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THE K-FUNCTIONAL OF CERTAIN PAIRS OF REARRANGEMENT INVARIANT SPACES

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Let X, Y be rearrangement invariant spaces and let $M = M(Y, X)$ be the space of all multipliers of Y into X. It is shown that if $X = YM$ and some technical conditions are satisfied, then the K-functional $K(t, f, X, Y)$ is equivalent to the expression

$$\|f^*\chi_{[0,\psi(t))}\|_X + t\|f^*\chi_{[\psi(t),\infty)}\|_Y$$

where ψ is the inverse of the fundamental function φ_M of M, defined by $\varphi_M(u) = \|\chi_{[0,u)}\|_M$.

Let X, Y be *rearrangement invariant spaces* over $[0, \infty)$ (see [3], §2.a, for basic facts on rearrangement invariant spaces). The *K-functional* of the pair (X, Y) is defined for every $f \in X+Y$ and $t > 0$ by

$$K(t, f, X, Y) = \inf\{\|g\|_X+t\|h\|_Y;\ g \in X,\ h \in Y,\ f = g+h\}$$

(see [1] for the application of the K-functional in interpolation theory).

In this note we compute, up to equivalence, the K-functional of certain pairs (X, Y) of rearrangement invariant spaces, using an auxiliary space of all multipliers from Y into X.

We denote by $M = M(Y, X)$ the space of all *multipliers from* Y *into*

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X , that is, all measurable functions f so that $fg \in X$ for every $g \in Y$, normed by

$$\|f\|_M = \sup\{\|fg\|_X;\ g \in Y,\ \|g\|_Y \le 1\} .$$

It is easily verified that M is a Banach lattice (under the pointwise almost everywhere ordering) and that its norm is rearrangement invariant in the sense that if τ is a measure automorphism of $[0, \infty)$ then $f \in M$ if and only if $f \circ \tau \in M$, and in this case $\|f\|_M = \|f \circ \tau\|_M$. In what follows we assume, furthermore, that

(a) M is an rearrangement invariant space in the sense of [3], §2.a; in particular, $L_1(0, \infty) \cap L_\infty(0, \infty) \subseteq M$,

(b) the fundamental function $\varphi_M(u) = \|\chi_{[0,u)}\|_M$ is strictly increasing, satisfying $\varphi_M(0+) = \lim_{u\to 0} \varphi_M(u) = 0$ and $\varphi_M(\infty-) = \lim_{u\to\infty} \varphi_M(u) = \infty$,

(c) every $f \in X$ has a representation $f = gh$ with $g \in Y$ and $h \in M$, that is, $X = YM$.

We remark that (b) implies that $M \ne L_\infty(0, \infty)$, and thus $X \ne Y$. Also, the fundamental function of every rearrangement invariant space is known to be continuous; thus φ_M is continuous by (a) (see, for example, [7]).

It follows that the inverse function $\psi = \varphi_M^{-1} : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing, and satisfies

$$\|\chi_{[0,\psi(t))}\|_M = t \ , \quad t > 0 .$$

As usual we denote by f^* the *decreasing rearrangement* of the measurable function f (see [3], §2.a for basic properties). Our main result is the following theorem.

THEOREM 1. *Let* X, Y *be rearrangement invariant spaces over* $[0, \infty)$, *and let* $M = M(Y, X)$ *be the space of all multipliers from* Y *into* X . *Suppose that properties* (a)-(c) *are satisfied. Then the* K-*functional of the pair* (X, Y) *is equivalent to the expression*

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$$F(t, f, X, Y) = \|f^*\chi_{[0,\psi(t))}\|_X + t\|f^*\chi_{[\psi(t),\infty)}\|_Y .$$

