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MODULES OVER ÉTALE GROUPOID ALGEBRAS AS SHEAVES

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Abstract

The author has previously associated to each commutative ring with unit $\Bbbk$ and étale groupoid $\mathscr{G}$ with locally compact, Hausdorff, totally disconnected unit space a $\Bbbk$-algebra $\Bbbk\mathscr{G}$. The algebra $\Bbbk\mathscr{G}$ need not be unital, but it always has local units. The class of groupoid algebras includes group algebras, inverse semigroup algebras and Leavitt path algebras. In this paper we show that the category of unitary $\Bbbk\mathscr{G}$-modules is equivalent to the category of sheaves of $\Bbbk$-modules over $\mathscr{G}$. As a consequence, we obtain a new proof of a recent result that Morita equivalent groupoids have Morita equivalent algebras.

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1. Introduction

In an effort to obtain a uniform theory for group algebras, inverse semigroup algebras and Leavitt path algebras [2], the author [26] associated to each commutative ring with unit $\Bbbk$ and étale groupoid $\mathscr{G}$ with locally compact, totally disconnected unit space a $\Bbbk$-algebra $\Bbbk\mathscr{G}$ (see also [6]). These algebras are discrete versions of groupoid C^*-algebras [20, 22] and a number of analogues of results from the operator theoretic setting have been obtained in this context. In particular, Cuntz–Krieger uniqueness theorems [4, 5], characterizations of simplicity [4, 5] and the connection of groupoid Morita equivalence to Morita equivalence of algebras [7] have been proven for groupoid algebras under the Hausdorff assumption.

In this paper, we prove a discrete analogue of Renault's disintegration theorem [23], which roughly states that representations of groupoid C^*-algebras are obtained by integrating representations of the groupoid. A representation of a groupoid consists of a field of Hilbert spaces over the unit space with an action of the groupoid by unitary transformations on the fibres [20, 22].

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Here we prove that the category of unitary $\Bbbk\mathscr{G}$-modules is equivalent to the category of sheaves of $\Bbbk$-modules over $\mathscr{G}$. This simultaneously generalizes the following two well-known facts: if X is a locally compact, totally disconnected space and $C_c(X, \Bbbk)$ is the ring of locally constant functions $X \to \Bbbk$ with compact support, then the category of unitary $C_c(X, \Bbbk)$-modules is equivalent to the category of sheaves on X (compare with [21]); and if $\mathscr{G}$ is a discrete groupoid, then the category of unitary $\Bbbk\mathscr{G}$-modules is equivalent to the category of functors from $\mathscr{G}$ to the category of $\Bbbk$-modules (compare with [15]). In the context of étale Lie groupoids and convolution algebras of smooth functions, analogous results can be found in [13]. However, the techniques in the totally disconnected case setting are quite different.

As a consequence of our results, we obtain a new proof that Morita equivalent groupoids have Morita equivalent groupoid algebras, which the author feels is more conceptual than the one in [7] (since it works with module categories rather than Morita contexts), and at the same time does not require the Hausdorff hypothesis.

We hope that this geometric realization of the module category will prove useful in the study of simple modules, primitive ideals and in other contexts analogous to those in which Renault's disintegration theorem is used in operator theory.

2. Étale groupoids

In this paper, a topological space will be called compact if it is Hausdorff and satisfies the property that every open cover has a finite subcover.

2.1. Topological groupoids. A topological groupoid $\mathscr{G}$ is a groupoid (that is, a small category each of whose morphisms is an isomorphism) whose unit space $\mathscr{G}^{(0)}$ and arrow space $\mathscr{G}^{(1)}$ are topological spaces and whose domain map $\boldsymbol{d}$, range map $\boldsymbol{r}$, multiplication map, inversion map and unit map $u : \mathscr{G}^{(0)} \to \mathscr{G}^{(1)}$ are all continuous. Since u is a homeomorphism with its image, we identify elements of $\mathscr{G}^{(0)}$ with the corresponding identity arrows and view $\mathscr{G}^{(0)}$ as a subspace of $\mathscr{G}^{(1)}$ with the subspace topology. We write $\mathscr{G}^{(2)}$ for the space of composable arrows (g, h) with $\boldsymbol{d}(g) = \boldsymbol{r}(h)$.

A topological groupoid $\mathscr{G}$ is *étale* if $\boldsymbol{d}$ is a local homeomorphism. This implies that $\boldsymbol{r}$ and the multiplication map are local homeomorphisms and that $\mathscr{G}^{(0)}$ is open in $\mathscr{G}^{(1)}$ [24]. Note that the fibres of $\boldsymbol{d}$ and $\boldsymbol{r}$ are discrete in the induced topology. A *local bisection* of $\mathscr{G}$ is an open subset $U \subseteq \mathscr{G}^{(1)}$ such that $\boldsymbol{d}|_U$ and $\boldsymbol{r}|_U$ are homeomorphisms to their images. The set of local bisections of $\mathscr{G}$, denoted $\mathrm{Bis}(\mathscr{G})$, is a basis for the topology on $\mathscr{G}^{(1)}$ [9, 20, 24]. If U, V are local bisections, then

$$UV = \{uv \mid u \in U, v \in V\},$$
$$U^{-1} = \{u^{-1} \mid u \in U\}$$

are local bisections. In fact, $\mathrm{Bis}(\mathscr{G})$ is an inverse semigroup [14].

An étale groupoid is said to be *ample* [20] if $\mathscr{G}^{(0)}$ is Hausdorff and has a basis of compact open sets. In this case $\mathscr{G}^{(1)}$ is locally Hausdorff but need not be Hausdorff. Let $\mathrm{Bis}_c(\mathscr{G})$ denote the set of compact open local bisections of $\mathscr{G}$. Then $\mathrm{Bis}_c(\mathscr{G})$ is an inverse subsemigroup of $\mathrm{Bis}(\mathscr{G})$ and is a basis for the topology of $\mathscr{G}^{(1)}$ [20].

