GENERIC EXPANSIONS OF GEOMETRIC THEORIES

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Abstract. As a continuation of ideas initiated in [19], we study bi-colored (generic) expansions of geometric theories in the style of the Fraïssé–Hrushovski construction method. Here we examine that the properties NTP_2, strongness, $NSOP_1$, and simplicity can be transferred to the expansions. As a consequence, while the corresponding bi-colored expansion of a red non-principal ultraproduct of p-adic fields is NTP_2, the expansion of algebraically closed fields with generic automorphism is a simple theory. Furthermore, these theories are strong with bdn("$x = x$") $= (\aleph_0)_-$.

§1. Introduction. Extending the methods of stability and applying them to a larger class of theories is a dominating theme in current research of pure model theory. This line of research shows the prevalence of these methods with potential applications beyond model theory. To instantiate general concepts in stability-hierarchy and perhaps examine some related open questions/conjectures one would need to look for some new examples, conceivably through adapting known model-theoretic methods. Our main aim in this paper is to study the Fraïssé–Hrushovski method beyond the realm of stability/simplicity. We aim to continue further, ideas started in [14] to use the Fraïssé–Hrushovski construction for studying bi-colored expansions of geometric theories which are either $NSOP_1$, simple, or NTP_2.

Our motivation mainly stems in the comprehensive studies of the expansion of algebraically closed fields with a unary predicate p—often called a color predicate—interpreted either by an arbitrary set (Black fields) [2, 3, 21], an additive subgroup (Red fields) [6, 22], or a multiplicative subgroup (Green fields) [5, 22]. All examples obtained by this constructions are ω-stable, either with Morley rank ω (non-collapsed constructions) [3, 21, 22] or finite Morley rank (collapsed constructions) [2, 5].

The other theme that our results are naturally connected to is the study of the generic expansions of models of geometric theories by a unary predicate which can be interpreted either by a submodel (Lovely pairs) [7, 9, 10, 20] or more generally by submodels of reducts [10, 11, 12, 17].

To explain our contributions in more technical terms, we assume that T is a complete geometric theory without finite models in a countable language $\mathcal{L}$. It is routine to assume that T admits elimination of quantifiers in $\mathcal{L}$. The theory T is geometric if it eliminates the quantifier $\exists^\infty$ and the algebraic closure operator gives

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rise to a pre-geometry. As a consequence, a notion of dimension function, dim, can be defined; where $\dim(\bar{a})$ for any finite tuple $\bar{a}$ is the size of a basis of $\bar{a}$. The theory T is further required to satisfy the free-amalgamation property (Definition 2.6). Subsequently, $\mathcal{L}$ is expanded to $\mathcal{L}_p = \mathcal{L} \cup \{p\}$ by adding a unary predicate p called the *coloring predicate*. We consider the class of all $\mathcal{L}_p$-structures whose universe M is a model of $T^\forall$, fix a *rational* number $0 < \alpha \leq 1$, and for $M \models T^\forall$ and a finite subset A of M, define the pre-dimension function

$$\delta_\alpha(A) = \dim(A) - \alpha|p(A)|.$$

Now, as T is geometric, the dimension function satisfies certain definability conditions which makes $\mathcal{K}_\alpha^+$, the class of $\mathcal{L}_p$-structures M such that $\delta_\alpha(A) \geq 0$ for all $A \subseteq_{\text{fin}} M$, $\mathcal{L}_p$-axiomatizable.

There is a notion $\leqslant_\alpha$ of closed substructures in $\mathcal{K}_\alpha^+$ associated with the pre-dimension function δ_α. The free-amalgamation of T implies that $(\mathcal{K}_\alpha^+, \leqslant_\alpha)$ has the amalgamation property which guaranties that $(\mathcal{K}_\alpha^+, \leqslant_\alpha)$ has Fraïssé limits for arbitrary cardinal $\lambda \geq \aleph_0$ denoted by λ-rich models.

Here we give a complete axiomatization $\mathbb{T}_\alpha$ for the class all $\aleph_0$-rich structures. This axiomatization together with a description of types enables us to prove that certain model theoretic properties of T can be transferred to $\mathbb{T}_\alpha$. More precisely, the following results are obtained in this paper.

THEOREM (Theorems 4.14, 4.22, and 4.25). *If T is NTP$_2$, strong, NSOP$_1$ and simple then so is $\mathbb{T}_\alpha$.*

THEOREM (Theorem 4.16). *If T is further strong and indecomposable (Definition 2.9) then* $\text{bdn}(\mathbb{T}_\alpha) = (\aleph_0)_-$.

COROLLARY (Corollaries 4.17 and 4.27).

1. *Let T be the theory of a non-principal ultraproduct of $\mathbb{Q}_p$'s. Then $\mathbb{T}_{\frac{1}{2}}$ is a strong theory with* $\text{bdn}(\mathbb{T}_{\frac{1}{2}}) = (\aleph_0)_-$.
2. *Let T be any complete theory of a pseudo finite field. Then $\mathbb{T}_{\frac{1}{2}}$ is a simple theory of unbounded weight.*

It is worth mentioning some technical differences between the present work and [14]. First, while in [14] α is assumed to be both rational and irrational, here we restrict ourselves only to rational α's. This restriction yields less technical difficulties in axiomatizing the class of $\aleph_0$-rich structures. On the other hand, in [14] in addition to quantifier elimination and free amalgamation properties, T is assumed to be a geometric indecomposable theory. This extra condition implies that there exists a simpler (in fact Π_2) axiomatization of $\mathbb{T}_\alpha$. However here we prefer not to impose the indecomposability condition and only use it to show Theorem 4.16.

The structure of the paper is as follows. In Section 2 after fixing the setting and reviewing the basic concepts, we introduce the class $(\mathcal{K}_\alpha^+, \leqslant_\alpha)$. In Section 3 we prove there is a complete axiomatization $\mathbb{T}_\alpha$ for its rich structures. Finally in Section 4 we prove the main theorems mentioned above.

§2. Preliminaries and conventions. Throughout this paper, $\mathcal{L}$ is a countable language and T is a complete geometric theory without finite models and has the quantifier elimination. Recall that T is *geometric* if it eliminates the quantifier $\exists^\infty$ and the algebraic closure, acl, satisfies the exchange property.

Convention. We use capital letters M, N, P, K for the $\mathcal{L}$-structures and A, B, C, D and X, Y, Z, show finite and infinite sets, respectively. Instead of $X \cup Y$ we would write XY. For tuples $\bar{a}, \bar{b}$ in a model of M of T (or even $T^\forall$) by the quantifier-free type of $\bar{a}$ over $\bar{b}$, denoted by $\mathrm{qftp}_{\mathcal{L}}(\bar{a}/\bar{b})$, we mean the set of all quantifier-free formulas $\varphi(\bar{x}, \bar{b})$ such that $M \models \varphi(\bar{a}, \bar{b})$.

The dimension obtained by acl in T is denoted by dim. So for a set Y, $\dim(\bar{a}/Y)$ is the size of the acl-base of $\{a_1, \dots, a_n\}$ over Y. If $\bar{b} = (b_1, \dots, b_n)$, then $\dim(\bar{a}/\bar{b}) = \dim(\bar{a}/\{b_1, \dots, b_m\})$, for $\bar{a} = (a_1, \dots, a_n)$. If $\varphi(\bar{x}, \bar{y})$ is an $\mathcal{L}$-formula and $\bar{b} \in M^{|\bar{y}|}$, then $\dim(\varphi(M, \bar{b})) = \max\{\dim(\bar{a}/\bar{b}) : \quad M \models \varphi(\bar{a}, \bar{b})\}$.

The set Y is called dim-independent from Z over X and denoted by $Y \mathop{\smile\!\!\!\!\!\mid}^{\dim}_X Z$ if for every $\bar{a} \in Y^{|\bar{a}|}$, $\dim(\bar{a}/X) = \dim(\bar{a}/XZ)$. Moreover, if $Y \cap Z = X$ we state that Y and Z are *free* over X. To emphasize that Y and Z are free over X, the set YZ is written as $Y \oplus_X Z$.

In the following fact, some properties of the dimension and dim-independence are expressed. As a convention, assume that all subsets and tuples are from (a sufficiently saturated) model M of T.

FACT 2.1 [24]. *The dimension has the following properties.*

- *Finite character.* $\dim(\bar{a}/Y) = \min\{\dim(\bar{a}/B) : \quad B \subseteq_{fin} Y\}$.
- *Additivity.* $\dim(\bar{a}\bar{b}/Z) = \dim(\bar{a}/Z) + \dim(\bar{b}/\bar{a}Z)$.
- *Monotonicity.* $\dim(\bar{a}/Y) \geq \dim(\bar{a}/Z)$ *for* $Y \subseteq Z$.
- *Definability. For each formula* $\varphi(\bar{x}, \bar{y})$ *and* $k \leq |\bar{x}|$, *the set*

$$\{\bar{b} \in M^{|\bar{y}|} : \quad \dim(\varphi(M, \bar{b})) = k\}$$

is definable by a formula $d_{\varphi,k}(\bar{y})$.
- $\bigvee$*-Definability. If* $\dim(\bar{a}/\bar{b}) \leq n$, *then there is a formula* $\psi(\bar{x}, \bar{y})$ *such that* $\psi(\bar{x}, \bar{b}) \in \mathrm{qftp}(\bar{a}/\bar{b})$ *and if* $M \models \psi(\bar{a}', \bar{b}')$ *then* $\dim(\bar{a}'/\bar{b}') \leq n$, *for every* $\bar{a}', \bar{b}'$.

By the above properties, one can prove the following lemma. The first part of the following statement appears in [11, Lemma 2.3] and the second part in [14, Lemma 2.2]. This lemma is later used in Section 3.

