

## BLASCHKE-TYPE MAPS AND HARMONIC MAJORATION ON RIEMANN SURFACES

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An analytic map  $h$  of type  $\mathcal{B}\mathcal{L}$  from a Riemann surface  $R$  into another  $S$ , both having Green's functions, behaves well near the "boundary" of  $R$ . Let  $X$  stand for a family of holomorphic functions, and let  $f$  be holomorphic on  $S$ . We shall show, for several  $X$ 's, the following:

- (i)  $f \in X(S) \Leftrightarrow f \circ h \in X(R)$ ;
- (ii)  $\|f \circ h\| = \|f\|$ .

Use is made of harmonic majoration of subharmonic functions on  $R$  and on  $S$ .

### 1. Introduction

A Riemann surface  $R$  is called hyperbolic if  $R$  admits a Green's function  $g_R(z, w)$  with pole  $w \in R$ . In the present paper,  $R$  and  $S$  denote hyperbolic Riemann surfaces. Let  $h: R \rightarrow S$  be a nonconstant analytic map of type  $\mathcal{B}\mathcal{L}$  in the sense of Heins [3, p. 440], namely, for each fixed  $w \in S$  the superharmonic function  $g_S(h(z), w)$  in  $R$  does not majorize any strictly positive and bounded harmonic function on  $R$ . Here, we say that a function  $f_1$  majorizes another  $f_2$  on  $R$  if  $f_1 \geq f_2$  on  $R$ . Let  $X(R)$  be a family of holomorphic functions on  $R$ . A motivation of the present paper arises from the following.

**PROPOSITION X.** *Let  $h: R \rightarrow S$  be as above and suppose that  $f$  is holomorphic on  $S$ . Then,  $f \in X(S)$  if and only if  $f \circ h \in X(R)$ , and in this case, the "norm" is invariant; symbolically,  $\|f \circ h\| = \|f\|$ .*

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Received 8 January 1985

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\$A2.00 + 0.00.

To prove that Proposition X is valid for some  $X$ 's , we need our main theorem. If a subharmonic function  $u$  on  $R$  is majorized by an harmonic function on  $R$  , then the least harmonic majorant  $u_{\hat{R}}$  of  $u$  , the smallest among all harmonic functions majorizing  $u$  , exists.

**THEOREM 1.** *Let  $h: R \rightarrow S$  be a nonconstant map of type Bl , and let  $u$  be a subharmonic function on  $S$  majorizing a harmonic function on  $S$  . Then,*

(I)  $u_{\hat{S}}$  exists if and only if  $(u \circ h)_{\hat{R}}$  exists; if this is the case, then  $u_{\hat{S}} \circ h = (u \circ h)_{\hat{R}}$  on  $R$  ;

(II) furthermore,

$$(1.1) \quad \sup_{z \in R} [(u \circ h)_{\hat{R}} - u \circ h](z) = \sup_{z \in S} (u_{\hat{S}} - u)(z).$$

Note that if  $h: R \rightarrow S$  is an arbitrary analytic map, and if  $u_{\hat{S}}$  exists, then  $(u \circ h)_{\hat{R}}$  exists with  $(u \circ h)_{\hat{R}} \leq u_{\hat{S}} \circ h$  .

**THEOREM 2.** *Proposition X is true for*

$$X = N, N^+, H^p, BMOA, H^p_{\sigma} \text{ and } BMOA_{\sigma} \quad (0 < p < \infty).$$

Detailed explanations of "norms" for  $X$ 's in Theorem 2 will be postponed. The class  $N(R)$  consists of  $f$  such that  $\log^+ |f| = \max(\log |f|, 0)$  is majorized by a harmonic function on  $R$ . Heins [4, Theorems 11.1 and 11.2, p. 440] shows that if  $f$  is meromorphic on  $S$  , then  $f$  is Lindelöfian on  $S$  if and only if  $f \circ h$  is Lindelöfian. As a consequence,  $f \in N(S)$  if and only if  $f \circ h \in N(R)$  . We shall give another proof of this. Theorem 2 for  $X = N$  asserts much more about the "norm".