2.2. $\mathscr{G}$-sheaves and Morita equivalence. Let $\mathscr{G}$ be an étale groupoid. References for this section are [8, 11, 12, 16–19]. A *(right) $\mathscr{G}$-space* consists of a space E, a continuous map $p : E \to \mathscr{G}^{(0)}$ and an action map

$$E \times_{\mathscr{G}^{(0)}} \mathscr{G}^{(1)} \to E$$

(where the fibre product is with respect to p and $\boldsymbol{r}$), denoted $(x, g) \mapsto xg$, satisfying the following axioms:

- $ep(e) = e$ for all $e \in E$;
- $p(eg) = \boldsymbol{d}(g)$ whenever $p(e) = \boldsymbol{r}(g)$;
- $(eg)h = e(gh)$ whenever $p(e) = \boldsymbol{r}(g)$ and $\boldsymbol{d}(g) = \boldsymbol{r}(h)$.

Left $\mathscr{G}$-spaces are defined dually.

A $\mathscr{G}$-space (E, p) is said to be *principal* if the natural map

$$E \times_{\mathscr{G}^{(0)}} \mathscr{G}^{(1)} \to E \times_{\mathscr{G}^{(0)}} E$$

given by $(e, g) \mapsto (eg, e)$ is a homeomorphism. A *morphism* $(E, p) \to (F, q)$ of $\mathscr{G}$-spaces is a continuous map $\varphi : E \to F$ such that

$$\begin{array}{ccc} E & \xrightarrow{\quad\varphi\quad} & F \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle q} & \\ & \mathscr{G}^{(0)} & \end{array}$$

commutes and $\varphi(eg) = \varphi(e)g$ whenever $p(e) = \boldsymbol{r}(g)$.

Morita equivalence plays an important role in groupoid theory. There are a number of different, but equivalent, formulations of the notion. See [8, 11, 12, 16–19] for details. Two topological groupoids $\mathscr{G}$ and $\mathscr{H}$ are said to be *Morita equivalent* if there is a topological space E with the structure of a principal left $\mathscr{G}$-space (E, p) and a principal right $\mathscr{H}$-space (E, q) such that p, q are open surjections and the actions commute, meaning $p(eh) = p(e)$, $q(ge) = q(e)$ and $(ge)h = g(eh)$ whenever $g \in \mathscr{G}^{(1)}$, $h \in \mathscr{H}^{(1)}$ and $\boldsymbol{d}(g) = p(e)$, $\boldsymbol{r}(h) = q(e)$.

A continuous functor $f : \mathscr{G} \to \mathscr{H}$ of étale groupoids is called an *essential equivalence* if $\boldsymbol{d}\pi_2 : \mathscr{G}^{(0)} \times_{\mathscr{H}^{(0)}} \times \mathscr{H}^{(1)} \to \mathscr{H}^{(0)}$ is an open surjection (where the fibre product is over f and $\boldsymbol{r}$) and the square

$$\begin{array}{ccc} \mathscr{G}^{(1)} & \xrightarrow{\qquad f \qquad} & \mathscr{H}^{(1)} \\ {\scriptstyle (\boldsymbol{d},\boldsymbol{r})}\big\downarrow & & \big\downarrow{\scriptstyle (\boldsymbol{d},\boldsymbol{r})} \\ \mathscr{G}^{(0)} \times \mathscr{G}^{(0)} & \xrightarrow[\quad f\times f \quad]{} & \mathscr{H}^{(0)} \times \mathscr{H}^{(0)} \end{array}$$

is a pullback. The first condition corresponds to being essentially surjective and the second to being fully faithful in the discrete context. Étale groupoids $\mathscr{G}$ and $\mathscr{H}$

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are Morita equivalent if and only if there are an étale groupoid $\mathscr{K}$ and essential equivalences $f : \mathscr{K} \to \mathscr{G}$ and $f' : \mathscr{K} \to \mathscr{H}$.

If $\mathscr{G}$ is an étale groupoid, then a $\mathscr{G}$*-sheaf* consists of a $\mathscr{G}$-space (E, p) such that $p : E \to \mathscr{G}^{(0)}$ is a local homeomorphism (the tradition to use right actions is standard in topos theory). The fibre $p^{-1}(x)$ of E over x is denoted E_x and is called the *stalk* of E at x. A morphism of $\mathscr{G}$-sheaves is just a morphism of $\mathscr{G}$-spaces; note, however, that the corresponding map of total spaces must necessarily be a local homeomorphism. The category $\mathcal{B}\mathscr{G}$ of all $\mathscr{G}$-sheaves is called the *classifying topos* of $\mathscr{G}$ [11, 12].

If A is a set, then the constant $\mathscr{G}$-sheaf $\Delta(A)$ is $(A \times \mathscr{G}^{(0)}, \pi_2)$ with action $(a, \boldsymbol{r}(g))g = (a, \boldsymbol{d}(g))$ (where A is endowed with the discrete topology). As a sheaf over $\mathscr{G}^{(0)}$, note that $\Delta(A)$ is nothing more than the sheaf of locally constant A-valued functions on $\mathscr{G}^{(0)}$. (Recall that a locally constant function from a topological space X to a set A is just a continuous map $X \to A$, where A is endowed with the discrete topology.) The functor $\Delta : \mathbf{Set} \to \mathcal{B}\mathscr{G}$ is exact and hence sends rings to internal rings of $\mathcal{B}\mathscr{G}$. If $\Bbbk$ is a commutative ring with unit, then a $\mathscr{G}$*-sheaf of* $\Bbbk$*-modules* is by definition an internal $\Delta(\Bbbk)$-module in $\mathcal{B}\mathscr{G}$. Explicitly, this amounts to a $\mathscr{G}$-sheaf (E, p) together a $\Bbbk$-module structure on each stalk E_x such that:

- the zero section, denoted 0, sending $x \in \mathscr{G}^{(0)}$ to the zero of E_x is continuous;
- addition $E \times_{\mathscr{G}^{(0)}} E \to E$ is continuous;
- scalar multiplication $K \times E \to E$ is continuous;
- for each $g \in \mathscr{G}^{(1)}$, the map $R_g : E_{\boldsymbol{r}(g)} \to E_{\boldsymbol{d}(x)}$ given by $R_g(e) = eg$ is $\Bbbk$-linear;

where $\Bbbk$ has the discrete topology in the third item. Note that the first three conditions are equivalent to (E, p) being a sheaf of $\Bbbk$-modules over $\mathscr{G}^{(0)}$.