LEMMA 2.2. *Let* M *be an* $\aleph_0$*-saturated model of* T, $\varphi(\bar{x}; \bar{y})$ *be an* $\mathcal{L}$*-formula and* k *be a natural number. Then,*

1. *there is a formula* $H_\varphi(\bar{y})$ *that defines the set*

$$\{\bar{b} \in M^{|\bar{y}|} : \quad \exists \bar{a} \;\; M \models \varphi(\bar{a}; \bar{b}) \;\&\; \bar{a} \cap \mathrm{acl}(\bar{b}) = \emptyset\},$$

2. *there exists an* $\mathcal{L}$*-formula* $D_{\varphi,k}(\bar{y})$ *which defines the set*

$$\{\bar{b} \in M^{|\bar{y}|} : \quad \exists \bar{a} \;\; M \models \varphi(\bar{a}; \bar{b}) \;\&\; \dim(\bar{a}/\bar{b}) \geq k \;\&\; \bar{a} \cap \mathrm{acl}(\bar{b}) = \emptyset\}.$$

The following definition characterizes the notion of a strong formula. Intuitively, such formula is called strong, since it has enough information about dim.

DEFINITION 2.3. An $\mathcal{L}$-formula $\varphi(\bar{x}, \bar{y})$ is called *strong* with respect to the distinct $\bar{a}, \bar{b}$, whenever $\varphi(\bar{a}, \bar{y}) \in \mathrm{qftp}(\bar{b}/\bar{a})$ and for every $\bar{a}', \bar{b}'$ in a sufficiently saturated model $M \models T$,

1. if $M \models \varphi(\bar{a}', \bar{b}')$ then $\bar{a}', \bar{b}'$ are distinct, and
2. if $M \models \varphi(\bar{a}', \bar{b}')$ and $\dim(\bar{b}'/\bar{a}') = \dim(\bar{b}/\bar{a})$ then for every partition $P = (\bar{y}_1, \bar{y}_2)$ of $\bar{y}$ we have $\dim(\bar{b}_2'/\bar{b}_1'\bar{a}') \leq \dim(\bar{b}_2/\bar{b}_1\bar{a})$.

Particularly, the second item of the above definition deduces $\dim(\bar{b}_1'/\bar{a}') \geq \dim(\bar{b}_1/\bar{a})$.

The $\bigvee$-Definability in Fact 2.1 proves that a strong formula exists for every pair of distinct finite sequences. The proof of the following lemma can be found in [14, Lemma 2.6].

LEMMA 2.4.

1. *For any pair of distinct tuples $\bar{a}, \bar{b}$, there exists a strong formula $\varphi(\bar{x}, \bar{y})$ with respect to $\bar{a}, \bar{b}$.*
2. *Let $\varphi(\bar{x}, \bar{y})$ be a strong formula with respect to $\bar{a}, \bar{b}$ and $T \models (\theta(\bar{x}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))$. If $\theta(\bar{a}, \bar{y}) \in \mathrm{qftp}(\bar{b}/\bar{a})$, then $\theta(\bar{x}, \bar{y})$ is strong with respect to $\bar{a}, \bar{b}$.*

The following fact presents fundamental properties of dim-independence.

FACT 2.5. [24] dim-*independence has the following properties.*

1. *Symmetry.* $Y \mathop{\smile\!\!\!\!\!\!|}_X^{\dim} Z$ *if and only if* $Z \mathop{\smile\!\!\!\!\!\!|}_X^{\dim} Y$.
2. *Transitivity.* $Y \mathop{\smile\!\!\!\!\!\!|}_X^{\dim} Z_1Z_2$ *if and only if* $Y \mathop{\smile\!\!\!\!\!\!|}_X^{\dim} Z_1$ *and* $Y \mathop{\smile\!\!\!\!\!\!|}_{XZ_1}^{\dim} Z_2$.
3. acl-*Preservation.* $Y \mathop{\smile\!\!\!\!\!\!|}_X^{\dim} Z$ *if and only if* $\mathrm{acl}(Y) \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(X)}^{\dim} \mathrm{acl}(Z)$.
4. dim-*Morley sequences. Any non-constant order indiscernible sequence* $\{a_i : i \in I\}$ *over* X *is a* dim-*Morley sequence, i.e., for any two disjoint subsequences* J_1 *and* J_2 *of* I *we have* $J_1 \mathop{\smile\!\!\!\!\!\!|}_X^{\dim} J_2$.
5. *Strong finite character. If* $\bar{a} \mathop{\not\smile\!\!\!\!\!\!|}_{\bar{c}}^{\dim} \bar{b}$ *then there exists a formula* $\varphi(\bar{z}, \bar{x}, \bar{y}) \in \mathrm{qftp}_{\mathcal{L}}(\bar{c}, \bar{a}, \bar{b})$ *which* dim-*forks, i.e., for each model* M *and tuples* $\bar{a}'$, $\bar{b}'$ *and* $\bar{c}' \in M$ *if* $M \models \varphi(\bar{c}', \bar{a}', \bar{b}')$ *then* $\bar{a}' \mathop{\not\smile\!\!\!\!\!\!|}_{\bar{c}'}^{\dim} \bar{b}'$.

Recall that an $\mathcal{L}$-embedding $f : M \rightarrow N$ is *algebraically closed* if $\mathrm{acl}(f(M)) = f(M)$.

DEFINITION 2.6. The theory T has the *free amalgamation property* over algebraically closed substructures whenever for every $M_0, M_1, M_2 \models T^{\forall}$ and every algebraically closed embeddings $f_1 : M_0 \rightarrow M_1$, $f_2 : M_0 \rightarrow M_2$, there exist $M \models T$ and embeddings $g_1 : M_1 \rightarrow M$, $g_2 : M_2 \rightarrow M$ such that:

1. $g_1 \circ f_1 = g_2 \circ f_2$, and
2. $g_1(M_1)$ and $g_2(M_2)$ are free over $g_1 \circ f_1(M_0)$ in M.

CONVENTION 2.7. *The structure M is called a free amalgam of M_1 and M_2 over M_0 and denoted by $M_1 \oplus_{M_0} M_2$. Note that M is not unique up to isomorphism.*

The following observation expresses a property of the free amalgamation which can be easily proved. Assume T has the free amalgamation property and a model M of T which is λ-saturated, for $\lambda \geq \aleph_0$.

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Observation 2.8. *Let $\Sigma(\bar{x})$ be a partial type over X which has a solution with no intersection with* $\operatorname{acl}(X)$. *Then, for every small set $Y \supseteq X$, there is a solution $\bar{d}'$ for Σ in M such that $\bar{d}' \cap Y = \emptyset$. Moreover, $\bar{d}'$ can be chosen such that $X\bar{d}'$ and Y are free over X.*

Proof. By the hypothesis, there is a solution in $M \oplus_{\operatorname{acl}(X)} M$ which has no intersection with $\operatorname{acl}(X)$. Since M is λ-saturated one can find a solution $\bar{d}'$ in M such that $X\bar{d}'$ and Y are free over X. ⊣

In the rest of this section, the notion of indecomposability is presented. This notion has already appeared in [1] with a different name, federated. While this notion is used in [14] for the axiomatization of bi-colored expansions, here it is only used in Section 4 to prove Theorem 4.16.

Definition 2.9. We call T is *indecomposable* if no finite-dimensional algebraically closed set X can be written as a finite union $X = Y_1 \dots Y_n$ with $\dim(Y_i) < \dim(X)$ for $i \leq n$.

The assumption of indecomposability provides the following desirable property for bases. The proof can be found in [14, Lemma 2.10].

Lemma 2.10. *Assume T is indecomposable and $M \models T$. Let $B = \{d_1, \dots, d_m\}$ be an independent set over $A \subseteq M$. Then, for each natural number n, there is a subset $D \subseteq$* $\operatorname{acl}(Ad_1, \dots, d_m)$ *with $|D| = n$ such that every m-element subset of BD is a base for* $\operatorname{acl}(Ad_1, \dots, d_m)$ *over A.*

Example 2.11. The class of geometric theories includes strongly minimal (see [12]), o-minimal theories (see [11]) (in particular, ACF_0, ACF_p, and RCF), generic expansion of algebraically closed fields of characteristic $p > 0$ by an additive subgroup (ACF_pG) (see [12]), (any completion of a) perfect bounded PAC fields and in particular any completion of a pseudo finite field (see [12]). Further this class also includes theories of valued fields $Th(\mathbb{Q}_p)$ and $Th(\mathbb{C}_p)$ and any (non-trivially valued) Henselian valued field of equi-characteristic 0 in the language of Denef–Pas [19]. Hence, in particular, the theory of a non-principal ultraproduct of all $\mathbb{Q}_p$'s is geometric.

Note that the algebraic closure of the mentioned theories are equal to the field-theoretic algebraic closure, (see [18, Theorem 4] and [11, Proposition 4.5]) and hence these theories satisfy the exchange property. On the other hand, by compactness it can be easily seen that the theory of a field with some extra structures eliminates the quantifier $\exists^\infty$ if the model-theoretic algebraic closure coincides with the field-theoretic algebraic closure, and hence these theories are geometric. Furthermore models of the mentioned theories are geometric fields in the sense of [12, 13], and therefore they also enjoy the free amalgamation property.

In all such fields, an algebraically closed subset cannot be decomposed into finitely many algebraically closed subsets of strictly smaller transcendence degree (to justify this, consider an algebraically closed set over a field K as a vector space, and observe that no vector space can be written as the union of finitely many proper sub-vector spaces). Hence the mentioned theories are also indecomposable.

