup_ T_p _of_ L _such that_ $\phi(T_p) \cap N = 1$.

Proof. The first part of the theorem is obvious. Let L be minimal with respect to $G = LN$ and $A = L \cap N$. Let U_p be a Sylow p-subgroup of A and V_p be a Sylow p-subgroup of L containing U_p. Since for each p there is a Sylow p-subgroup T_p of L such that $\phi(T_p) \cap N = 1$ and since the Sylow p-subgroups of L are conjugate to each other it follows that $\phi(V_p) \cap N = 1$. But $V_p \cap N = U_p$. Thus U_p is a nilpotent normal subgroup of V_p. Moreover $\phi(V_p) \cap U_p = 1$. Hence, by theorem 1.2, $V_p = C_p . U_p$, where C_p is a complement of U_p in V_p. Now let W_p be a Sylow p-subgroup of N containing U_p. Clearly V_p and W_p generate a Sylow p-subgroup S_p of G with $S_p \cap N = W_p$. Moreover $S_p = C_p . W_p$ and $C_p \cap W_p = 1$. Thus C_p is a complement of $S_p \cap N$ in S_p. Now $C_p \subset V_p \subset L$ and L is minimal with respect to $G = LN$. Hence by theorem 1 of [2] G splits over N. Indeed L is a complement of N in G.

In theorems 1.3, 1.4 and 1.5, L turns out to be a complement of N. However D.G. Higman in [2] pointed out that in general the minimality of L with respect to $G = LN$ does not necessarily mean that L is a complement of N even in the case when N is abelian.

In the following theorem, we shall show that, if N is a nilpotent normal subgroup of G, then the minimality of L with respect to $G = LN$ is characterized by $L \cap N \subset \phi(L)$.

THEOREM 1.6. Let N be a nilpotent normal subgroup of G. L is minimal with respect to $G = LN$ if and only if $L \cap N \subset \phi(L)$.

Proof. Let L be minimal with respect to $G = LN$. Let $A = L \cap N$ and $B = \phi(L) \cap A$. Clearly B is normal in L and A. Let $\bar{L} = L/B$ and $\bar{A} = A/B$. Since $B \subset \phi(L)$ we have $\phi(\bar{L}) = \phi(L)/B$. Therefore $\bar{A} \cap \phi(\bar{L}) = 1$. But A is a nilpotent normal subgroup of L. It follows that $A' \subset \phi(A) \subset \phi(L)$, where A' is the commutator subgroup of A. Thus $\bar{A}$ is abelian. Hence by Gaschütz's theorem $\bar{L}$ splits over $\bar{A}$. Let $\bar{C}$ be a complement of $\bar{A}$ in $\bar{L}$. Let C be the set of all preimages of $\bar{C}$ in L. If $\bar{A} \neq 1$ then C is a proper subgroup of L. But clearly $G = CN$. Hence $\bar{A} = 1$ in view of the fact that L is minimal with respect to $G = LN$. It follows therefore $A = L \cap N \subset \phi(L)$.

Conversely, let $L \cap N \subset \phi(L)$. We shall show that L must be minimal with respect to $G = LN$. Let Q be any subgroup of L such that $G = QN$. Let $B = \phi(L) \cap N$. Clearly B is normal in G. Let $\bar{G} = G/B$, $\bar{L} = L/B$ and $\bar{N} = N/B$. Thus $\bar{G} = \bar{L}.\bar{N}$ where $\bar{L} \cap \bar{N} = 1$. Since $G = QN$, we have $\bar{G} = \overline{QN}$ where $\bar{Q} = QB/B$. But $\bar{Q} \subset \bar{L}$. Therefore $\bar{Q} \cap \bar{N} = 1$. Hence $\bar{Q} = \bar{L}$. Since $QB/B \approx Q/Q \cap B$, it follows that $L = QB$. But $B \subset \phi(L)$. Hence $L = Q$. This completes the proof.

2. In general if a group G splits over a normal subgroup N the subgroups of G may not necessarily split over their intersections with N. In this section we shall characterize normal subgroups of G having the property that every subgroup S of G splits over $S \cap N$.

DEFINITION 2.1. A subgroup N of G is said to be hereditarily non-Frattini in G if, for every non-trivial subgroup S of G, $S \cap N \not\subset \phi(S)$ unless $S \cap N = 1$.

LEMMA 2.2. Let G be any finitely generated group and N be a normal subgroup of G such that $N \not\subset \phi(G)$. Then G splits over N if every maximal subgroup S of G splits over $S \cap N$.

Proof. Since $N \not\subset \phi(G)$, there exists a maximal subgroup S of G such that $N \not\subset S$. By hypothesis, S splits over $S \cap N = A$. Let C be a complement of A in S. Let x be an element of N not contained in S. Then,

$$G = \{x, S\} = \{x, CA\} = \{x, C\}\{x, A\} .$$

We shall show that $G = CN$.

Let $g \in G$. Then $G = uv$, where u is a word in x and elements of C and $v \in \{x, A\} \subset N$. Since $x \in N$, u can be expressed

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as $u = cw$, where $c \in C$ and $w \in N$. Thus $g = cwv$, whence $G = CN$. Moreover $C \cap N = C \cap A = 1$. Hence C is a complement of N in G.

THEOREM 2.3. _Let_ G _be a finite group and_ N _be a normal subgroup of_ G. _Then every subgroup_ S _of_ G _splits over_ $S \cap N$ _if and only if_ N _is hereditarily non-Frattini in_ G.

Proof. Suppose every subgroup S of G splits over $S \cap N$. Then clearly $S \cap N$ cannot be contained in $\phi(S)$ unless $S \cap N = 1$. Hence N is hereditarily non-Frattini in G.

To prove the converse we shall apply induction on the order of G. Let S be any proper subgroup of G and let $A = S \cap N$. Moreover, since N is hereditarily non-Frattini, it follows that A is hereditarily non-Frattini in S. Since S is a proper subgroup of G, therefore, by induction every subgroup T of S splits over $T \cap A$. In particular S splits over A. Thus every maximal subgroup of G splits over its intersection with N. Since $N \not\subset \phi(G)$, therefore by Lemma 2.2 G splits over N.

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