Internal $\Delta(\Bbbk)$-module homomorphisms are just $\mathscr{G}$-sheaf morphisms

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathscr{G}^{(0)} & \end{array}$$

which restrict to $\Bbbk$-module homomorphisms on the stalks. The category of $\mathscr{G}$-sheaves of $\Bbbk$-modules will be denoted $\mathcal{B}_{\Bbbk}\mathscr{G}$.

It follows from standard topos theory that if $\mathcal{B}\mathscr{G}$ is equivalent to $\mathcal{B}\mathscr{H}$, then the equivalence commutes with the constant functor (up to isomorphism) and hence yields an equivalence of categories $\mathcal{B}_{\Bbbk}\mathscr{G}$ and $\mathcal{B}_{\Bbbk}\mathscr{H}$. Indeed, Δ is left adjoint to the hom functor out of the terminal object and equivalences preserve terminal objects.

Moerdijk proved that if $\mathscr{G}$ and $\mathscr{H}$ are étale groupoids, then $\mathcal{B}\mathscr{G}$ is equivalent to $\mathcal{B}\mathscr{H}$ if and only if $\mathscr{G}$ and $\mathscr{H}$ are Morita equivalent groupoids [8, 11, 12, 16–18]. Hence, Morita equivalent groupoids have equivalent categories of sheaves of $\Bbbk$-modules for any base ring $\Bbbk$.

2.3. Groupoid algebras. Let $\mathscr{G}$ be an ample groupoid and $\Bbbk$ a commutative ring with unit. Define $\Bbbk\mathscr{G}$ to be the $\Bbbk$-submodule of $\Bbbk^{\mathscr{G}^{(1)}}$ spanned by the characteristic

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functions χ_U with $U \in \mathrm{Bis}_c(\mathscr{G})$. If $\mathscr{G}^{(1)}$ is Hausdorff, then $\Bbbk\mathscr{G}$ consists precisely of the locally constant functions $\mathscr{G}^{(1)} \to \Bbbk$ with compact support; otherwise, it is the $\Bbbk$-submodule spanned by those functions $f : \mathscr{G}^{(1)} \to \Bbbk$ that vanish outside some Hausdorff open subset U with $f|_U$ locally constant with compact support. See [6, 25, 26] for details.

The convolution product on $\Bbbk\mathscr{G}$, defined by

$$f_1 * f_2(g) = \sum_{\boldsymbol{d}(h)=\boldsymbol{d}(g)} = f_1(gh^{-1})f_2(h),$$

turns $\Bbbk\mathscr{G}$ into a $\Bbbk$-algebra. Note that the sum is finite because the fibres of $\boldsymbol{d}$ are closed in $\mathscr{G}^{(1)}$ and discrete in the induced topology, and f_1, f_2 are linear combinations of functions with compact support. We often just write for convenience $f_1 f_2$ instead of $f_1 * f_2$. One has that $\chi_U\chi_V = \chi_{UV}$ for $U, V \in \mathrm{Bis}_c(\mathscr{G})$; see [26]. If $\mathscr{G}^{(1)} = \mathscr{G}^{(0)}$, then the convolution product is just pointwise multiplication and so $\Bbbk\mathscr{G}$ is just the usual ring of locally constant functions $\mathscr{G}^{(0)} \to \Bbbk$ with compact support.

The ring $\Bbbk\mathscr{G}$ is unital if and only if $\mathscr{G}^{(0)}$ is compact [26]. However, it is very close to being unital in the following sense. A ring R is said to have *local units* if it is a direct limit of unital rings in the category of not necessarily unital rings (that is, the homomorphisms in the directed system do not have to preserve the identities). Equivalently, R has local units if, for any finite subset $r_1, \ldots, r_n$ of R, there is an idempotent $e \in R$ with $er_i = r_i = r_ie$ for $i = 1, \ldots, n$ [1, 3]. Denote by $E(R)$ the set of idempotents of R.

Proposition 2.1. *Let $\mathscr{G}$ be an ample groupoid and $\Bbbk$ a commutative ring with units. Let $\mathscr{B}$ denote the generalized boolean algebra of compact open subsets of $\mathscr{G}^{(0)}$. If $U \in \mathscr{B}$, let $\mathscr{G}|_U = (U, \boldsymbol{d}^{-1}(U) \cap \boldsymbol{r}^{-1}(U))$.*

(1) *$\mathscr{B}$ is directed.*
(2) *If $U \in \mathscr{B}$, then $\mathscr{G}|_U$ is an open ample subgroupoid of $\mathscr{G}$.*
(3) *If $U \in \mathscr{B}$, then $\chi_U \cdot \Bbbk\mathscr{G} \cdot \chi_U \cong \Bbbk\mathscr{G}|_U$.*
(4) *$\Bbbk\mathscr{G} = \bigcup_{U\in\mathscr{B}} \chi_U \cdot \Bbbk\mathscr{G} \cdot \chi_U = \varinjlim_{U\in\mathscr{B}} \Bbbk\mathscr{G}|_U$.*

In particular, $\Bbbk\mathscr{G}$ has local units.