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2.1. Bi-colored expansions. In the following subsection, the theory T is geometric with the quantifier elimination and the free-amalgamation properties. First, we fix some more notation and conventions.

Unary predicate p called the *coloring predicate* is added to the language $\mathcal{L}$ and we denote the new language by $\mathcal{L}_p$. Also, the class $\mathcal{K}$ is defined as the class of all $\mathcal{L}_p$-structures whose universe M is a model of $T^{\forall}$. For every $X \subseteq M$, the $\mathcal{L}_p$-structure generated by X in M is denoted by $\langle X \rangle_M$, or $\langle X \rangle$. Therefore, M is *finitely generated* if there is a finite set $A \subseteq M$ such that $M = \langle A \rangle$. As a convention, also $\emptyset$ is a finitely generated structure in $\mathcal{K}$. The element x is called *colored* if $M \models p(x)$. If there is no ambiguity we may write $p(x)$ instead of $M \models p(x)$. Moreover, $p(\bar{x})$ is used instead of $\bigwedge_{i=1}^{n} p(x_i)$, when $\bar{x} = (x_1, \ldots, x_n)$. Additionally $p(X/Y)$ denotes the set of colored elements of $X \setminus Y$. Throughout this paper, by $\text{tp}(X/Y)$ we mean the type of X over Y in $\mathcal{L}_p$. Moreover for the rest of the paper, a rational number $0 < \alpha \leq 1$ is fixed. Then for every structure $M \in \mathcal{K}$ and a finite subset A of M, a *pre-dimension* map δ_α is defined as

$$\delta_\alpha(A) = \dim(A) - \alpha|p(A)|.$$

Furthermore, for every $X \subseteq M$ we define $\delta_\alpha(A/X) = \dim(A/X) - \alpha|p(A/X)|$.

Note that by the quantifier elimination, $\delta_\alpha(A)$ is independent from the choice of M. For any finite subsets A, B and C it is not hard to check that

$$\delta_\alpha(AB/C) = \delta_\alpha(A/BC) + \delta_\alpha(B/C).$$

The pre-dimension δ_a is submodular, i.e.,

$$\delta_\alpha(AB) + \delta_\alpha(A \cap B) \leq \delta_\alpha(A) + \delta_\alpha(B).$$

For the class $\mathcal{K}$, one can define the subclass $\mathcal{K}_\alpha^+$ as

$$\mathcal{K}_\alpha^+ := \{M \in \mathcal{K} | \ \delta_\alpha(A) \geq 0 \ \text{ for every } \ A \subseteq_{\text{fin}} M\}.$$

For simplicity, if $M \in \mathcal{K}_\alpha^+$ then for every $X \subseteq M$, we say $X \in \mathcal{K}_\alpha^+$ if $\langle X \rangle \in \mathcal{K}_\alpha^+$. Therefore, an embedding $f : X \to Y$ means the embedding $\hat{f} : \langle X \rangle \to \langle Y \rangle$ where $X, Y \in \mathcal{K}_\alpha^+$. In the rest of the paper, every structure is assumed to be in $\mathcal{K}_\alpha^+$, unless we emphasize otherwise.

Clearly, by the previous conventions $\emptyset \in \mathcal{K}_\alpha^+$. Moreover, $\bigvee$-Definability implies that $\mathcal{K}_\alpha^+$ is axiomatizable in $\mathcal{L}_p$ by the $\mathcal{L}_p$-sentences

$$\neg \exists x_1, \ldots, x_n \big(\psi_l(x_1, \ldots, x_n) \wedge \bigvee_{\substack{Y \subseteq \{x_1, \ldots, x_n\} \\ |Y| \geq k}} p(Y)\big),$$

where ψ_l asserts that the dimension of $(x_1, \ldots, x_l)$ is bounded by l, and $l < \alpha k$.

In the following definition, the notion of closedness is presented.

Definition 2.12.

1. For $A \subseteq_{\text{fin}} M$ and $X \subseteq M$, A is said to be closed in X and denoted by $A \leqslant_\alpha X$ if $A \subseteq X$ and $\delta_\alpha(B/A) \geq 0$ for every $A \subseteq B \subseteq_{\text{fin}} X$.
2. For arbitrary subsets X and Y of M, X is called closed in Y and denoted by $X \leqslant_\alpha Y$ if $X \subseteq Y$ and $\delta_\alpha(A/X) \geq 0$ for every $A \subseteq_{\text{fin}} Y$.

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3. The structure M is called closed in the structure N and denoted by $M \leqslant_\alpha N$ if M is a substructure of N and $M \leqslant_\alpha N$ in the sense of (1) and (2).

One can relax the first item of the above definition by taking arbitrary finite subsets A and B of X, since $\delta_\alpha(B/A) = \delta_\alpha(AB/A)$. Hence the first item is a special case of item 2.

For simplicity, we will omit α in $\leqslant_\alpha$ and δ_α.

Note that $\alpha > 0$ implies that all algebraic points over $\emptyset$ are non-colored. By the definition of δ_α and since $\alpha > 0$ whenever $X \leqslant M$, then $\neg p(x)$ for all $x \in \mathrm{acl}_M(X) \setminus X$. So, $\langle X \rangle_M$ and $\mathrm{acl}_M(X)$ are closed in M.

Remark 2.13. $(\mathcal{K}_\alpha^+, \leqslant)$ is a smooth class, i.e., for every M, M_1, M_2, X,

1. $\emptyset, M \leqslant M$.
2. If $M \leqslant M_1$ and $M_1 \leqslant M_2$, then $M \leqslant M_2$.
3. If $M \leqslant M_2$ then $M \leqslant M_1$ for all $M \subseteq M_1 \subseteq M_2$.
4. If $M \leqslant M_1$ then $M \cap X \leqslant X$ for all $X \subseteq M_1$.

By 2 and 4 of the above remark, one can conclude if $M_1, M_2 \leqslant M$ then $M_1 \cap M_2 \leqslant M$.

Next, we introduce the concept of closure and an intrinsic extension of a set.

Definition 2.14. Let $X, A, B \subseteq M$,

1. The *closure* of X in M, denoted by $\mathrm{cl}_M(X)$, is the smallest subset Y of M containing X such that $Y \leqslant M$.
2. The set B is called an *intrinsic extension* of A and shown by $A \leqslant_i B$ if $A \subseteq B$ but there is no $A' \neq B$ with $A \subseteq A' \leqslant B$. Equivalently, $\delta(B) \lneqq \delta(A')$ for all $A \subseteq A' \subsetneqq B$.
3. A pair (A, B) is called *minimal* if $A \subseteq B$, $A \nleqslant B$ but $A \leqslant C$ for all $A \subseteq C \subsetneqq B$. It is clear that if (A, B) is a minimal pair, then $A \leqslant_i B$. Moreover if B is an intrinsic extension of A then it is possible to find a tower $B_0 = A \subseteq B_1 \subseteq \cdots \subseteq B_n = B$ where each (B_i, B_{i+1}) is minimal.

The following statements are well-known facts about basic properties of cl_M.

Fact 2.15 [23, Notation 3.14].

1. $\mathrm{cl}_M(X)$ *is the intersection of all closed subsets of M that contain X.*
2. $\mathrm{cl}_M(A) = \bigcup\{B \subseteq M : A \leqslant_i B\}$.
3. $\mathrm{cl}_M(X) = \bigcup_{A \subseteq X} \mathrm{cl}_M(A)$.

Since α is rational, the values of δ take place in a discrete set. Hence $\mathrm{cl}(A)$ is finite for each finite set A. Therefore $\mathrm{cl}(A) \subseteq \mathrm{acl}(A)$, where acl denotes the algebraic closure in the sense of $\mathcal{L}_p$.

The following definition separates two different types of closed extensions.

Definition 2.16. Let $A \leqslant B$.

1. The set B is *algebraic* over A if $\dim(b/A) = 0$ for every $b \in B \setminus A$.
2. B is *transcendental* over A if $\dim(b/A) = 1$ for every $b \in B \setminus A$.

Remark 2.17. It can be easily seen that if $A \leqslant B$ then there exists B_1 such that $A \leqslant B_1 \leqslant B$ with B_1 is algebraic over A and B is transcendental over B_1. Furthermore, if B is an algebraic extension of A, $A \leqslant B$, and $f : B \to M$ is an $\mathcal{L}_p$-embedding, $M \in \mathcal{K}_\alpha^+$ then, we have $\mathrm{cl}_M(f(B)) = \mathrm{cl}_M(f(A)) \oplus_{f(A)} f(B)$.

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§3. Theory of rich structures. This section is devoted to study the class of λ-rich structures in $\mathcal{K}_\alpha^+$, for $\lambda \geq \aleph_0$. These structures are obtained as Fraïssé limits of $\mathcal{K}_\alpha^+$ which are also called λ-generic or λ-ultra-homogeneous. It is subsequently proved that there is a complete theory $\mathbb{T}_\alpha$ which axiomatizes the class of rich structures.

To this end we, first of all, need to define the notion of strong $\mathcal{L}$-embedding.

Definition 3.1. An $\mathcal{L}$-embedding $f : M \to N$ is *strong* if $f(M) \leqslant N$. Also for every two sets A, B, the embedding $f : A \to B$ is strong whenever $\hat{f} : \langle A\rangle \to \langle B\rangle$ is a strong embedding.

The following definition introduces the amalgamation property for the class $(\mathcal{K}_\alpha^+, \leqslant)$.

Definition 3.2. The class $(\mathcal{K}_\alpha^+, \leqslant)$ has the amalgamation property (AP) if for each M_0, M_1, M_2 and each pair of strong embeddings $f_1 : M_0 \to M_1$, $f_2 : M_0 \to M_2$, there exist M and strong embeddings $g_1 : M_1 \to M$, $g_2 : M_2 \to M$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

The next lemma establishes the amalgamation property for $(\mathcal{K}_\alpha^+, \leqslant)$. The proof is straightforward and can be found in [14, Lemma 3.2].