The other  $X$ 's are:

$N^+$  : the Smirnov class [10];

$H^p$  : the Hardy class [7], [8];

$BMOA$  : the family of holomorphic functions of bounded mean oscillation [6];

$H^p_{\sigma}$  : the hyperbolic Hardy class [12], [13];

$BMOA_{\sigma}$  : the family of holomorphic functions of hyperbolically bounded mean oscillation [11].

2. Proof of Theorem 1

The core of the proof is to establish (I) for the case  $R = S = \Delta \equiv \{|z| < 1\}$ .

LEMMA 2.1. *Let  $u$  be subharmonic in  $\Delta$  and majorize a harmonic function there. Let  $h: \Delta \rightarrow \Delta$  be of type Bl, or, equivalently, a nonconstant inner function [1, p. 24], [3, p. 454]. Then  $u^\wedge_\Delta$  exists if and only if  $(u \circ h)^\wedge_\Delta$  exists; if this is the case, then*

$$(2.1) \quad u^\wedge_\Delta \circ h = (u \circ h)^\wedge_\Delta .$$

Proof. For simplicity we write  $v^\wedge = v^\wedge_\Delta$ . We may assume that  $u \geq 0$  and  $u$  is nonconstant. Actually, let  $w$  be harmonic in  $\Delta$  with  $w \leq u$ . Then

$$(u-w)^\wedge + w = u^\wedge \quad \text{and} \quad ((u-w) \circ h)^\wedge + w \circ h = (u \circ h)^\wedge ,$$

whence (2.1) holds if and only if

$$(u-w)^\wedge \circ h = ((u-w) \circ h)^\wedge .$$

First of all, if  $u^\wedge$  exists, then  $(u \circ h)^\wedge$  exists because  $u \circ h \leq u^\wedge \circ h$ . Thus, we must show that if  $(u \circ h)^\wedge$  exists, then  $u^\wedge$  exists and (2.1) holds. Furthermore, it suffices to show that if  $(u \circ h)^\wedge$  exists, then

$$(2.2) \quad u^\wedge \circ h(0) = (u \circ h)^\wedge(0) .$$

For arbitrary  $w \in \Delta$  we set  $T_w(z) = (z+w)/(1+\bar{w}z)$ . Then,

$$\begin{aligned} (u \circ h)^\wedge &= (u \circ h \circ T_w \circ T_{-w})^\wedge \leq (u \circ h \circ T_w)^\wedge \circ T_{-w} \\ &\leq (u \circ h)^\wedge \circ T_w \circ T_{-w} = (u \circ h)^\wedge \end{aligned}$$

so that

$$(u \circ h)^\wedge \circ T_w = (u \circ h \circ T_w)^\wedge .$$

Since  $h \circ T_w$  is inner and  $h \circ T_w(0) = h(w)$ , it follows that

$$(u \circ h)^\wedge(w) = (u \circ (h \circ T_w))^\wedge(0) = u^\wedge \circ (h \circ T_w)(0) = u^\wedge \circ h(w) .$$

For the proof of (2.2) we may assume that  $h(0) = 0$ . In fact,  $H = T_{-h(0)} \circ h$  is inner and  $H(0) = 0$ . Since

$$u \circ h = u \circ T_{h(0)} \circ H ,$$

and since  $u \circ T_{h(0)}^T$  is subharmonic in  $\Delta$ , and since  $(u \circ T_{h(0)}^T \circ H)^\wedge$  exists, it follows that

$$\begin{aligned} u^\wedge \circ h(0) &= u^\wedge \circ T_{h(0)}^T \circ H(0) = (u \circ T_{h(0)}^T)^\wedge \circ H(0) = \\ &= (u \circ T_{h(0)}^T \circ H)^\wedge(0) = (u \circ h)^\wedge(0) . \end{aligned}$$

Set  $\Delta_r = \{|z| < r\}$ ,  $0 < r < 1$ , and set

$$\begin{aligned} u_r &= u_{\Delta_r}^\wedge \text{ in } \Delta_r , \\ &= u \text{ on } |z| = r . \end{aligned}$$

If the inequality

$$(2.3) \quad (u_r \circ h)(0) \leq (u \circ h)^\wedge(0)$$

is true for all  $r$ ,  $0 < r < 1$ , and for  $h(0) = 0$ , then letting  $r \uparrow 1$  we obtain, since  $u_r(0) = (u_r \circ h)(0)$ , that

$$u^\wedge(0) \leq (u \circ h)^\wedge(0) ,$$

which, together with the obvious relation,

$$(u \circ h)^\wedge(0) \leq u^\wedge \circ h(0) = u^\wedge(0) ,$$

yields (2.2).