Proof. Clearly, $\mathscr{B}$ is directed since the union of two elements is their join. Also, $\mathscr{G}|_U$ is an open ample subgroupoid of $\mathscr{G}$. It follows that $\Bbbk\mathscr{G}|_U$ can be identified with a subalgebra of $\Bbbk\mathscr{G}$ by extending functions on $\mathscr{G}|_U^{(1)}$ to be 0 outside of $\mathscr{G}|_U^{(1)}$. Since χ_U is the identity of $\Bbbk\mathscr{G}|_U$ (compare with [26]), $\Bbbk\mathscr{G}|_U$ is a unital subring of $\chi_U \cdot \Bbbk\mathscr{G} \cdot \chi_U$. But, if $f \in \Bbbk\mathscr{G}$ and $g \notin \mathscr{G}|_U^{(1)}$, then $(\chi_U \cdot f \cdot \chi_U)(g) = 0$. Thus, $\chi_U \cdot \Bbbk\mathscr{G} \cdot \chi_U = \Bbbk\mathscr{G}|_U$.

Let $R = \bigcup_{U\in\mathscr{B}} \chi_U \cdot \Bbbk\mathscr{G}|_U \cdot \chi_U$. Then R is a $\Bbbk$-subalgebra of $\Bbbk\mathscr{G}$ because $\mathscr{B}$ is directed. To show that R is the whole ring, we just need to show that it contains the spanning set χ_U with $U \in \mathrm{Bis}_c(\mathscr{G})$. Put $V = U^{-1}U \cup UU^{-1}$. Then $V \in \mathscr{B}$ and $\chi_V \cdot \chi_U \cdot \chi_V = \chi_{VUV} = \chi_U$. □

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Examples of groupoid algebras of ample groupoids include group algebras, Leavitt path algebras [6, 7] and inverse semigroup algebras [26], as well as discrete groupoid algebras and certain cross product and partial action cross product algebras. In general, groupoid algebras allow one to construct discrete analogues of a number of classical C^*-algebras that can be realized as C^*-algebras of ample groupoids [9, 20, 22].

3. The equivalence theorem

Fix an ample groupoid $\mathscr{G}$ and a commutative ring with unit $\Bbbk$. Our goal is to establish an equivalence between the category mod-$\Bbbk\mathscr{G}$ of unitary right $\Bbbk\mathscr{G}$-modules and the category $\mathcal{B}_{\Bbbk}\mathscr{G}$ of $\mathscr{G}$-sheaves of $\Bbbk$-modules. Let us recall the missing definitions.

If R is a ring with local units, a right R-module M is *unitary* if $MR = M$ or, equivalently, for each $m \in M$, there is an idempotent $e \in E(R)$ with $me = m$. We write mod-R for the category of unitary right R-modules. Two rings R, S with local units are *Morita equivalent* if mod-R is equivalent to mod-S [1, 3, 10]. One can equivalently define Morita equivalence in terms of unitary left modules and in terms of Morita contexts [1, 3, 10].

Suppose that R is a $\Bbbk$-algebra with local units. Then we note that every unitary R-module is a $\Bbbk$-module and the $\Bbbk$-module structure is compatible with the $\Bbbk$-algebra structure. Indeed, if $e \in E(R)$, then Me is a unital eMe-module and hence a $\Bbbk$-module in the usual way. As M is unitary, it follows that M is the directed union $\bigcup_{e\in E(M)} Me$ and hence a $\Bbbk$-module. More concretely, the $\Bbbk$-module structure is given as follows: if $c \in \Bbbk$ and $m \in M$, then $cm = m(ce)$, where e is any idempotent such that $me = m$. The $\Bbbk$-module structure is then automatically preserved by any R-module homomorphism, as in the case of unital rings.

Define a functor $\Gamma_c : \mathcal{B}_{\Bbbk}\mathscr{G} \to \text{mod-}\Bbbk\mathscr{G}$ as follows. If (E, p) is a $\mathscr{G}$-sheaf of $\Bbbk$-modules, then $\Gamma_c(E, p)$ is the set of all compactly supported (global) sections $s : \mathscr{G}^{(0)} \to E$ of p with pointwise addition. We define a $\Bbbk\mathscr{G}$-module structure by

$$(sf)(x) = \sum_{\boldsymbol{d}(g)=x} f(g)s(\boldsymbol{r}(g))g = \sum_{\boldsymbol{d}(g)=x} f(g)R_g(s(\boldsymbol{r}(g))).$$

As usual, the sum is finite because f is a finite sum of functions with compact support and the fibres of $\boldsymbol{d}$ are closed and discrete. It is easy to check that this makes $\Gamma_c(E, p)$ into a $\Bbbk\mathscr{G}$-module and that the induced $\Bbbk$-module structure is just the pointwise one. The following observation is so fundamental that we shall often use it without comment throughout.

Proposition 3.1. *If $U \in \mathrm{Bis}_c(\mathscr{G})$ and $s \in \Gamma_c(E, p)$, then*

$$(s\chi_U)(x) = \begin{cases} s(\boldsymbol{r}(g))g & \text{if } g \in U,\ \boldsymbol{d}(g) = x, \\ 0 & \text{if } x \notin U^{-1}U. \end{cases}$$

In particular, if $U \subseteq \mathscr{G}^{(0)}$ is compact open, then $(s\chi_U) = \chi_U(x)s(x)$.

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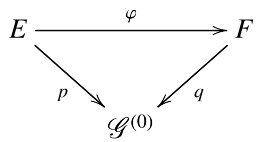
The module $\Gamma_c(E, p)$ is unitary because if $s : \mathscr{G}^{(0)} \to E$ has compact support, then we can find a compact open set U containing the support of s (just cover the support by compact open sets and take the union of a finite subcover). Then one readily checks that $s\chi_U = s$ using Proposition 3.1.

If



is a morphism of $\mathscr{G}$-sheaves of $\Bbbk$-modules and $s \in \Gamma_c(E, p)$, then we define $\Gamma_c(\varphi)(s) = \varphi \circ s$. It is straightforward to verify that Γ_c is a functor.