Lemma 3.3. *The class $(\mathcal{K}_\alpha^+, \leqslant)$ has the amalgamation property. Moreover, if both f_1 and f_2 are algebraically closed then the structure $M \in \mathcal{K}_\alpha^+$ can be chosen in such a way that $g_1(M_1)$ and $g_2(M_2)$ are free over $g_1 \circ f_1(M_0)$ in M.*

Since $\emptyset$ is in $\mathcal{K}_\alpha^+$ and $\emptyset \leqslant M$ for each $M \in \mathcal{K}_\alpha^+$, the amalgamation property implies the joint embedding property, that is for every $M_1, M_2 \in \mathcal{K}_\alpha^+$ there exist $M \in \mathcal{K}_\alpha^+$ and closed embeddings $f_1 : M_1 \to M$ and $f_2 : M_2 \to M$. One can easily check that if M is in $\mathcal{K}_\alpha^+$ then so are all of its substructures.

Definition 3.4. An $\mathcal{L}_p$-structure M in $\mathcal{K}_\alpha^+$ is called λ-rich, for a cardinal $\lambda \geq \aleph_0$, if:

1. $M \models T$.
2. If $M_1 \leqslant M_2$ and M_1, M_2 are generated by sets of cardinality $< \lambda$, then every strong embedding $f : M_1 \to M$ extends to a strong embedding $g : M_2 \to M$.

It is clear from the above definition by letting $M_1 = \emptyset$ that there is a strong embedding $g : M_2 \to M$ for each M_2. This property of M is called λ-universality.

The reason for existence of a λ-rich structure is the amalgamation and joint embedding properties, with the closedness of $\mathcal{K}_\alpha^+$ under the union of $\leqslant$-chains of models (of T). This property is summarised in the following fact.

Fact 3.5. *The class $(\mathcal{K}_\alpha^+, \leqslant)$ contains λ-rich structures, for all $\lambda \geq \aleph_0$.*

The following theorem shows that richness implies $\mathcal{L}$-saturation.

Theorem 3.6. *Each λ-rich structure in $\mathcal{K}_\alpha^+$ is λ-saturated as a model of T.*

Proof. Let M be λ-rich. Assume that $\Sigma(x)$ is a partial 1-type over a set $X \subseteq M$, where $|X| < \lambda$ and without loss of generality we assume that X is closed in M. Let $d \notin M$ be a solution of $\Sigma(x)$ in an $\mathcal{L}$-elementary extension N of M. Extend the coloring of M to N by letting $\neg p(x)$ for each $x \in N - M$, so that now $N \in \mathcal{K}_\alpha^+$. Observe that $\mathrm{cl}_M(X) \leqslant M, N$, hence we keep the notation $\mathrm{cl}(X)$. Note that

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$\langle \mathrm{cl}(X)d\rangle \in \mathcal{K}_\alpha^+$, $\langle \mathrm{cl}(X)\rangle \leqslant \langle \mathrm{cl}(X)d\rangle$ and $\langle \mathrm{cl}(X)\rangle \leqslant M$. Now since M is λ-rich, there is a strong embedding $f : \langle \mathrm{cl}(X)d\rangle \to M$, and hence $f(d)$ is the solution of $\Sigma(x)$ in M. ⊣

3.1. Axiomatization of rich structures. As we mentioned earlier, there exists a complete theory $\mathbb{T}_\alpha$ that axiomatizes the class of λ-rich structures of $(\mathcal{K}_\alpha^+, \leqslant)$. The following notion of intrinsic formulas is utilized in this axiomatization.

Definition 3.7. Let $A \leqslant_i B$ and $\bar{a}$ and $\bar{b}$ be enumerations of A and $B \setminus A$, respectively. A formula $\psi_{A,B}(\bar{x};\bar{y}) \in \mathrm{qftp}(\bar{a}\bar{b})$ is called *intrinsic* if each realization $\bar{a}'$ and $\bar{b}'$ of $\psi_{A,B}(\bar{x};\bar{y})$ in a model M of T implies $A' \leqslant_i B'$, where $\bar{a}'$ and $\bar{b}'$ are enumerations of A' and $B' \setminus A'$, respectively. Assume $\Delta_{A,B;C}(\bar{x},\bar{y})$ is the collection of all intrinsic formulas in $\mathrm{qftp}(\bar{a}\bar{b})$.

Let $A \leqslant B$ and B is a transcendental extension of A. Set

$$\mathcal{C}_{A,B} := \{C : \ A \leqslant B \subseteq C \ \ \& \ \ B \leqslant_i C \ \ \& \ \ B \not\perp_A^{\mathrm{dim}} (C \setminus B)\}.$$

Respectively for any $C \in \mathcal{C}_{A,B}$ let $\mathcal{F}_C$ be the collection of the formulas $\varphi(\bar{x},\bar{y};\bar{z}) \in \Delta_{A,B;C}$ which also dim-forks. Note that by strong finite character of Fact 2.5 the set $\mathcal{F}_C$ is non-empty.

Below for an $\mathcal{L}$-formula $\varphi(\bar{x},\bar{y})$, let $\varphi^*(\bar{y},\bar{x}) := \varphi(\bar{x},\bar{y})$. Respectively, for a natural number k, consider $D_{\varphi^*,k}(\bar{x})$ and $d_{\varphi^*,k}(\bar{x})$ as the formulas introduced in Lemma 2.2 and Fact 2.1.

Also, for $A \subseteq B$ we denote $\bar{a}$ and $\bar{b}$ to be enumerations for A and $B \setminus A$, respectively.

Definition 3.8. Let $\mathbb{T}_\alpha$ be an $\mathcal{L}_p$-theory whose models $\mathbb{M}$ satisfies the followings.

1. $\mathbb{M} \models T$.
2. $\mathbb{M} \in \mathcal{K}_\alpha^+$ (that is $\delta(A) \geq 0$ for all $A \subseteq_{\mathrm{fin}} \mathbb{M}$).
3. For each transcendental extension B over A and enumerations $\bar{a}$ and $\bar{b}$ with $\dim(\bar{b}/\bar{a}) = k$ if $\varphi(\bar{x},\bar{y})$ is strong with respect to $\bar{a},\bar{b}$ then for a given finite subset Φ_0 of $\mathcal{C}_{A,B}$,

$$\mathbb{M} \models \Big[\forall \bar{x} \ \ \Big(D_{\varphi^*,k}(\bar{x}) \wedge d_{\varphi^*,k}(\bar{x}) \to \exists \bar{y} \ \ \big(\varphi(\bar{x},\bar{y}) \wedge \bigwedge_{p(b_i)} p(y_i) \wedge \bigwedge_{\neg p(b_i)} \neg p(y_i)$$
$$\wedge \bigwedge_{C \in \Phi_0} \neg \exists \bar{z}_C \ \ \psi_{A,B,C}(\bar{x},\bar{y};\bar{z}_C)\big)\Big)\Big],$$

where $\psi_{A,B,C}(\bar{x},\bar{y};\bar{z}_C) \in \mathcal{F}_C$, for each $C \in \Phi_0$.

The above items can be expressed as first-order axiom-schemes; the second item is mentioned before Definition 2.12.

Remark 3.9. Notice that if $|B \setminus A| = 1$ the item 3 states that if there are infinitely many y that satisfies $\varphi(\bar{x},y)$ then there exists y with the same color as b such that $\varphi(\bar{x},y) \wedge \bigwedge_{C\in\Phi_0} \neg\exists \bar{z}_C \ \ \psi_{A,B,C}(\bar{x},y;\bar{z}_C)$ holds.

Lemma 3.10. *Any $\aleph_0$-rich structure $\mathbb{M}$ is a model of $\mathbb{T}_\alpha$ (and hence $\mathbb{T}_\alpha$ is consistent).*

PROOF. By Theorem 3.6, $\mathbb{M}$ is $\aleph_0$-saturated as a model of T. Now we just prove that the third item in Definition 3.8 holds. Consider $\bar{a}, \bar{b}$, φ, Φ_0 and $\mathcal{C}_{A,B}$ as in the hypothesis of item 3. Let $\bar{a}'$ in $\mathbb{M}$ be such that $\mathbb{M} \models D_{\varphi^*,k}(\bar{a}') \wedge d_{\varphi*,k}(\bar{a}')$. So there is $\bar{b}' \in \mathbb{M}^{|\bar{b}|}$ with $\mathbb{M} \models \varphi(\bar{b}', \bar{a}')$, $\bar{b}' \cap \operatorname{acl}(\bar{a}') = \emptyset$, and $\dim(\bar{b}'/\bar{a}') = k$. Since $\operatorname{cl}(\bar{a}')$ is finite, by ω-saturation of $\mathbb{M}$ and Observation 2.8 we may assume that $\langle \operatorname{cl}(\bar{a}')\rangle \cap \bar{b}' = \emptyset$ and $\bar{b}'$ and $\langle \operatorname{cl}(\bar{a}')\rangle$ are free over $\bar{a}'$. Consider the $\mathcal{L}$-structure generated by $\operatorname{cl}(\bar{a}')\bar{b}'$ in $\mathbb{M}$ and make it into an $\mathcal{L}_p$-structure N by coloring $\operatorname{cl}(\bar{a}')\bar{b}'$ with the same colors as $\operatorname{cl}(\bar{a})\bar{b}$ and leaving the rest of the elements non-colored. It is clear that $\langle \operatorname{cl}(\bar{a}')\rangle \leqslant \mathbb{M}$. We claim that also $\langle \operatorname{cl}(\bar{a}')\rangle \leqslant N$. By this claim, since $\mathbb{M}$ is $\aleph_0$-rich, there is a strong embedding $f : N \to \mathbb{M}$ which fixes $\langle \operatorname{cl}(\bar{a}')\rangle$ pointwise. Let $f(\bar{b}') = \bar{e}$. Then

$$\mathbb{M} \models \Big(\varphi(\bar{a}', \bar{e}) \wedge \bigwedge_{p(b_i)} p(e_i) \wedge \bigwedge_{\neg p(b_i)} \neg p(e_i)\Big).$$