For the proof of (2.3) we fix  $r$  and we set

$$M = \max_{|z|=r} u(z) .$$

Then  $M > 0$  and

$$(2.4) \quad u_r(z) \leq M \text{ for all } z \in \overline{\Delta_r} .$$

Now, for a.e.  $\zeta \in \partial\Delta$ , the limit exists,

$$h(\zeta) = \lim_{t \rightarrow 1-0} h(t\zeta) , \text{ and } |h(\zeta)| = 1 .$$

By Egorov's theorem, for each  $\epsilon > 0$ , there exists an open set  $E \Subset \overline{E}(\epsilon)$  on  $\partial\Delta$ , and  $t$ ,  $0 < t < 1$ , such that the linear Lebesgue measure  $m(E) < \epsilon/M$  and

$$(2.5) \quad |h(t\zeta)| > r \text{ for all } \zeta \in \partial\Delta \setminus E .$$

Let  $G_t$  be the component of the open set  $h^{-1}(\Delta_r) \cap \Delta_t$ , which

contains 0 . Then, for  $z \in \Delta_t \cap \partial G_t$  ,

$$(2.6) \quad u_r \circ h(z) - (u \circ h)^\wedge(z) = u \circ h(z) - (u \circ h)^\wedge(z) \leq 0$$

because  $|h(z)| = r$  . On the other hand, since  $|h(z)| \leq r$  for  $z \in A \equiv \partial \Delta_t \cap \partial G_t$  , (2.4) yields

$$(2.7) \quad u_r \circ h(z) \leq M \quad \text{for } z \in A .$$

Furthermore, by (2.5) we have  $z/t \in E$  for  $z \in A$  , whence

$$(2.8) \quad m(A) \leq tm(E) < \varepsilon/M .$$

Let  $\omega$  be the harmonic measure of  $A$  in  $\Delta_t$  , that is, the harmonic function in  $\Delta_t$  , which is continuously equal to 1 on  $A$  and 0 on  $\partial \Delta_t \setminus A$  . Then,

$$(2.9) \quad \omega(0) = m(A) < \varepsilon/M$$

by (2.8). Note that  $\omega(z) > 0$  for  $z \in \Delta_t \cap \partial G_t$  . Now, the maximum principle applied to the harmonic function

$$u_r \circ h - (u \circ h)^\wedge - M\omega$$

in  $\dot{G}_t$  , together with (2.6) and (2.7), shows that this function is nonpositive on the whole  $G_t$  because  $u \geq 0$  . In particular, the evaluation at 0 yields

$$u_r \circ h(0) \leq (u \circ h)^\wedge(0) + \varepsilon$$

by (2.9). Since  $\varepsilon > 0$  is arbitrary we obtain (2.3).

LEMMA 2.2. *Let  $\pi$  be a universal covering map from  $\Delta$  onto  $R$  , and let  $u$  be subharmonic on  $R$  . Then  $u_R^\wedge$  exists if and only if  $(u \circ \pi)_\Delta^\wedge$  exists. In this case  $u_R^\wedge \circ \pi = (u \circ \pi)_\Delta^\wedge$  in  $\Delta$  and*

$$(2.10) \quad \sup_{w \in \Delta} [(u \circ \pi)_\Delta^\wedge - u \circ \pi](w) = \sup_{z \in R} (u_R^\wedge - u)(z) .$$

We do not assume that  $u$  majorizes a harmonic function.