Precisely, there exists a constant $A > 0$ *depending only on* X *and* Y *, so that, for every* $f \in X+Y$ *and every* $t > 0$ *,*

$$A^{-1} \cdot F(t, f, X, Y) \le K(t, f, X, Y) \le F(t, f, X, Y) .$$

Before we proceed with the proof, we consider the most important case where $X = L_p(0, \infty)$ and $Y = L_q(0, \infty)$. If $1 \le p < q \le \infty$ then $M = M(L_q, L_p) = L_r$, where $r^{-1} + q^{-1} = p^{-1}$. Properties (a)-(c) hold, and $\varphi_M(u) = u^{1/r}$. Thus $\psi(t) = t^r$, and Theorem 1 says that

$$K(t, f, L_p, L_q) \approx \left(\int_0^{t^r} f^*(s)^p ds\right)^{1/p} + t\left(\int_{t^r}^{\infty} f^*(s)^q ds\right)^{1/q} ,$$

which is the well known result of Holmstedt [2, Theorem 4.1].

If $q = \infty$, then one gets easily (see [1])

$$K(t, f, L_p, L_\infty) \approx \left(\int_0^{t^p} f^*(s)^p ds\right)^{1/p} .$$

If $1 \le q < p \le \infty$ then $M = M(L_q, L_p) = \{0\}$, so (a)-(c) fail. But in this case we can use the general formula

$$K(t, f, X, Y) = tK(t^{-1}, f, Y, K)$$

to compute the K-functional of (L_p, L_q) in terms of that of (L_q, L_p) .

Next we shall need the following facts.

PROPOSITION 2. *Let* $f_1, f_2 \in L_1(0, \infty) + L_\infty(0, \infty)$ *, and let* $0 < t_1, t_2$. *Then*

(i) $(f_1+f_2)^*(t_1+t_2) \le f^*(t_1) + f^*(t_2)$,

(ii) $(f_1 \cdot f_2)^*(t_1+t_2) \le f^*(t_1) \cdot f^*(t_2)$.

Indeed, this follows from the obvious formulas

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$$\text{(i')}\quad \{s;\ |f_1(s)+f_2(s)| > f_1^*(t_1)+f_2^*(t_2)\}$$
$$\subseteq \{s;\ |f_1(s)| > f_1^*(t_1)\} \cup \{s;\ |f_2(s)| > f_2^*(t_2)\}$$

and

$$\text{(ii')}\quad \{s;\ |f_1(s)f_2(s)| > f_1^*(t_1)\cdot f_2^*(t_2)\}$$
$$\subseteq \{s;\ |f_1(s)| > f_1^*(t_1)\} \cup \{s;\ |f_2(s)| > f_2^*(t_2)\}$$

and the definition of the decreasing rearrangement in terms of the distribution function; see [3], §2.a.

PROPOSITION 3. *Let* X, Y *and* M *be as above, and suppose that* $X = YM$. *Then there exists a constant* $1 \le B < \infty$ *so that for every non-negative, non-increasing function* $f \in X$ *there exist non-negative, non-increasing functions* $g \in Y$ *and* $h \in M$ *so that* $f \le gh$ *and* $\|g\|_Y \cdot \|h\|_M \le B\|f\|_X$.

We shall prove Proposition 3 later on.

Finally, recall that for every $0 < s < \infty$ the *dilation operator* D_s is defined by

$$(D_s f)(t) = f(t/s) \ , \quad 0 \le t < \infty \ ,$$

and is bounded in any rearrangement invariant space X . We denote by $\|D_s\|_X$ the norm of D_s as an operator on X .

Proof of Theorem 1. Fix $0 < t$ and $f \in X+Y$. Let

$$K(t,\ f) = K(t,\ f,\ X,\ Y) \quad\text{and}\quad F(t,\ f) = F(t,\ f,\ X,\ Y) \ .$$

Clearly $K(t,\ f) = K(t,\ f^*)$ and $F(t,\ f) = F(t,\ f^*)$ so there is no loss of generality in assuming that $f = f^*$; that is, $f \ge 0$ and f is non-increasing. For every measurable set E we have $f = f\chi_E + F\chi_{\sim E}$ and thus

$$K(t,\ f) \le \|f\chi_E\|_X + t\|f\chi_{\sim E}\|_Y \ .$$

Using this with $E = [0,\ \psi(t))$ we get $K(t,\ f) \le F(t,\ f)$ (where the right hand side may, *a priori*, be infinite). For the converse inequality, suppose that $f = g + h$ is an arbitrary decomposition with $g \in X$ and $h \in Y$. Then, using Proposition 2 *(i)*, we get, for every $0 < \alpha < 1$,