Furthermore we prove that

$$\mathbb{M} \models \neg\exists \bar{z}\ \psi_{A,B,C}(\bar{a}', \bar{e}; \bar{z}),$$

for every $C \in \Phi_0$. Otherwise suppose there exists $\bar{c}'$ in $\mathbb{M}$ such that

$$\mathbb{M} \models \psi_{A,B,C}(\bar{a}', \bar{e}; \bar{c}'),$$

for some $C \in \Phi_0$. As $\bar{c}'$ is disjoint from $\bar{a}'$ and $\bar{e}$, it follows that $\bar{c}' \subseteq \operatorname{cl}(\bar{a}'\bar{e}) \setminus \bar{a}'\bar{e}$. Let A' and B' be the union of elements of tuples $\bar{a}'$ and $\bar{a}'\bar{e}$, respectively. Since $\bar{b}'$ and $\langle \operatorname{cl}(\bar{a}')\rangle$ are free over $\bar{a}'$ it follows that $\operatorname{cl}(B') = \operatorname{cl}(A') \oplus_{A'} B'$. As $\psi_{A,B,C}$ is an intrinsic formula, we have that $\bar{c}' \subseteq \operatorname{cl}(B')$ and $\bar{c}' \cap B' = \emptyset$. Now since $\operatorname{cl}(A') \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\dim}_{A'} B'$, it follows that $\bar{c}' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\dim}_{A'} B'$. But this contradicts the fact that $\psi_{A,B,C}(\bar{x}, \bar{y}; \bar{z})$ dim-forks.

Now to prove the claim, let $\bar{b}_1' \subseteq \bar{b}'$ and $\bar{d} \subseteq N \setminus \operatorname{cl}(\bar{a}')\bar{b}'$. Notice that $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{\dim}_{\operatorname{cl}(\bar{a}')} \bar{a}'$ implies $\dim(\bar{b}_1'/\langle \operatorname{cl}(\bar{a}')\rangle) = \dim(\bar{b}_1'/\bar{a}')$. Moreover as φ is strong, by Definition 2.3 we have $\dim(\bar{b}_1'/\bar{a}') \geq \dim(\bar{b}_1/\bar{a})$. Therefore

$$\begin{aligned}
\delta(\bar{b}_1'\bar{d}/\langle \operatorname{cl}(\bar{a}')\rangle) &= \dim(\bar{b}_1'\bar{d}/\langle \operatorname{cl}(\bar{a}')\rangle) - \alpha|p(\bar{b}_1'\bar{d})| \\
&= \dim(\bar{b}_1'/\langle \operatorname{cl}(\bar{a}')\rangle) - \alpha|p(\bar{b}_1')| \\
&= \dim(\bar{b}_1'/\bar{a}') - \alpha|p(\bar{b}_1')| \\
&\geq \dim(\bar{b}_1/\bar{a}) - \alpha|p(\bar{b}_1)| \geq 0.
\end{aligned}$$

⊣

In the following, we first prove that any $\aleph_0$-saturated model of $\mathbb{T}_\alpha$ is $\aleph_0$-rich. This result, together with the fact that any two $\aleph_0$-rich structures of $\mathcal{K}_\alpha^+$ are back and forth equivalent, implies that $\mathbb{T}_\alpha$ is complete.

THEOREM 3.11. *Suppose that $\mathbb{M}$ is an $\aleph_0$-saturated model of $\mathbb{T}_\alpha$ then $\mathbb{M}$ is $\aleph_0$-rich.*

PROOF. Let $M \leqslant N$ be two finitely generated structures in $\mathcal{K}_\alpha^+$ and $f : M \to \mathbb{M}$ be a strong $\mathcal{L}_p$-embedding. We claim that there is a strong $\mathcal{L}_p$-embedding $g : N \to \mathbb{M}$ extending f.

Since M and N are finitely generated, there are $A \subseteq_{\text{fin}} M$ and $B \subseteq_{\text{fin}} N$ such that $M = \langle A\rangle$, $N = \langle B\rangle$ and $A \leqslant B \leqslant N$. By Remark 2.17 we have two specific cases to consider.

Case 1. B is algebraic over A.

Since $A \leqslant N \in \mathcal{K}_\alpha^+$, any $x \in B \setminus A$ is non-colored. So by Remark 2.17 it is clear that there is a strong $\mathcal{L}_p$-embedding $g : B \to \mathbb{M}$ extending f.

Case 2. B is transcendental over A. By axiomatization and ω-saturation there is an embedding $f : B \to \mathbb{M}$ fixing A pointwise and there is no embedding of C over $f(B)$ in $\mathbb{M}$, for each $C \in \mathcal{C}_{A,B}$. We claim that $f(B) \leqslant \mathbb{M}$. Otherwise there should be a tuple $\bar{d} \in \mathbb{M}$ disjoint from $f(B)$ with $f(B) \leqslant_i f(B)\bar{d}$ therefore $\delta(\bar{d}/f(B)) < 0$. But since all structures in $\mathcal{C}_{A,B}$ are omitted over $f(B)$, it follows that $\bar{d} \mathop{\smile\!\!\!\!\!\mid}_A^{\dim} f(B)$. So $\delta(\bar{d}/A) = \delta(\bar{d}/f(B)) < 0$. But this is a contradiction, since $A \leqslant \mathbb{M}$. ⊣

THEOREM 3.12. *$\mathbb{T}_\alpha$ is complete.*

PROOF. Clearly any two $\aleph_0$-rich models of $\mathbb{T}_\alpha$ are back and forth equivalent. So the above theorem would imply that any $\aleph_0$-saturated models of $\mathbb{T}_\alpha$ are back and forth equivalent. Hence $\mathbb{T}_\alpha$ is complete. ⊣

COROLLARY 3.13. *Let $\mathfrak{C}$ be a monster model of $\mathbb{T}_\alpha$.*

1. *Assume that $\bar{a}_1, \bar{a}_2$ in $\mathfrak{C}$ are small tuples and X is a closed small subset of $\mathfrak{C}$. Then,*
$$\operatorname{tp}(\bar{a}_1/X) = \operatorname{tp}(\bar{a}_2/X) \Leftrightarrow \langle \operatorname{cl}(X\bar{a}_1)\rangle \cong_{\langle X\rangle} \langle \operatorname{cl}(X\bar{a}_2)\rangle.$$
2. *Any $\mathcal{L}_p$-formula is equivalent to a Boolean combination of formulas of the form $\exists \bar{y}\ \varphi(\bar{x};\bar{y})$, where $\varphi(\bar{x};\bar{y})$ is an $\mathcal{L}_p$-intrinsic formula.*
3. *$\mathfrak{C}$ is λ-rich, for all $\lambda < |\mathfrak{C}|$.*

§4. Classification properties of $\mathbb{T}_\alpha$. In this section we study certain classification properties of $\mathbb{T}_\alpha$ and show that if T is NTP_2, strong, $NSOP_1$, and simple then so is $\mathbb{T}_\alpha$. From now on we suppose that $\mathfrak{C}$ is a monster model of $\mathbb{T}_\alpha$ and all tuples and subsets are considered in $\mathfrak{C}$ and small.

We first need to define the notion of D-independence to analyse the forking independence for $\mathbb{T}_\alpha$.

DEFINITION 4.1. Let $\mathbb{M}$ be an arbitrary model of $\mathbb{T}_\alpha$, $A, B \subseteq_{\text{fin}} \mathbb{M}$ and $X \subseteq \mathbb{M}$. Define:

1. $D(A) = \min\{\delta(C) |\quad A \subseteq C \subseteq_{\text{fin}} \mathbb{M}\}$,
2. $D(B/A) = D(BA) - D(A)$,
3. $D(A/X) = \inf\{D(A/X_0),\quad X_0 \subseteq_{\text{fin}} X\}$.

It can be easily seen that $D(A) = \delta(\operatorname{cl}(A))$ and $D(B/A) = D(B/\operatorname{cl}(A)) = D(\operatorname{cl}(B)/\operatorname{cl}(A)) = \delta(\operatorname{cl}(BA)/\operatorname{cl}(A))$. Therefore the set V_D of values $D(B/A)$ forms a discrete set of positive real numbers, that is it does not have any limit point. So in the third item of the above definition the infimum is attained and $D(A/X) = D(A/X_0)$ for some finite subset X_0 of X.

DEFINITION 4.2. Let $\mathbb{M} \models \mathbb{T}_\alpha$, $A, B \subseteq_{\text{fin}} \mathbb{M}$ and $Z \subseteq \mathbb{M}$. We say that A, B are D-independent over Z and write $A \mathop{\smile\!\!\!\!\!\mid}_Z^D B$ whenever $D(A/Z) = D(A/ZB)$ and $\operatorname{cl}(AZ) \cap \operatorname{cl}(BZ) = \operatorname{cl}(Z)$. Moreover for arbitrary subsets X, Y of $\mathbb{M}$ we say that X and Y are D-independent over Z and write $X \mathop{\smile\!\!\!\!\!\mid}_Z^D Y$ if $C \mathop{\smile\!\!\!\!\!\mid}_Z^D E$ for any $C \subseteq_{\text{fin}} X$ and $E \subseteq_{\text{fin}} Y$.