Proof. Since  $z = \pi(w)$  ranges over all  $R$  as  $w$  ranges over all  $\Delta$  , (2.10) is apparent if  $u_R^\wedge \circ \pi = (u \circ \pi)_\Delta^\wedge$  is established. We use again  $(u \circ \pi)^\wedge = (u \circ \pi)_\Delta^\wedge$  , etc. Obviously,  $(u \circ \pi)^\wedge$  exists if  $u_R^\wedge$  exists; in this case  $(u \circ \pi)^\wedge \leq u_R^\wedge \circ \pi$  . Suppose that  $(u \circ \pi)^\wedge$  exists. Since  $(u \circ \pi)^\wedge$  is

automorphic with respect to the cover transformation group consisting of Möbius transformations of  $\Delta$  onto  $\Delta$ ,  $v = (u \circ \pi)^{\wedge} \circ \pi^{-1}$  is well defined on  $R$ . Since  $(u \circ \pi)^{\wedge} \geq u \circ \pi$  we obtain  $v \geq u$ , whence  $u^{\wedge}_R$  exists and  $v \geq u^{\wedge}_R$ . Thus,  $(u \circ \pi)^{\wedge} \geq u^{\wedge}_R \circ \pi$ .

Proof of Theorem 1. Let  $\pi_R: \Delta \rightarrow R$  and  $\pi_S: \Delta \rightarrow S$  be universal covering maps, and apply Lemma 2.2 to  $u \circ h$  on  $R$ . Then  $(u \circ h \circ \pi_R)^{\wedge}$  exists if and only if  $(u \circ h)^{\wedge}_R$  exists and

$$(2.11) \quad (u \circ h)^{\wedge}_R \circ \pi_R = (u \circ h \circ \pi_R)^{\wedge}.$$

A single-valued branch of  $\pi_S^{-1} \circ h \circ \pi_R$  in  $\Delta$ , which we denote by  $H = \pi_S^{-1} \circ h \circ \pi_R$ , is locally of type  $\mathcal{B}\ell$ , whence of type  $\mathcal{B}\ell$  by [3, Corollary, p. 472]. Then

$$(2.12) \quad (u \circ h \circ \pi_R)^{\wedge} = (u \circ \pi_S \circ H)^{\wedge},$$

so that, Lemma 2.1, applied to the subharmonic function  $u \circ \pi_S$  in  $\Delta$ , and to the inner function  $H$ , asserts the existence of  $(u \circ \pi_S)^{\wedge}$  and

$$(2.13) \quad (u \circ \pi_S)^{\wedge} \circ H = (u \circ \pi_S \circ H)^{\wedge} = (u \circ h)^{\wedge}_R \circ \pi_R$$

by (2.11) and (2.12). Summing up these arguments, we know that  $(u \circ h)^{\wedge}_R$  exists if and only if  $(u \circ \pi_S)^{\wedge}$  exists; (2.13) holds in this case. On the other hand, by Lemma 2.2, again,  $(u \circ \pi_S)^{\wedge}$  exists if and only if  $u^{\wedge}_S$  exists, and in this case,  $u^{\wedge}_S \circ \pi_S = (u \circ \pi_S)^{\wedge}$ , whence, by (2.13),

$$(u \circ h)^{\wedge}_R \circ \pi_R = u^{\wedge}_S \circ \pi_S \circ H = u^{\wedge}_S \circ h \circ \pi_R \quad \text{on } \Delta.$$

Consequently, the equality  $(u \circ h)^{\wedge}_R = u^{\wedge}_S \circ h$  on  $R$  is established.

To prove (1.1) in (II) we first observe that

$$K \equiv \sup_{z \in R} [(u \circ h)^{\wedge}_R - u \circ h](z) = \sup_{w \in \Delta} [(u \circ h \circ \pi_R)^{\wedge} - u \circ h \circ \pi_R](w)$$

by (2.10) for  $u \circ h$  on  $R$ . Since  $u \circ h \circ \pi_R = u \circ \pi_S \circ H$ , and since  $(u \circ h \circ \pi_R)^{\wedge} = (u \circ \pi_S)^{\wedge} \circ H$  by (2.12) and (2.13), it follows that

$$(2.14) \quad K = \sup_{w \in \Delta} [(u \circ \pi_S)^\wedge - u \circ \pi_S] \circ H(w) = \sup_{z \in H(\Delta)} v(z)$$

where  $v = (u \circ \pi_S)^\wedge - u \circ \pi_S$  in  $\Delta$ . Now, by the theorem of O. Frostman [2, p. 111],  $\Delta \setminus H(\Delta)$  is of capacity zero, whence  $H(\Delta)$  is dense in  $\Delta$ . For each  $z \in \Delta$ , we then choose a sequence  $\{z_n\}$  with  $z_n \in H(\Delta)$  and  $z_n \rightarrow z$ . Since  $v$  is lower-semicontinuous, it follows that