Conversely, let M be a unitary right $\Bbbk\mathscr{G}$-module. We define a $\mathscr{G}$-sheaf $\mathrm{Sh}(M) = (\widetilde{M}, p_M)$ in steps. Recall that we have been using $\mathcal{B}$ to denote the generalized boolean algebra of compact open subsets of $\mathscr{G}^{(0)}$. For each $U \in \mathcal{B}$, we can consider the $\Bbbk$-submodule $M(U) = M\chi_U$. If $U \subseteq V$, then $M(U) = M\chi_U = M\chi_{UV} = M\chi_U\chi_V \subseteq M\chi_V = M(V)$. Note that $M(U)$ is a $\Bbbk\mathscr{G}|_U = \chi_U \cdot \mathscr{G} \cdot \chi_U$-module and, since $\Bbbk\mathscr{G} = \bigcup_{U\in\mathcal{B}} \chi_U \cdot \mathscr{G} \cdot \chi_U = \varinjlim_{U\in\mathcal{B}} \Bbbk\mathscr{G}|_U$, it follows that $M = \bigcup_{U\in\mathcal{B}} M(U) = \varinjlim_{U\in\mathcal{B}} M(U)$, where the latter has the obvious module structure coming from $\Bbbk\mathscr{G} = \varinjlim_{U\in\mathcal{B}} \Bbbk G|_U$.

Let $x \in \mathscr{G}^{(0)}$. If $x \in V \subseteq U$ with $U, V \in \mathcal{B}$, then we have a $\Bbbk$-module homomorphism $\rho^U_V : M(U) \to M(V)$ given by $m \mapsto m\chi_V$. Observe that $\rho^U_U = 1_{M(U)}$ and that if $W \subseteq V \subseteq U$, then we have $\rho^V_W \circ \rho^U_V = \rho^U_W$. It follows that the $M(U)$ with $U \in \mathcal{B}$, together with the connecting maps ρ^U_V, form a directed system of $\Bbbk$-modules. Therefore, we can form the direct limit $M_x = \varinjlim_{x\in U} M(U)$. If $m \in M(U)$, we let $[m]_x$ denote the equivalence class of m in M_x. Since $M = \bigcup_{U\in\mathcal{B}} M(U)$ and each element of $\mathcal{B}$ is contained in an element which contains x, it follows that $[m]_x$ is defined for all $m \in M$ and $m \mapsto [m]_x$ gives a $\Bbbk$-linear map $M \to M_x$.

Put $\widetilde{M} = \coprod_{x\in\mathscr{G}^{(0)}} M_x$ and let $p_M(M_x) = x$ for $x \in \mathscr{G}^{(0)}$. Let U be a compact open subset of $\mathscr{G}^{(0)}$ and let $m \in M$. Define

$$(U, m) = \{[m]_x \mid x \in U\} \subseteq \widetilde{M}.$$

Suppose $[m]_x \in (U, m_1) \cap (V, m_2)$. Then there is a compact open neighbourhood $W \subseteq U \cap V$ of x such that $m\chi_W = m_1\chi_W = m_2\chi_W$. It follows that $[m]_x \in (W, m) \subseteq (U, m_1) \cap (V, m_2)$ and hence the sets (U, m) form a basis for a topology on $\widetilde{M}$. Continuity of p_M follows because if U is a compact open subset of $\mathscr{G}^{(0)}$, then $p_M^{-1}(U) = \bigcup_{m\in M}(U, m)$ is open. Trivially, p_M takes (U, m) bijectively to U and is thus a local homeomorphism.

Each stalk M_x is a $\Bbbk$-module. We must show continuity of the $\Bbbk$-module structure. To establish continuity of the zero section $x \mapsto [0]_x$, suppose that (U, m) is a basic neighbourhood of $[0]_x$. Then there is a compact open neighbourhood W of x with $W \subseteq U$ and $m\chi_W = 0$. Then, for all $z \in W$, one has $[m]_z = [0]_z$ and so the zero section maps W into (U, m). Thus, the zero section is continuous.

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To see that scalar multiplication is continuous, let $k \in \Bbbk$ and suppose $[kn]_x = k[n]_x \in (U, m)$. Then there is a compact open neighbourhood W of x with $W \subseteq U$ and $kn\chi_W = m\chi_W$. If $(k, [n]_z) \in \{k\} \times (W, n)$, then $k[n]_z = [kn]_z = [m]_z$ because $z \in W$ and $kn\chi_W = m\chi_W$. This yields continuity of scalar multiplication.

Continuity of addition is proved as follows. Suppose that (U, m) is a basic neighbourhood of $[m_1]_x + [m_2]_x = [m_1 + m_2]_x$. Then there is a compact open neighbourhood W of x with $W \subseteq U$ and $(m_1 + m_2)\chi_W = m\chi_W$. Therefore, if $([m_1]_z, [m_2]_z) \in ((m_1, W) \times (m_2, W)) \cap (\widetilde{M} \times_{\mathscr{G}^{(0)}} \widetilde{M})$, then $[m_1]_z + [m_2]_z = [m_1 + m_2]_z = [m]_z \in (U, m)$. Therefore, addition is continuous.

Next, we must define the $\mathscr{G}$-action. Define, for $g \in \mathscr{G}^{(1)}$, a mapping $R_g : M_{\boldsymbol{r}(g)} \to M_{\boldsymbol{d}(g)}$ by $R_g([m]_{\boldsymbol{r}(g)}) = [m\chi_U]_{\boldsymbol{d}(x)}$, where U is a compact local bisection containing g. We also write $R_g([m]_{\boldsymbol{r}(g)}) = [m]_{\boldsymbol{r}(g)}g$.