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FACT 4.3 [4]. *The relation* $\mathop{\smile\!\!\!\!\!\!\,|}^{D}$ *has the following properties.*

1. *D-symmetry. If* $A \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{B} C$, *then* $C \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{B} A$.
2. *D-transitivity.* $A \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{B} C$ *and* $A \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{BC} E$ *if and only if* $A \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{B} CE$.
3. *D-monotonicity. If* $AA' \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{B} CC'$ *then* $A \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{B} C$.
4. *D-local character. For any* X *there exists a finite set* $X_0 \subseteq X$ *such that* $A \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{X_0} X$.
5. *D-existence. For all sets* X, Y, Z *with* $Y \subseteq Z$ *and* X *is finite there exists a finite* X' *such that* $\mathrm{tp}(X/Y) = \mathrm{tp}(X'/Y)$ *and* $X' \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{Y} Z$.
6. *D-closure preservation. For every* X, Y, Z *if* $X \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{Y} Z$ *then* $\mathrm{cl}(XY) \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{\mathrm{cl}(Y)} \mathrm{cl}(YZ)$.

The following lemma can be proved using techniques available in [4, Section 3].

LEMMA 4.4. *Let* $Z \leqslant \mathbb{M}$, $X, Y \subseteq \mathbb{M}$. *Then* $X \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{Z} Y$ *if and only if*:

1. $\mathrm{cl}(XZ) \cap \mathrm{cl}(YZ) = Z$,
2. $\mathrm{cl}(XY) = \mathrm{cl}(XZ) \cup \mathrm{cl}(YZ)$,
3. $\mathrm{cl}(XZ) \mathop{\smile\!\!\!\!\!\!\,|}^{\dim}_{Z} \mathrm{cl}(YZ)$.

In other words $X \mathop{\smile\!\!\!\!\!\!\,|}^{D}_{Z} Y$ *if and only if* $\mathrm{cl}(XY) = \mathrm{cl}(XZ) \oplus_Z \mathrm{cl}(YZ)$.

The following lemma can be easily proved using the properties of D-independence. This lemma is particularly needed for proving strong finite character of the independence that is introduced to show the $NSOP_1$ for $\mathbb{T}_\alpha$.

LEMMA 4.5. *For every* $\mathbb{M} \models \mathbb{T}_\alpha$ *and rational number* $\gamma \in V_D$, *there is a partial type* $\Sigma_\gamma(\bar{x}, \bar{y})$ *over* $\mathbb{M}$ *such that for every* $\bar{a}$ *and* $\bar{b}$ *we have that*

$$\Sigma_\gamma(\bar{a}, \bar{b}) \text{ if and only if } D(\bar{a}/\mathbb{M}\bar{b}) \geq \gamma.$$

PROOF. For finite tuples $\bar{a}$ and $\bar{b}$ of fixed length if $D(\bar{a}/\mathbb{M}\bar{b}) < \gamma$ then there is a finite subset M_0 of $\mathbb{M}$ with $\delta(\mathrm{cl}(\bar{a}\bar{b}M_0)/\mathrm{cl}(\bar{a}M_0)) < \gamma$. So by $\bigvee$-definability one can find an existential $\mathcal{L}_p$-formula $\psi(\bar{x}, \bar{y}, \bar{m}_0) \in \mathrm{tp}(\bar{a}\bar{b}/M_0)$ describing the closure of $\bar{a}\bar{b}M_0$ and witnessing $\delta(\mathrm{cl}(\bar{a}\bar{b}M_0)/\mathrm{cl}(\bar{a}M_0)) < \gamma$. Hence $\Sigma_\gamma(\bar{x}, \bar{y})$ is a partial type which consists of $\neg\psi(\bar{x}, \bar{y}, \bar{m}_0)$ for every formula witnessing $D(\bar{a}/\mathbb{M}\bar{b}) < \gamma$. ⊣

4.1. NTP_2 **and strongness of** $\mathbb{T}_\alpha$**.** We begin this section by showing that $\mathbb{T}_\alpha$ is NTP_2 (respectively strong), provided that T is NTP_2 (respectively strong). To this end we show that the burden of $\mathbb{T}_\alpha < \infty$ under the assumption that T is NTP_2. We first review the basic related concepts and facts all of which can be found in [8].

DEFINITION 4.6 (In a monster model $\mathfrak{M}$ of a theory Γ). A formula $\varphi(\bar{x}, \bar{y})$ is TP_2 if there is an array $(\bar{a}_{ij})_{i,j\in\omega}$ and $k \in \omega$ such that:

1. $\{\varphi(\bar{x}, \bar{a}_{ij})\}_{j\in\omega}$ is k-inconsistent for each $i \in \omega$, i.e., any of its k-element subset is inconsistent.
2. $\{\varphi(\bar{x}, \bar{a}_{if(i)})\}_{i\in\omega}$ is consistent for each $f : \omega \to \omega$.
   A formula is NTP_2 if it is not TP_2. The theory Γ is NTP_2 if it implies that every formula is NTP_2.

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Definition 4.7.

1. An array $(\bar{a}_{ij})_{i\in\alpha, j\in\beta}$ is *A-indiscernible* if the sequence of its rows and the sequence of its columns are A-indiscernible.
2. The rows of an array $(\bar{a}_{ij})_{i\in\alpha, j\in\beta}$ are *mutually indiscernible* over A if each row $\bar{a}_i = (\bar{a}_{ij} : \quad j \in \beta)$ is indiscernible over $A \cup \bar{a}_{\neq i}$ where $\bar{a}_{\neq i} := \{\bar{a}_{lj} : \quad l \neq i,\ j \in \beta\}$.
3. The rows of an array $(\bar{a}_{ij})_{i\in\alpha, j\in\beta}$ are *almost mutually indiscernible* over A if each row $\bar{a}_i = (\bar{a}_{ij} \quad j \in \beta)$ is indiscernible over $A\bar{a}_{<i}(\bar{a}_{l0})_{l>i}$ where $\bar{a}_{<i} := \{\bar{a}_{lj} : \quad l < i,\ j \in \beta\}$.

Next we recall the notion of burden. As in [8], for notational convenience, we consider an extension Card* of the linear order on cardinals by adding a new maximum element ∞ and replacing every limit cardinal κ by two new elements κ_- and κ_+ with $\kappa_- < \kappa_+$. The standard embedding of cardinals into Card* identifies κ with κ_+. In the following, whenever we take a supremum of a set of cardinals, we will be computing it in Card*.

Definition 4.8. Let $p(\bar{x})$ be a (partial) type.

(i) An inp-pattern of depth κ in $p(\bar{x})$ consists of $(\bar{a}_i, \varphi_i(\bar{x}, \bar{y}_i), k_i)_{i\in\kappa}$ with $\bar{a}_i = (\bar{a}_{ij})_{j\in\omega}$ and $k_i \in \omega$ such that:
  - $\{\varphi_i(\bar{x}, \bar{a}_{ij})\}_{j\in\omega}$ is k_i-inconsistent for every $i \in \kappa$,
  - $p(\bar{x}) \cup \{\varphi_i(\bar{x}, \bar{a}_{if(i)})\}_{i\in\kappa}$ is consistent for every $f : \kappa \to \omega$.

(ii) The burden of a partial type $p(\bar{x})$, denoted by $\mathrm{bdn}(p)$, is the supremum of the depths of inp-patterns in Card*.

(iii) A theory Γ is called strong if $\mathrm{bdn}(p) \leq (\aleph_0)_-$ for every finitary type p (equivalently, there is no inp-pattern of infinite depth).

Fact 4.9. *A theory Γ is NTP$_2$ if and only if* $\mathrm{bdn}(p) < |\Gamma|^+$ *for every finitary type p if and only if* $\mathrm{bdn}(\text{"}x = x\text{"}) < |\Gamma|^+$.

For a theory Γ put $\mathrm{bdn}(\Gamma) = \mathrm{bdn}(\text{"}x = x\text{"})$.

Fact 4.10. *Let Γ be an arbitrary theory and $\kappa \in$ Card*. Then the following are equivalent.*

(a) $\mathrm{bdn}(p) < \kappa$.

(b) *If $b \models p(\bar{x})$ and the rows of the array $(\bar{a}_{ij})_{i\in\kappa, j\in\omega}$ are almost mutually indiscernible over A then there are some $i \in \kappa$ and indiscernible sequence $\bar{a}' := (\bar{a}'_j)_{j\in\omega}$ such that $\bar{a}'$ is indiscernible over $\bar{b}A$ and $(\bar{a}'_j)_{j\in\omega} \equiv_{\bar{a}_{i0}A} (\bar{a}_{ij})_{j\in\omega}$.*

(c) *For any mutually indiscernible array $(\bar{a}_{ij})_{i\in\kappa, j\in\omega}$ over A and $\bar{b} \models p$, there are some $i \in \kappa$ and indiscernible sequence $\bar{a}' := (\bar{a}'_j)_{j\in\omega}$ such that $\bar{a}'$ is indiscernible over $\bar{b}A$ and $(\bar{a}'_j)_{j\in\omega} \equiv_{\bar{a}_{i0}A} (\bar{a}_{ij})_{j\in\omega}$.*

From now on, as before, we work in the monster model $\mathfrak{C}$ of $\mathbb{T}_\alpha$.

Lemma 4.11. *Let $I = \{\bar{a}_i : \quad i < \kappa\}$ be an $\mathcal{L}_p$-indiscernible sequence over A. Then I is $\mathcal{L}_p$-indiscernible also over* $\mathrm{cl}(A)$.