$$K \geq \liminf_{n \rightarrow \infty} v(z_n) \geq v(z).$$

We thus have

$$K \leq \sup_{z \in \Delta} v(z) \leq K,$$

whence

$$K = \sup_{\zeta \in \Delta} [(u \circ \pi_S)^\wedge - u \circ \pi_S](\zeta),$$

which, together with (2.10) for  $u \circ \pi_S$  on  $\Delta$ , yields

$$K = \sup_{z \in S} (u_S^\wedge - u)(z).$$

### 3. Proof of Theorem 2

(i)  $X = N$ . This is a consequence of Theorem 1 for  $u = \log^+ |f|$ . The "norm" of  $f \in N(R)$  is

$$\|f\|_{w, N(R)} = (\log^+ |f|)_R^\wedge(w).$$

where  $w \in R$  is a fixed point. Then, for  $f \in N(S)$ ,

$$\|f \circ h\|_{w, N(R)} = \|f\|_{h(w), N(S)}.$$

(ii)  $X = N^+$ . The Smirnov class  $N^+(R) (= S(R)$  in [10]) consists of all  $f \in N(S)$  such that  $\log^+ |f|$  is majorized by a quasibounded harmonic function in the sense of Parreau [7] on  $R$ , or, equivalently,  $(\log^+ |f|)^\wedge$  is quasibounded, that is, the limiting function of a nondecreasing sequence of nonnegative and bounded harmonic functions on  $R$ . Note that  $N^+(\Delta) = N^+$  in [1, p. 26]. Some observations must be added. We claim that if  $v \geq 0$  is harmonic on  $S$  and if  $v \circ h$  is quasibounded on  $R$ , then  $v$  is quasibounded on  $S$ . For the proof of this, we let

$v = v_b + v_*$  be the Parreau decomposition of  $v$ , where  $v_b \geq 0$  is quasibounded, and  $v_* \geq 0$  is singular; see [7], [3], [10]. According to [3, Theorem 20.1, p. 468],  $v_* \circ h$  is singular on  $R$ . Since  $v \circ h \geq v_* \circ h$ , and since the singular part  $(v \circ h)_*$  of the decomposition of  $v \circ h$  on  $R$  is zero,  $0 = (v \circ h)_* \geq v_* \circ h$ , so that  $v_* \circ h = 0$ , whence  $v_* = 0$ .

Now, Theorem 2 for  $X = N^+$ . Let  $f$  be holomorphic on  $S$ . If  $(\log^+ |f|)_{\hat{S}}$  is quasibounded, then  $[(\log^+ |f|) \circ h]_{\hat{R}} = (\log^+ |f|)_{\hat{S}} \circ h$  is quasibounded on  $R$ . The converse is true by the observation in the preceding paragraph. Thus,  $f \in N^+(S)$  if and only if  $f \circ h \in N^+(R)$ . As the "norm" of  $F \in N^+(R)$  we use  $\|F\|_{w, N(R)}$  as in (i).

(iii)  $X = H^p$ . The Hardy class  $H^p(R)$  ( $0 < p < \infty$ ) consists of  $f$  holomorphic on  $R$  such that  $(|f|^p)^\wedge$  exists. The "norm" with the reference point  $w \in R$  is

$$\|f\|_{w, H^p(R)} = [(|f|^p)^\wedge(w)]^{1/p}$$

see [7, p. 137], [8, p. 50]; this is actually a norm in case  $p \geq 1$ . Theorem 1 with  $u = |f|^p$  establishes the present case. The norm identity is

$$\|f \circ h\|_{w, H^p(R)} = \|f\|_{h(w), H^p(S)}$$

(iv)  $X = H^p_\sigma$ . The class  $H^p_\sigma(R)$  ( $0 < p < \infty$ ) consists of  $f$  holomorphic and bounded,  $|f| < 1$ , on  $R$  such that the subharmonic function

$$\sigma(f)^p \equiv (\tanh^{-1} |f|)^p$$

admits an harmonic majorant on  $R$ . The "norm" with the reference point  $w \in R$  is

$$(3.1) \quad [(\sigma(f)^p)^\wedge(w)]^{1/p}$$

It is now easy to establish this case with the aid of Theorem 1 with  $u = \sigma(f)^p$ . It is known that  $H^p_\sigma(R)$  is a complete metric space with metric relating to (3.1); see [13].