Proposition 3.2. *Let $\mathscr{G}$ be an ample groupoid and $\Bbbk$ a commutative ring with unit. Then the following hold.*

(1) *R_g is a well-defined $\Bbbk$-module homomorphism.*
(2) *If $(g, h) \in \mathscr{G}^{(2)}$, then $([m]_{\boldsymbol{r}(g)}g)h = [m]_{\boldsymbol{r}(gh)}(gh)$.*
(3) *If $x \in \mathscr{G}^{(0)}$, then $[m]_x x = [m]_x$.*

Proof. Suppose $g : y \to x$. To show that R_g is well defined, let $[m]_x = [n]_x$ and let $U, V \in \mathrm{Bis}_c(\mathscr{G})$ with $g \in U \cap V$. Then there exist a compact open neighbourhood W of x with $m\chi_W = n\chi_W$ and $Z \in \mathrm{Bis}_c(\mathscr{G})$ such that $g \in Z \subseteq U \cap V$. Note that $g \in WZ \subseteq U \cap V$ and so $y \in Z^{-1}WZ \subseteq \mathscr{G}^{(0)}$. Also, we compute

$$\begin{aligned} m\chi_U\chi_{Z^{-1}WZ} &= m\chi_{UZ^{-1}WZ} = m\chi_{U(WZ)^{-1}WZ} = m\chi_{WZ} = m\chi_W\chi_Z \\ &= n\chi_W\chi_Z = n\chi_{WZ} = n\chi_{V(WZ)^{-1}WZ} = n\chi_{VZ^{-1}WZ} \\ &= n\chi_V\chi_{Z^{-1}WZ}, \end{aligned}$$

which shows that $[m\chi_U]_y = [n\chi_V]_y$, that is, R_g is well defined. Clearly, R_g is $\Bbbk$-linear.

Suppose now $(g, h) \in \mathscr{G}^{(2)}$. Choose $U, V \in \mathrm{Bis}_c(\mathscr{G})$ such that $g \in U$ and $h \in V$. Then $gh \in UV$ and so if $g : y \to x$ and $h : z \to y$, then

$$([m]_x g)h = [m\chi_U]_y h = [m\chi_U\chi_V]_z = [m\chi_{UV}]_z = [m]_z(gh),$$

as required.

Finally, if $x \in \mathscr{G}^{(0)}$ and U is a compact open neighbourhood of x in $\mathscr{G}^{(0)}$, then $[m]_x x = [m\chi_U]_x = [m]_x$ by definition of M_x. □

Proposition 3.3. *Let $\mathscr{G}$ be an ample groupoid and $\Bbbk$ a commutative ring with unit. Then the assignment $M \mapsto \mathrm{Sh}(M)$ constitutes the object part of the functor $\mathrm{Sh} : \text{mod-}\Bbbk\mathscr{G} \to \mathscr{B}_{\Bbbk}\mathscr{G}$.*

Proof. In light of Proposition 3.2, there is an action map

$$\widetilde{M} \times_{\mathscr{G}^{(0)}} \mathscr{G}^{(1)} \to \widetilde{M}$$

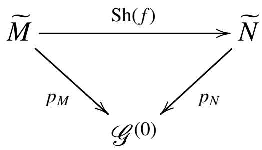
given by $([m]_{\boldsymbol{r}(g)}, g) \mapsto [m]_{\boldsymbol{r}(g)}g$ and satisfying $p_M([m]_{\boldsymbol{r}(g)}g) = \boldsymbol{d}(g)$. To prove that $\mathrm{Sh}(M)$ is a $\mathscr{G}$-sheaf of $\Bbbk$-modules, it remains to show that the action map is continuous. Let $g : y \to x$ and let (m, U) be a basic neighbourhood of $[n]_x g$. Then $y \in U$

and $[m]_y = [n]_x g$. Let $V \in \mathrm{Bis}_c(\mathscr{G})$ with $g \in V$. Then $[n\chi_V]_y = [m]_y$ and so there exists a compact open neighbourhood $W \subseteq U$ of y with $n\chi_{VW} = n\chi_V\chi_W = m\chi_W$. Note that $g \in VW$ and $x \in VWV^{-1} \subseteq \mathscr{G}^{(0)}$. Consider the neighbourhood

$$N = ((n, VWV^{-1}) \times VW) \cap (\widetilde{M} \times_{\mathscr{G}^{(0)}} \mathscr{G}^{(1)})$$

of $([n]_x, g)$. If $([n]_z, h) \in N$, with $h : z' \to z$, then, because $h \in VW$, we have $[n]_z h = [n\chi_{VW}]_{z'} = [m\chi_W]_{z'} = [m]_{z'} \in (m, U)$ as $z' \in V^{-1}VW \subseteq W \subseteq U$. This establishes that $(\widetilde{M}, p)$ is a $\mathscr{G}$-sheaf of $\Bbbk$-modules.

Next, suppose $f : M \to N$ is a $\Bbbk\mathscr{G}$-module homomorphism. Then $f(M(U)) = f(M\chi_U) = f(M)\chi_U \subseteq N\chi_U = N(U)$. Thus, there is an induced $\Bbbk$-linear map $f_x : M_x \to N_x$ given by $f_x([m]_x) = [f(m)]_x$.



by $\mathrm{Sh}(f)([m]_x) = f_x([m]_x)$. First, we check that $\mathrm{Sh}(f)$ preserves the action. Suppose $g : y \to x$ and $U \in \mathrm{Bis}_c(\mathscr{G})$ with $g \in U$. Then

$$f_y([m]_x g) = f_y([m\chi_U]_y) = [f(m\chi_U)]_y = [f(m)\chi_U]_y = [f(m)]_x g = f_x([m]_x)g,$$

as required.