Proof. Let $\{\bar{c}_i : \quad i < \kappa\}$ be an $\mathcal{L}_p$-indiscernible sequence over $\mathrm{cl}(A)$ that realizes the EM-type of I. This sequence has the same type over A as the type of I, and therefore, there is an $\mathcal{L}_p$-automorphism $\sigma : \mathfrak{C} \to \mathfrak{C}$ fixing A pointwise such that

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$\sigma(\bar{c}_i) = \bar{a}_i$, for each $i < \kappa$. So, $\{\bar{a}_i : \ i < \kappa\}$ is $\mathcal{L}_p$-indiscernible over $\sigma(\mathrm{cl}(A)) = \mathrm{cl}(A)$. ⊣

The following notions of D-Morley sequence and mutually D-Morley sequence as well as Lemma 4.13 are needed to prove the main Theorem 4.14.

Definition 4.12.

1. An indiscernible sequence $(\bar{a}_i : \ i \in \alpha)$ is D-Morley over A if $\bar{a}_{i_n} \downarrow^D_A \bar{a}_{i_0}, \dots, \bar{a}_{i_{n-1}}$ for each $i_0 < \dots < i_n \in \alpha$.
2. The mutually indiscernible array $(\bar{a}_{ij})_{i\in\alpha, j\in\beta}$ over A is mutually D-Morley over A if each row is a D-Morley sequence over A and the sequence of its rows $(\bar{a}_i)_{i\in\alpha}$ is also D-Morley over A.

Lemma 4.13. *Assume that a mutually indiscernible array $I = (\bar{a}_{ij})_{i\in\kappa, j\in\omega}$ over A is given. Then there is a closed set Z including A such that I is mutually indiscernible and mutually D-Morley over Z.*

Proof. We extend the array $I := (\bar{a}_{ij})_{i\in\kappa, j\in\omega}$ to a mutually indiscernible array $(\bar{a}_{is})_{i\in\kappa, s\in\mathbb{Q}}$ over A. Put $Z_1 := (\bar{a}_{is})_{i\in\kappa, s<0}$.

It is clear that $(\bar{a}_{ij})_{i\in\kappa, j\in\omega}$ is mutually indiscernible over $Z_1 A$.

To prove that $(\bar{a}_{ij})_{i\in\kappa, j\in\omega}$ is mutually D-Morley over $Z_1 A$, we first show that the row $\bar{a}_i$ is a D-Morley sequence over $Z_1 A$, for each $i \in \kappa$. So we prove that $\bar{a}_{ij} \downarrow^D_{Z_1 A} \{\bar{a}_{il} : \ l < j\}$, for each $j \in \omega$. Put $\bar{b} := (\bar{a}_{il})_{l<j}$. Hence it is enough to verify that

$$\bar{a}_{ij} \downarrow^D_{Z_1 A} \bar{b}. \quad (*)$$

Take a finite subset Z_0 of $Z_1 A$ with $\bar{a}_{ij}\bar{b} \downarrow^D_{Z_0} Z_1 A$. So the properties of D-independence implies that $\bar{a}_{ij} \downarrow^D_{Z_0} Z_1 A$ and $\bar{a}_{ij} \downarrow^D_{Z_0\bar{b}} Z_1 A\bar{b}$. By D-transitivity, to show $(*)$, it is enough to prove that $\bar{a}_{ij} \downarrow^D_{Z_0} \bar{b}$. Put $B = Z_0 \cap \bar{a}_i$ and find $\bar{b}' = \{\bar{a}_{il_0}, \dots, \bar{a}_{il_{j-1}}\}$ such that $s < l_0 < \dots < l_{j-1} < 0$ for each s with $\bar{a}_{is} \in B$. As I is a mutually indiscernible array, $\bar{b} \equiv_{Z_1 A\bar{a}_{ij}} \bar{b}'$. But since $\bar{b}' \subseteq Z_1 A$, it follows that $\bar{a}_{ij} \downarrow^D_{Z_0} \bar{b}'$. Hence $\bar{a}_{ij} \downarrow^D_{Z_0} \bar{b}$.

By a similar argument we can prove that the sequence of rows of I is also D-Morley over $Z_1 A$. Take $Z = \mathrm{cl}(Z_1 A)$. By Lemma 4.11 we have that the sequence I is also indiscernible over Z. ⊣

Theorem 4.14. *If T is NTP$_2$ then so is $\mathbb{T}_\alpha$.*

Proof. Let $p(\bar{x})$ be an arbitrary finitary type. We prove that $\mathrm{bdn}(p) < |\mathbb{T}_\alpha|^+ = \aleph_1$. By Fact 4.10 it is enough to show that for any given mutually indiscernible $I = (\bar{a}_i)_{i\in\aleph_1}$ over $\emptyset$ and $\bar{b} \models p$, there are some $i \in \aleph_1$ and sequence $\bar{a}' = (\bar{a}'_j)_{j\in\omega}$ such that $\bar{a}'$ is indiscernible over $\bar{b}$ and $(\bar{a}'_j)_{j\in\omega} \equiv_{\bar{a}_{i0}} (\bar{a}_{ij})_{j\in\omega}$.

By Lemma 4.13 we may find a closed set Z for which $(\bar{a}_i)_{i\in\aleph_1}$ is mutually D-Morley over Z. Since each row $\bar{a}_i$ is D-independent over Z, it implies that $\mathrm{cl}(\bar{a}_i Z) = \bigcup_{k\in\omega} \mathrm{cl}(\bar{a}_{ik} Z)$. Furthermore for any $i_1, \dots, i_n \in \aleph_1$ and $j_1, \dots, j_n \in \omega$, we have

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$$\mathrm{cl}(\bar{a}_{i_1},\dots,\bar{a}_{i_n}Z)=\mathrm{cl}(\bar{a}_{i_1}Z)\oplus_Z\cdots\oplus_Z\mathrm{cl}(\bar{a}_{i_n}Z),\quad (*)$$

$$\mathrm{cl}(\bar{a}_{i_1j_1},\dots,\bar{a}_{i_nj_n}Z)=\mathrm{cl}(\bar{a}_{i_1j_1}Z)\oplus_Z\cdots\oplus_Z\mathrm{cl}(\bar{a}_{i_nj_n}Z).\quad (**)$$

Now by an easy application of Ramsey's theorem we may assume that the array $J=(\mathrm{cl}(\bar{a}_{ij}Z))_{i\in\aleph_1,j\in\omega}$ is mutually indiscernible and mutually D-Morley over Z. Let $\bar{b}_{ij}=\mathrm{cl}(\bar{a}_{ij}Z)$.

Take a finite tuple $\bar{e}\models p$. By D-local character (Fact 4.3) there is a finite subset I_0 of components of I such that $\bar{e}\mathop{\smile\!\!\!\!\!\!|}^D_{I_0} I$. Choose a countable ordinal ν such that $I_0\subseteq\bigcup_{i<\nu}\bar{a}_i$. Put $I':=(\bar{a}_{ij})_{i>\nu,j\in\omega}$, $J'=(\bar{b}_{ij})_{i>\nu,j\in\omega}$ and $\bar{e}'=\mathrm{cl}(\bar{e}I_0)$. Then $\bar{e}'$ is finite and both I' and J' are mutually indiscernible and mutually D-Morley over $\mathrm{cl}(I_0Z)$. Furthermore $I_0\mathop{\smile\!\!\!\!\!\!|}^D_Z I'$. Hence by D-closure preservation (item 6 of Fact 4.3) $\bar{e}'\mathop{\smile\!\!\!\!\!\!|}^D_{I_0} J'$ and $I_0\mathop{\smile\!\!\!\!\!\!|}^D_Z J'$.

So for any $i>\nu$ and $j_1<j_2<\cdots<j_k$ we have that

$$\mathrm{cl}(\bar{b}_{ij_1},\dots,\bar{b}_{ij_k}I_0)=\big(\bar{b}_{ij_1}\oplus_Z\cdots\oplus_Z\bar{b}_{ij_k}\big)\oplus_Z\mathrm{cl}(I_0Z),\quad (\dagger)$$

$$\mathrm{cl}(\bar{b}_{ij_1},\dots,\bar{b}_{ij_k}I_0\bar{e}')=\mathrm{cl}(\bar{b}_{ij_1},\dots,\bar{b}_{ij_k}I_0)\oplus_{\mathrm{cl}(I_0)}\bar{e}'.\quad (\dagger\dagger)$$

Since T is NTP_2 and $\bar{e}'$ is finite, there are a countable ordinal $t>\nu$ and $\bar{b}'_t=(\bar{b}'_{tj})_{j\in\omega}$ such that $\bar{b}_t=(\bar{b}_{tj})_{j\in\omega}\equiv_{\bar{b}_{t0}\,\mathrm{cl}(I_0Z)}(\bar{b}'_{tj})_{j\in\omega}$ and $\bar{b}'_t$ is an $\mathcal{L}$-indiscernible over $\mathrm{cl}(I_0Z)\bar{e}'$. Subsequently if we (re-)color $\langle\bar{b}'_tI_0\rangle$ the same as $\langle\bar{b}_tI_0\rangle$ then we have that $\langle\bar{b}'_tI_0\rangle\in\mathcal{K}^+_\alpha$ and $\bar{b}_{t0}\,\mathrm{cl}(I_0Z)\leqslant\mathfrak{C}$. So $\langle\bar{b}'_tI_0\rangle\oplus_{\langle\bar{b}_{t0}\,\mathrm{cl}(I_0Z)\rangle}\langle\bar{b}_{t0}\bar{e}'I_0Z\rangle\in\mathcal{K}^+_\alpha$. Now by richness of monster model we may find a closed isomorphic copy $\langle\bar{b}''_t\rangle$ of $\langle\bar{b}'_t\rangle$ over $\langle\mathrm{cl}(I_0Z)\bar{e}'\rangle$. Note that since $\bar{b}''_t$ and $\bar{b}_t$ have the same $\mathcal{L}$-types and colors, $\bar{b}''_t$ is a dim-Morley and $\mathcal{L}$-indiscernible sequence over $\mathrm{cl}(I_0Z)\bar{e}'$. Hence $\bar{b}''_{tj_1}\dots\bar{b}''_{tj_n}=\bar{b}''_{tj_1}\oplus_Z\cdots\oplus_Z\bar{b}''_{tj_n}\leqslant\bar{b}''_t\leqslant\mathfrak{C}$, for each $j_1<\cdots<j_n$. Furthermore $\bar{e}'\mathop{\smile\!\!\!\!\!\!|}^D_{I_0}\bar{b}''_t$, $I_0\mathop{\smile\!\!\!\!\!\!|}^D_Z\bar{b}''_t$ and $(\bar{b}''_{tj_1}\oplus_Z\cdots\oplus_Z\bar{b}''_{tj_n})\oplus_Z\mathrm{cl}(I_0Z)\leq\mathfrak{C}$, for each $j_1<\cdots<j_n$. Therefore $(\dagger)$ and $(\dagger\dagger)$ also hold for $\bar{b}''_t$.