(v) A subharmonic function  $u$  on  $R$  is said to be of bounded mean oscillation on  $R$ ,  $u \in BMOS(R)$  in notation, if  $u_{\hat{R}}$  exists and



$$\|u\|_{BMOS(R)} \equiv \sup_{z \in R} (u_R^\wedge - u)(z) < \infty .$$

This means that the potential  $p$  in the Riesz decomposition  $u = u_R^\wedge - P$ , is bounded on  $R$ . Let  $u$  be a subharmonic function on  $S$  majorizing a harmonic function there. Then  $u \in BMOS(S)$  if and only if  $u \circ h \in BMOS(R)$ , and further, in this case,

$$\|u \circ h\|_{BMOS(R)} = \|u\|_{BMOS(S)} .$$

This is a consequence of Theorem 1 with the emphasis on (1.1).

(vi)  $X = BMOA$ . The terminology in (v) is justified by the following observations. According to Metzger [6] a holomorphic function  $f$  on  $R$  is said to be of bounded mean oscillation,  $f \in BMOA(R)$ , if

$$\|f\|_{BMOA(R)} = \sup_{w \in R} 2\pi^{-1} \iint_R g_R(z, w) |f'(z)|^2 dx dy < \infty .$$

In [11] we find the relation for  $f$  holomorphic on  $R$ :

$$(3.2) \quad (|f|^2)_R^\wedge(w) - |f|^2(w) = 2\pi^{-1} \iint_R g_R(z, w) |f'(z)|^2 dx dy .$$

Thus,  $f \in BMOA(R)$  if and only if  $|f|^2 \in BMOS(R)$ ; in this case,

$$\| |f|^2 \|_{BMOS(R)} = \|f\|_{BMOA(R)} .$$

Theorem 2 for  $X = BMOA$  now follows from (v) above. The quantity  $\|f\|_{BMOA(R)}$  is called BMOA pseudo-norm of  $f \in BMOA(R)$ .

(vii)  $X = BMOA_\sigma$ . The situation is the same on replacing  $|f'|^2$  by  $|f'|^2 / (1 - |f|^2)^2$  and  $|f|^2$  by  $\lambda(f) = -\log(1 - |f|^2)$  for  $f$  holomorphic and bounded,  $|f| < 1$ , on  $R$ . Thus,  $f \in BMOA_\sigma(R)$  if  $\|f\|_{BMOA_\sigma(R)} = \|\lambda(f)\|_{BMOS(R)} < \infty$ . The equality follows from the analogue

of (3.2),

$$\lambda(f)^\wedge(w) - \lambda(f)(w) = 2\pi^{-1} \iint_R g_R(z, w) |f'(z)|^2 / (1 - |f(z)|^2)^2 dx dy ;$$

see [11]. Again (v) proves the case  $X = BMOA_\sigma$ .

Remark. Let  $L(R)$  be the family of meromorphic and Lindelöfian functions on  $R$ . It should be noted that Theorem 2 for  $X = N$  yields

the following result of Heins cited in the introduction. For  $f$  meromorphic on  $S$ , we have  $f \in L(S) \Leftrightarrow f \circ h \in L(R)$ . For the proof we may suppose that  $f$  is nonconstant. Let  $E$  be the set of all the poles of  $f$  on  $S$ . Then  $S_E = S \setminus E$  and  $R_E = R \setminus h^{-1}(E)$  both are hyperbolic Riemann surfaces. It then follows from [3, Theorem 16.1, p. 466] that the restriction of  $h$ , that is,  $h: R_E \rightarrow S_E$  is again of type  $\mathcal{B}\ell$ . On the other hand, it follows from Parreau's theorem [7, Théorème 20, p. 182] (this theorem is valid for  $\alpha = 0$ ) that

$$\begin{aligned} f \in L(S) &\Leftrightarrow f \in N(S_E) ; \\ f \circ h \in L(R) &\Leftrightarrow f \circ h \in N(R_E) . \end{aligned}$$

Therefore,  $f \in L(S) \Leftrightarrow f \circ h \in L(R)$ .

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