This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to I. G. Connell, Department of Mathematics, McGill University, Montreal, P.Q.

A FUNCTIONAL EQUATION FOR THE COSINE

PL. Kannappan

It is known [3], [5] that, the complex-valued solutions of

(B) $$f(x+y) + f(x-y) = 2 f(x)f(y), \quad \text{for } x, y \text{ real}$$

(apart from the trivial solution $f(x) \equiv 0$) are of the form

(C) $$f(x) = \frac{\phi(x) + \phi(x)^{-1}}{2}, \quad \text{where}$$

(D) $$\phi(x+y) = \phi(x)\, \phi(y).$$

In case f is a measurable solution of (B), then f is continuous [2], [3] and the corresponding ϕ in (C) is also continuous and ϕ is of the form [1],

(E) $$\phi(x) = \exp(c\, x), \quad c, \text{ a complex constant.}$$

In this paper, the functional equation

(P) $$f(x+y+2A) + f(x-y+2A) = 2 f(x) f(y)$$

where f is a complex-valued, measurable function of the real variable and $A \neq 0$ is a real constant, is considered. It is shown that f is continuous and that (apart from the trivial solutions $f \equiv 0, 1$), the only functions which satisfy (P) are the cosine functions $\cos ax$ and $-\cos bx$, where a and b belong to a denumerable set of real numbers.

Equation (P) is similar to the equation

(Q) $$f(x-y+A) - f(x+y+A) = 2 f(x) f(y)$$

considered by E.B. Van Vleck [4], where f is assumed real and

https://doi.org/10.4153/CMB-1968-059-8 Published online by Cambridge University Press

continuous and the general solution is $f(x) = \sin c x$, for a sequence $c = \frac{(4j+1)\pi}{2A}$, $(j = 0, 1, \ldots)$.

THEOREM. Let f be a complex-valued function of the reals R, satisfying (P) for every x, y in R, where $A \neq 0$ is a real constant. Then the general solution of (P) is given by either $f \equiv 0$ or $f(x) = g(x-2A)$, where g is an arbitrary solution of (B) with period $4A$.

Proof. First, setting

$$(1) \qquad f(x) = g(x-2A),$$

where g is a solution of (B) with period $4A$, we have

$$\begin{aligned} f(x+y+2A) + f(x-y+2A) &= g(x+y) + g(x-y) \\ &= g(x+y-4A) + g(x-y) \\ &= 2\, g(x-2A)\, g(y-2A) \\ &= 2\, f(x) f(y), \text{ which is (P).} \end{aligned}$$

Conversely, every solution of (P) is of the form (1). Indeed, interchanging x and y in (P) and comparing it with (P), we obtain

$$(2) \qquad f(x-y+2A) = f(y-x+2A), \quad \text{for all } x, y \text{ in } R.$$

Putting $x = A$, $y = 3A$ in (2) and $x = 0$, $y = 0$ in (P), we get

$$(3) \qquad f(0) = f(4A) \quad \text{and}$$

$$(4) \qquad f(2A) = f(0)^2.$$

From (3), (4) and (P) with $x = 0$, $y = 2A$, we deduce that either $f(0) = 0$ or $f(2A) = 1$. It is easily seen that $f(0) = 0$ implies $f(x) \equiv 0$ (by setting $y = 0$ in (P)). If

$$(5) \qquad f(2A) = 1,$$

we get $f(0)^2 = 1$ from (4) and from (5) and (P) with $y = 2A$, we see that

$$(6) \qquad f(x+4A) = f(x), \quad \text{for all } x \text{ in } R.$$

That is, f is a periodic function with period $4A$. We remark here that f is even (which can be deduced from (2) and (6)).

Replacing x by $x+2A$ and y by $y+2A$ in (P), we get

$$f(x+y+6A) + f(x-y+2A) = 2 f(x+2A)f(y+2A),$$

and from (6),

$$(7) \qquad f(x+y+2A) + f(x-y+2A) = 2f(x+2A) f(y+2A).$$

If g is defined as in (1) for all x in R, then g satisfies the equation

$$(B) \qquad g(x+y)+g(x-y) = 2 g(x) g(y).$$

One sees from (1) and (6), that g is periodic with the period $4A$, that is,

$$(8) \qquad g(x+4A) = g(x).$$

This completes the proof of this theorem.

COROLLARY. The only measurable solutions of (P) are $f \equiv 0, 1$ and $f(x) = \cos(\frac{n\pi x}{2A} - n\pi)$ where $n = 0, \pm 1, \ldots$ or equivalently, $f(x) \equiv 0, 1$, $f(x) = \cos ax$ and $f(x) = -\cos bx$ where $a = \frac{k\pi}{A}$ and $b = \frac{(2k+1)\pi}{2A}$ $(k = 0, 1, \ldots)$ is the complete set of measurable solutions of (P).

Proof. From (1) and the introduction we conclude that both f and g are continuous. Since g is periodic, from (C), (E) and (8), it is easy to see that $4cA = 2n\pi i$, $(n = 0, \pm 1, \ldots)$, thus $g(x) \equiv 0$ or $g(x) = \cosh \frac{n\pi ix}{2A}$. Hence from (1) we obtain, $f(x) \equiv 0$ or $f(x) = \cos(\frac{n\pi x}{2A} - n\pi)$, $n = 0, \pm 1, \ldots$. This contains, for $n = 0$, $f(x) \equiv 1$. When $n = 2k$, $(k = 0, 1, \ldots)$, we have $f(x) = \cos ax$, with $a = \frac{k\pi}{A}$, $(k = 0, 1, \ldots)$. This corresponds to the case $f(0) = 1$. When $n = 2k+1$, $(k = 0, 1, \ldots)$, we have $f(x) = -\cos bx$, with $b = \frac{(2k+1)\pi}{2A}$, $(k = 0, 1, \ldots)$. This corresponds to the case $f(0) = -1$. Thus the proof is complete.

Finally I express my sincere thanks to Prof. J. Aczel for his critical remarks during the preparation of this paper and to the referee for the useful remarks.

https://doi.org/10.4153/CMB-1968-059-8 Published online by Cambridge University Press

REFERENCES

1. J. Aczel, Lectures on functional equations and their applications. (Academic Press, 1966).

2. S. Kaczmarz, Sur l'equation fonctionelle $f(x) + f(x+y) = \phi(y)f(x + y/_2)$. Fund. Math., 6 (1924) 122-129.

3. PL. Kannappan, The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups. Proc. Amer. Math. Soc. 19 (1968) 69-74.

4. B. Van Vleck, A functional equation for the sine. Ann. Math. 7 (1910) 161-165.

5. W.H. Wilson, On certain related functional equations. Bull. Amer. Math. Soc., 26 (1919-1920) 300-312.

University of Waterloo

https://doi.org/10.4153/CMB-1968-059-8 Published online by Cambridge University Press