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$$f(s) = f^*(s) \le g^*\big((1-\alpha)s\big) + h^*(\alpha s) = D_{1/(1-\alpha)}(g^*)(s) + D_{1/\alpha}(h^*)(s) .$$

So, with $u = \psi(t)$, we have

$$\begin{aligned}(1)\qquad \|f\chi_{[0,u)}\|_X &\le \|D_{1/(1-\alpha)}(g^*)\cdot\chi_{[0,u)}\|_X + \|D_{1/\alpha}(h^*)\cdot\chi_{[0,u)}\|_X \\ &\le \|D_{1/(1-\alpha)}\|_X\|g\|_X + \|D_{1/\alpha}\|_Y\|h\|_Y\|\chi_{[0,u)}\|_M \\ &\le \max\{\|D_{1/(1-\alpha)}\|_X, \|D_{1/\alpha}\|_Y\} \cdot \big(\|g\|_X+t\|h\|_Y\big) .\end{aligned}$$

Also

$$\begin{aligned}(2)\qquad t\|f\chi_{[u,\infty)}\|_Y &\le t\big(\|D_{1/(1-\alpha)}(g^*)\chi_{[u,\infty)}\|_Y+\|D_{1/\alpha}(h^*)\chi_{[u,\infty)}\|_Y\big) \\ &\le \|\chi_{[0,u)}\|_M\|D_{1/(1-\alpha)}(g^*)\chi_{[u,\infty)}\|_Y + \|D_{1/\alpha}\|_Y t\|h\|_Y .\end{aligned}$$

Notice that $g_0 = D_{1/(1-\alpha)}(g^*) \in X$ is non-negative and non-increasing. So by Proposition 3 there exist non-negative, non-increasing functions $g_1 \in Y$ and $g_2 \in M$ with $g_0 \le g_1g_2$ and $\|g_1\|_Y\|g_2\|_M \le B\|g_0\|_X$.

It follows that

$$\begin{aligned}(3)\qquad \|\chi_{[0,u)}\|_M\|g_0\chi_{[u,\infty)}\|_Y &\le \|\chi_{[0,u)}\|_M\|g_1g_2\chi_{[u,\infty)}\|_Y \\ &\le \|\chi_{[0,u)}\|_M g_2(u)\|g_1\chi_{[u,\infty)}\|_Y \\ &\le \|g_2\chi_{[0,u)}\|_M\|g_1\|_Y \\ &\le \|g_2\|_M\|g_1\|_Y \le B\|g_0\|_X \le B\|D_{1/(1-\alpha)}\|_X\|g\|_X .\end{aligned}$$

Combining (2) and (3) we get

$$t\|f\chi_{[u,\infty)}\|_Y \le B \cdot \max\{\|D_{1/(1-\alpha)}\|_X, \|D_{1/\alpha}\|_Y\} \cdot \big(\|g\|_X+t\|h\|_Y\big) ,$$

and so, by (1),

$$\begin{aligned}F(t, f) &= \|f\chi_{[0,u)}\|_X + t\|f\chi_{[u,\infty)}\|_Y \\ &\le 2B \cdot \max\{\|D_{1/(1-\alpha)}\|_X, \|D_{1/\alpha}\|_Y\} \cdot \big(\|g\|_X+t\|h\|_Y\big) .\end{aligned}$$

Taking infimum over all representations $f = g + h$ with $g \in X$ and $h \in Y$ we get

$$F(t, f) \le AK(t, f)$$

where

$$A = 2B \cdot \inf_{0<\alpha<1} \big(\max\{\|D_{1/(1-\alpha)}\|_X, \|D_{1/\alpha}\|_Y\}\big) .$$

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This completes the proof of Theorem 1. □