It remains to check continuity of $\mathrm{Sh}(f)$. Let $[m]_x \in \widetilde{M}$ and let (U, n) be a basic neighbourhood of $f_x([m]_x)$. Then $x \in U$ and $f_x([m]_x) = [f(m)]_x = [n]_x$. Choose a compact open neighbourhood W of x contained in U such that $f(m)\chi_W = n\chi_W$. Consider the neighbourhood (W, m) of $[m]_x$. If $[m]_z \in (W, m)$, then $f_z([m]_z) = [f(m)]_z = [f(m)\chi_W]_z = [n\chi_W]_z = [n]_z$ because $z \in W$. Thus, $\mathrm{Sh}(f)(W, m) \subseteq (U, n)$, yielding the continuity of $\mathrm{Sh}(f)$. It is obvious that Sh is a functor. □

The following lemma will be useful for proving that these functors are quasi-inverse.

Lemma 3.4. *Let* $M \in$ mod-$\Bbbk\mathscr{G}$. *If* $U \in \mathrm{Bis}_c(\mathscr{G})$ *and* $x \notin U^{-1}U$, *then* $[m\chi_U]_x = 0$.

Proof. Since $U^{-1}U$ is compact and $\mathscr{G}^{(0)}$ is Hausdorff, we can find a compact open neighbourhood W of x with $W \cap U^{-1}U = \emptyset$. Then $m\chi_U\chi_W = m\chi_{UW} = 0$. □

Theorem 3.5. *Let* $\mathscr{G}$ *be an ample groupoid and* $\Bbbk$ *a commutative ring with unit. Then there are natural isomorphisms* $\Gamma_c \circ \mathrm{Sh} \cong 1_{\text{mod-}\Bbbk\mathscr{G}}$ *and* $\mathrm{Sh} \circ \Gamma_c \cong 1_{\mathcal{B}_{\Bbbk}\mathscr{G}}$. *Hence, the categories* mod-$\Bbbk\mathscr{G}$ *and* $\mathcal{B}_{\Bbbk}\mathscr{G}$ *are equivalent.*

Proof. Let M be a unitary $\Bbbk\mathscr{G}$-module and define $\eta_M : M \to \Gamma_c(\mathrm{Sh}(M))$ by $\eta_M(m) = s_m$, where $s_m(x) = [m]_x$ for all $x \in X$. We claim that s_m is continuous with compact support. Continuity is easy: if $s_m(x) \in (U, n)$, then $x \in U$ and $[m]_x = [n]_x$. So, there is a compact open neighbourhood W of x with $W \subseteq U$ and $m\chi_W = n\chi_W$. Then, if $z \in W$, we have $s_m(z) = [m]_z = [m\chi_W]_z = [n\chi_W]_z = [n]_z \in (U, n)$. Thus, s_m is continuous. We claim

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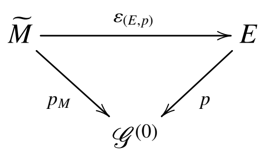
that the support of s_m is compact. Let $U \in \mathcal{B}$ with $m\chi_U = m$. Suppose $x \notin U$. Then Lemma 3.4 implies $[m]_x = [m\chi_U]_x = 0$. Thus, the support of s_m is a closed subset of U and hence compact.

We claim that η_M is an isomorphism (it is clearly natural in M). Let us first show that η_M is a module homomorphism. It is clearly $\Bbbk$-linear and hence it suffices to show that if $U \in \mathrm{Bis}_c(\mathscr{G})$, then $\eta_M(m\chi_U) = \eta_M(m)\chi_U$. Note that $\eta_M(m\chi_U) = s_{m\chi_U}$. If $x \notin U^{-1}U$, then $s_{m\chi_U}(x) = [m\chi_U]_x = 0$ by Lemma 3.4. If $x = \boldsymbol{d}(g)$ with $g \in U$, then we have $s_{m\chi_U}(x) = [m\chi_U]_x = [m]_{\boldsymbol{r}(g)}g = s_m(\boldsymbol{r}(g))g$. Therefore, in light of Proposition 3.1, we conclude that $s_{m\chi_U} = s_m\chi_U$. This shows that η_M is a $\Bbbk\mathscr{G}$-module homomorphism.

Suppose $0 \neq m \in M$. Then $m\chi_U = m$ for some $U \in \mathcal{B}$. Let $\mathcal{B}_U$ be the boolean ring of compact open subsets of U. Let I be the ideal of $\mathcal{B}_U$ consisting of those V with $m\chi_V = 0$. This is a proper ideal (since $U \notin I$) and hence contained in a maximal ideal $\mathfrak{m}$. Let x be the point of U corresponding to $\mathfrak{m}$ under Stone duality. Then the ultrafilter of compact open neighbourhoods of x is $\mathcal{B}\backslash\mathfrak{m}$ and hence does not intersect I. Therefore, $m\chi_V \neq 0$ for all compact open neighbourhoods of x, that is, $s_m(x) = [m]_x \neq 0$. Therefore, $\eta_M(m) = s_m \neq 0$ and so η_M is injective.

To see that η_M is surjective, let $s \in \Gamma_c(\mathrm{Sh}(M))$ and let K be the support of s. For each $x \in K$, we can find a compact open neighbourhood U_x of x and an element $m_x \in M$ such that $s(z) = [m_x]_z$ for all $z \in U_x$ (choose U_x mapping under s into a basic neighbourhood of $s(x)$ of the form (V_x, m_x)). By compactness of K, we can find a finite subcover of the U_x with $x \in X$. Since $\mathscr{G}^{(0)}$ is Hausdorff, we can refine the subcover by a partition into compact open subsets; that is, we can find disjoint compact open sets $V_1, \ldots, V_n$ and elements $m_1, \ldots, m_n \in M$ such that $K \subseteq V_1 \cup \cdots \cup V_n$ and $s(x) = [m_i]_x$ for all $x \in V_i$. Consider $m = m_1\chi_{V_1} + \cdots + m_n\chi_{V_n}$. Then $m\chi_{V_i} = m_i\chi_{V_i}$ and so $[m]_x = [m_i]_x = s(x)$ for all $x \in V_i$. We conclude that $[m]_x = s(x)$ for all $x \in V_1 \cup \cdots \cup V_n$. If $x \notin V_1 \cup \cdots \cup V_n$, then $x \notin K$ and so $s(x) = 0$. But also $[m]_x = \sum_{i=1}^{n}[m_i\chi_{V_i}]_x = 0$ by Lemma 3.4. Thus, $s(x) = [m]_x = s_m(x)$ for all $x \in \mathscr{G}^{(0)}$ and hence $s = \eta_M(m)$. This concludes the proof that η_M is an isomorphism.