Proof of Proposition 3. We show first that there exists a constant $B_1 > 0$ so that every $f \in X$ admits a representation $f = gh$ with $g \in Y$ and $h \in M$ and

$$\|g\|_Y \|h\|_M \le B_1 \|f\|_X \, .$$

Indeed if there is no such constant then there exist a sequence $\{E_n\}_{n=1}^{\infty}$ of disjoint measurable sets of positive measure, and a sequence $\{f_n\}_{n=1}^{\infty}$ of functions in X satisfying $f_n = f_n \chi_{E_n}$, $\|f_n\|_X = 1$, and so that if $f_n = gh$ with $g = g\chi_{E_n} \in Y$ and $h = h\chi_{E_n} \in M$ then $\|g\|_Y \|h\|_M \ge n^3$. Let $f = \sum_{n=1}^{\infty} f_n/n^2$. Then $f \in X$ and so $f = gh$ with $g \in Y$ and $h \in M$. Since $f_n/n^2 = g\chi_{E_n} \cdot h\chi_{E_n}$, we get, for every n,

$$n^3 \le n^2 \|g\chi_{E_n}\|_Y \|h\chi_{E_n}\|_M \le n^2 \|g\|_Y \|h\|_M$$

which is obviously a contradiction.

Next suppose that $f \in X$ is non-negative and on-increasing. By the first step of the proof, $f = gh$, with $g \in Y$, $h \in M$ and $\|g\|_Y \|h\|_M \le B_1 \|f\|_X$. By Proposition 2 *(ii)*,

$$f(s) \le \left(D_{\frac{1}{2}} g^*\right)(s) \cdot D_{\frac{1}{2}}(h^*)(s) \, .$$

Clearly $g_1 = D_{\frac{1}{2}}(g^*) \in Y$ and $h_1 = D_{\frac{1}{2}}(h^*) \in M$ are both non-negative and non-increasing, and

$$\begin{aligned} \|g_1\|_Y \|h_1\|_M &\le \left(\|D_{\frac{1}{2}}\|_Y \|g\|_Y\right) \left(\|D_{\frac{1}{2}}\|_M \|h\|_M\right) \\ &\le \left(B_1 \|D_{\frac{1}{2}}\|_Y \cdot \|D_{\frac{1}{2}}\|_M\right) \|f\|_X \, . \end{aligned}$$

This completes the proof of Proposition 3, with $B = B_1 \|D_{\frac{1}{2}}\|_Y \cdot \|D_{\frac{1}{2}}\|_M$. (Notice that since $\|gh\|_X \le \|g\|_Y \|h\|_M$ for every $g \in Y$ and $h \in M$, we have $B \ge 1$.) □

Let us apply Theorem 1 to an important special case, generalizing

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Holmstedt's result [2]. Let X be a minimal rearrangement invariant space over $[0, \infty)$ so that $\varphi_X(u) = \|\chi_{[0,u)}\|_X$ is strictly increasing and satisfies $\lim_{u\to 0} \varphi_X(u) = 0$, $\lim_{u\to\infty} \varphi_X(u) = \infty$. For every $0 < \alpha \le 1$ let

$$X^\alpha = \{f;\ |f|^{1/\alpha} \in X\}$$

normed by

$$\|f\|_{X^\alpha} = \||f|^{1/\alpha}\|_X^\alpha .$$

It is well known that X^α is a rearrangement invariant space (identified with the p-convexification of X, where $\alpha = 1/p$). Moreover, if X is reflexive, then $X^\alpha = \left(L_\infty(0, \infty), X\right)_\alpha$. We define $X^0 = L_\infty(0, \infty)$.

PROPOSITION 4. *Let X be as above and let $0 \le \beta < \alpha \le 1$. Then*

(i) $M = M\left(X^\beta, X^\alpha\right) = X^{\alpha-\beta}$, *and properties* (a)-(c) *hold for the pair* $\left(X^\alpha, X^\beta\right)$,

(ii) $K\left(t, f, X^\alpha, X^\beta\right)$ *is equivalent to the expression*

$$F\left(t, f, X^\alpha, X^\beta\right) = \|f^*\chi_{[0,\psi(t))}\|_{X^\alpha} + t\|f^*\chi_{[\psi(t),\infty)}\|_{X^\beta}$$

$$= \left\|(f^*)^{1/\alpha}\chi_{[0,\psi(t))}\right\|_X^\alpha + t\left\|(f^*)^{1/\beta}\chi_{[\psi(t),\infty)}\right\|_X^\beta$$

where $\psi(t) = \varphi_X^{-1}\left(t^{1/(\alpha-\beta)}\right)$.

Proof. We prove only the fact that $M = X^{\alpha-\beta}$ in the case $\beta > 0$; the rest follows easily from our assumptions on X and Theorem 1 (notice that if $\beta = 0$ then $M = X^\alpha$). Let $f \in X^{\alpha-\beta}$ and $g \in X^\beta$. Then $f_1 = |f|^{1/(\alpha-\beta)}$ and $g_1 = |g|^{1/\beta}$ belong to X, and thus (see [3], p. 43)

$$|fg|^{1/\alpha} = f_1^{1-(\beta/\alpha)} \cdot g_1^{\beta/\alpha} \in X ,$$

that is, $fg \in X^\alpha$, and

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$$\|fg\|_{X^\alpha} = \left\|f_1^{1-(\beta/\alpha)}\cdot g_1^{\beta/\alpha}\right\|_X^\alpha$$
$$\le \|f_1\|_X^{\alpha-\beta}\cdot\|g_1\|_X^\beta = \|f\|_{X^{\alpha-\beta}}\cdot\|g\|_{X^\beta} .$$

This shows that $f \in M$ and $\|f\|_M \le \|f\|_{X^{\alpha-\beta}}$. For the converse inclusion, let $f \in L_1(0, \infty) \cap L_\infty(0, \infty)$ be a non-negative function. Let $s > 0$ be defined by $(1+s)/\alpha = s/\beta$. Then

$$\|f^{1+s}\|_{X^\alpha} \le \|f\|_M\|f^s\|_{X^\beta} = \|f\|_M\|f^{s/\beta}\|_X^\beta .$$

Thus

$$\|f^{(1+s)/\alpha}\|_X^\alpha \le \|f\|_M\|f^{(1+s)/\alpha}\|_X^\beta$$

so, by the definition of s,

$$\|f^{\alpha-\beta}\|_X^{\alpha-\beta} = \|f\|_{X^{\alpha-\beta}} \le \|f\|_M .$$

By minimality of X this implies that $M \subseteq X^{\alpha-\beta}$, and thus $M = X^{\alpha-\beta}$ (with equality of norms). □

REMARK 1. Theorem 1 holds, with obvious modifications, for rearrangement invariant spaces over $\mathbb{N} = \{0, 1, 2, \ldots\}$, that is, for symmetric sequence spaces.

REMARK 2. After completing the first draft of the present paper, we learned from M. Cwikel that some (slightly weaker) versions of our Theorem 1 have already been established by different methods in [6], [4], and [5]. Our proof avoids the use of the auxiliary spaces of the form $M(X)$ and $\Lambda(X)$, as well as the hypothesis that $\varphi_M(u)/u^r$ is increasing for some $r > 0$.

We conclude the paper with the following problem.

PROBLEM. *Find an explicit formula for the* K*-functional of a general pair* (X, Y) *of rearrangement invariant spaces on* Ω*, where* Ω *is either* $\mathbb{N}$ *or* $[0, 1]$ *or* $[0, \infty)$. *In particular, does there exist for every* $t > 0$ *a measurable set* $A = A_t \subseteq \Omega$*, so that*

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$$K(t, f, X, Y) \approx \|f^*\chi_A\|_X + t\|f^*\chi_{\Omega\setminus A}\|_Y$$

for every $f \in X+Y$?

We suggest the use of the function $\|D_u\|_X/\|D_u\|_Y$ instead of our $\varphi_M(u)$, or Milman's $s(u) = \varphi_X(u)/\varphi_Y(u)$.

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