Next, let (E, p) be a $\mathscr{G}$-sheaf of $\Bbbk$-modules and put $M = \Gamma_c(E, p)$. We define an isomorphism



of $\mathrm{Sh}(\Gamma_c(E, p))$ and (E, p) as follows. Define $\varepsilon_{(E,p)}([s]_x) = s(x)$ for $s \in M$ and $x \in \mathscr{G}^{(0)}$. This is well defined because if $s\chi_U = s'\chi_U$ for some compact open neighbourhood U of x, then $s(x) = s'(x)$ by Proposition 3.1. Also, $p(s(x)) = x = p_M([s]_x)$, whence $p \circ \varepsilon_{(E,p)} = p_M$. Clearly, $\varepsilon_{(E,p)}$ restricts to a $\Bbbk$-module homomorphism on each fibre. Let $g : y \longrightarrow x$ and suppose $U \in \mathrm{Bis}_c(\mathscr{G})$ with $g \in U$; then $\varepsilon_{(E,p)}([s]_x g) = \varepsilon_{(E,p)}([s\chi_U]_y) = (s\chi_U)(y) = s(x)g = \varepsilon_{(E,p)}([s]_x)g$ (using Proposition 3.1). It therefore remains to prove that $\varepsilon_{(E,p)}$ is a homeomorphism.

To see that $\varepsilon_{(E,p)}$ is continuous, let $[s]_x \in \widetilde{M}$ and let U be a neighbourhood of $\varepsilon_{(E,p)}([s]_x) = s(x)$. Let W be a compact open neighbourhood of x with $s(W) \subseteq U$.

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Consider the neighbourhood (W, s) of $[s]_x$. Then, for $[s]_z \in (W, s)$, we have $\varepsilon_{(E,p)}([s_z]) = s(z) \in U$. Thus, $\varepsilon_{(E,p)}$ is continuous. As p, p_M are local homeomorphisms, we deduce from $p \circ \varepsilon_{(E,p)} = p_M$ that $\varepsilon_{(E,p)}$ is a local homeomorphism and hence open. It remains to prove that $\varepsilon_{(E,p)}$ is bijective.

Suppose $s(x) = \varepsilon_{(E,p)}([s]_x) = \varepsilon_{(E,p)}([t]_x) = t(x)$. Choose a neighbourhood U of $s(x) = t(x)$ such that $p|_U$ is a homeomorphism onto its image. Let W be a compact open neighbourhood of x such that both $s(W) \subseteq U$ and $t(W) \subseteq U$. Then, if $z \in W$, we have $p(s(z)) = z = p(t(z))$ and $s(z), t(z) \in U$ and hence $s(z) = t(z)$. Thus, $s\chi_W = t\chi_W$ (compare with Proposition 3.1) and so $[s]_x = [t]_x$. This yields injectivity of $\varepsilon_{(E,p)}$. Next, let $e \in E_x$. Let U be a neighbourhood of e such that $p|_U : U \to p(U)$ is a homeomorphism. Let W be a compact open neighbourhood of x contained in $p(U)$ and define $s \in \Gamma_c(E, p)$ to agree with $(p|_U)^{-1}$ on W and be 0 outside of W. Then $\varepsilon_{(E,p)}([s]_x) = s(x) = e$. Clearly, $\varepsilon_{(E,p)}$ is natural in (E, p). This completes the proof. □

As a corollary, we recover the main result of [7], and moreover we do not require the Hausdoff assumption.

Corollary 3.6. *Let $\mathscr{G}$ and $\mathscr{H}$ be Morita equivalent ample groupoids. Then $\Bbbk\mathscr{G}$ is Morita equivalent to $\Bbbk\mathscr{H}$ for any commutative ring with unit $\Bbbk$.*

By restricting to the case where $\mathscr{G}^{(1)} = \mathscr{G}^{(0)}$, we also have the following folklore result.

Corollary 3.7. *Let X be a Hausdorff space with a basis of compact open subsets and $\Bbbk$ a commutative ring with unit. Let $C_c(X, \Bbbk)$ be the ring of locally constant functions $X \to \Bbbk$ with compact support. Then the category of sheaves of $\Bbbk$-modules on X is equivalent to the category of unitary $C_c(X, \Bbbk)$-modules.*

If $\mathscr{G}$ is a discrete groupoid, then $\mathscr{B}\mathscr{G}$ is equivalent to the category $\mathbf{Set}^{\mathscr{G}^{op}}$ of contravariant functors from $\mathscr{G}$ to the category of sets [11, 12]. Therefore, $\mathscr{B}_{\Bbbk}\mathscr{G}$ is equivalent to the category $(\text{mod-}\Bbbk)^{\mathscr{G}^{op}}$ of contravariant functors from $\mathscr{G}$ to mod-$\Bbbk$. It is well known that $(\text{mod-}\Bbbk)^{\mathscr{G}^{op}}$ is equivalent to mod-$\Bbbk\mathscr{G}$ when $\mathscr{G}^{(0)}$ is finite [15] and presumably the following extension is also well known, although the author does not know a reference.

Corollary 3.8. *Let $\mathscr{G}$ be a discrete groupoid and $\Bbbk$ a commutative ring with unit. Then mod-$\Bbbk\mathscr{G}$ is equivalent to the category $(\text{mod-}\Bbbk)^{\mathscr{G}^{op}}$ of contravariant functors $\mathscr{G} \to$ mod-$\Bbbk$. Hence, naturally equivalent discrete groupoids have Morita equivalent algebras.*

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