Claim. *$\bar{b}''_t$ is also $\mathcal{L}_p$-indiscernible over* $\mathrm{cl}(I_0Z)\bar{e}'$.

Proof of claim. We prove that for any $i_1<\cdots<i_k$ and $j_1<\cdots<j_k$

$$\langle\mathrm{cl}(\bar{b}''_{ti_1},\dots,\bar{b}''_{ti_k}I_0\bar{e}')\rangle\cong_{\langle\mathrm{cl}(I_0Z)\bar{e}'\rangle}\langle\mathrm{cl}(\bar{b}''_{tj_1},\dots,\bar{b}''_{tj_k}I_0\bar{e}')\rangle.\tag{4.1}$$

Now by $(\dagger)$ and $(\dagger\dagger)$ for $\bar{b}''_t$ and by Corollary 3.13 in order for (4.1) to hold it is sufficient to have $\mathrm{tp}_{\mathcal{L}}(\bar{b}''_{ti_1},\dots,\bar{b}''_{ti_k}/\mathrm{cl}(I_0Z)\bar{e}')=\mathrm{tp}_{\mathcal{L}}(\bar{b}''_{tj_1},\dots,\bar{b}''_{tj_k}/\mathrm{cl}(I_0Z)\bar{e}')$. This is because by $(\dagger)$ and $(\dagger\dagger)$ any $\mathcal{L}$-isomorphism between $\mathrm{cl}(\bar{b}''_{ti_1},\dots,\bar{b}''_{ti_k}I_0\bar{e}')$ and $\mathrm{cl}(\bar{b}''_{tj_1},\dots,\bar{b}''_{tj_k}I_0\bar{e}')$ which maps $\bar{b}''_{ti_1},\dots,\bar{b}''_{ti_k}$ to $\bar{b}''_{tj_1},\dots,\bar{b}''_{tj_k}$ and fixing $\mathrm{cl}(I_0Z)\bar{e}'$ pointwise also preserves the coloring. But the latter equality holds, since $\bar{b}''_t$ is an $\mathcal{L}$-indiscernible sequence over $\mathrm{cl}(I_0Z)\bar{e}'$. ⊣

The next corollary to show strongness of $\mathbb{T}_\alpha$ can be easily deduced from (the proof of) the above theorem. To this end we verify that $\mathrm{bdn}(p)<(\aleph_0)_+=\aleph_0$ for every finitary type p.

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COROLLARY 4.15. *If T is strong, then so is $\mathbb{T}_\alpha$.*

The following theorem shows that if T is indecomposable then $\mathrm{bdn}(\mathbb{T}_{\frac{1}{2}}) = (\aleph_0)_-$. This result remains true for every rational α. However for simplicity and to have some concrete examples we restrict ourselves only to $\alpha = \frac{1}{2}$. Note that by the indecomposability of T we may use Lemma 2.10 to show that for every independent set $B = \{b_1, \dots, b_m\}$ over a set A and for every $n \geq m$ one can find $b_{m+1}, \dots, b_{m+n} \in \mathrm{acl}(B)$ such that each m-element subset of $\{b_1, \dots, b_m, b_{m+1}, \dots, b_{m+n}\}$ forms a basis for $\mathrm{acl}(AB)$. Below we take $A = \{b\}$ where b is a non-algebraic element, $m = k$ and $n = k + 1$, for $k \geq 1$.

THEOREM 4.16. *Let T be an indecomposable theory. Then* $\mathrm{bdn}(\mathbb{T}_{\frac{1}{2}})$ *is not finite.*

PROOF. To show $\mathrm{bdn}(\mathbb{T}_{\frac{1}{2}})$ is not finite, in the light of item (b) of Fact 4.10, for each natural number k we must construct an almost $\mathcal{L}_p$-mutually indiscernible array $(a_{ij})_{i\in k, j\in\omega}$ over $\emptyset$ and find b' such that for each $i \in k$ and for each $\mathcal{L}_p$-indiscernible sequence $\bar{a}' := (a'_j)_{j\in\omega}$ with $\bar{a}'$ is indiscernible over b' we have $(a'_j)_{j\in\omega} \not\equiv_{a_{i0}} (a_{ij})_{j\in\omega}$.

Let $b \in \mathfrak{C}$ be non-algebraic over $\emptyset$. Choose elements $\{b_1, \dots, b_k\}$ of $\mathfrak{C}$ such that $\{b_1, \dots, b_k\}$ is independent over b. Now by Lemma 2.10 we may choose $b_{k+1}, \dots, b_{2k+1} \in \mathrm{acl}(b, b_1, \dots, b_k)$ such that any k-element subset of $\{b_1, \dots, b_{2k+1}\}$ is a base for $\mathrm{acl}(b, b_1, \dots, b_{2k+1}) = \mathrm{acl}(b, b_1, \dots, b_k)$. Let $c_i = b_{k+2+i}$ for each $0 \leq i \leq k-1$. Hence $C = \{c_0, \dots, c_{k-1}\}$ is independent over $\emptyset$. Choose an $\mathcal{L}$-indiscernible sequence $\{c_{0j} : \ j \in \omega\}$ over $\{c_1, \dots, c_{k-1}\}$ with $c_{00} = c_0$. Respectively by induction over $i \in k$, choose an $\mathcal{L}$-indiscernible sequence $\{c_{ij} : \ j \in \omega\}$ over $\bar{c}_{<i}(c_{l0})_{l>i}$ with $c_{i0} = c_i$. Since $\{c_0, \dots, c_{k-1}\}$ is independent over $\emptyset$, by Ramsey theorem and compactness there exists an array $I := (c_{ij})_{i\in k, j\in\omega}$ such that c_{ij}'s are distinct. Furthermore by free amalgamation of T and Observation 2.8 we may choose I in such a way that $I \mathop{\smile\!\!\!\!\!\!\,\mid}^{\dim}_{C} \{b, b_1, \dots, b_{k+1}\}$. It is easy to see that I is an almost $\mathcal{L}$-mutually indiscernible sequence over $\emptyset$.

Now we color $\langle Ibb_1 \dots b_{k+1}\rangle$ by letting the elements of $I \cup \{b, b_1, \dots, b_{k+1}\}$ colored while leaving the rest of elements non-colored. By exchange property of T and since I is almost mutually indiscernible, each row $\bar{c}_i$ is dim-independent from $\bar{c}_{\neq i}$. So for each subset X of $\bar{c}_i$ we have that $X \leq I$. Moreover since $I \mathop{\smile\!\!\!\!\!\!\,\mid}^{\dim}_{C} \{b, b_1, \dots, b_{k+1}\}$, we have $I \leq Ibb_1 \dots b_{k+1}$. Therefore $\langle Ibb_1 \cdots_{k+1}\rangle \in \mathcal{K}^+_\alpha$. Take $f : \langle Ibb_1 \dots b_{k+1}\rangle \to \mathfrak{C}$ to be a strong $\mathcal{L}_p$-embedding with $b' = f(b)$, $b'_t = f(b_t)$ and $a_{ij} = f(c_{ij})$ for each $t \in k+2$, $i \in k$ and $j \in \omega$. Then it is clear that $(a_{ij})_{i\in k, j\in\omega}$ is an almost $\mathcal{L}_p$-mutually indiscernible array over $\emptyset$. Further we have that,

$$\delta(a_{00}, \dots, a_{(k-1)0}, b'_1, \dots, b'_{k+1}/b') = k - \frac{1}{2}(2k+1) =- \frac{1}{2}.$$

So $(b', \{b', a_{00}, \dots, a_{(k-1)0}, b'_1, \dots, b'_{k+1}\})$ is a minimal pair and hence by Fact 2.15 we have that $a_{i0} \in \mathrm{cl}(b')$ for each $i \in k$. We claim that for each $i \in k$ and $\mathcal{L}_p$-indiscernible sequence $(a'_j)_{j\in\omega}$ over b' we have that $(a'_j)_{j\in\omega} \not\equiv_{a_{i0}} (a_{ij})_{j\in\omega}$.

Assume not. Then for some $i \in k$ we have $(a'_j)_{j\in\omega} \equiv_{a_{i0}} (a_{ij})_{j\in\omega}$. Note that since $a'_0 = a_{i0} \in \mathrm{cl}(b')$ and $(a'_j)_{j\in\omega}$ is an $\mathcal{L}_p$-indiscernible sequence over b', we have that $a'_j \in \mathrm{cl}(b')$ for each $j \in \omega$.

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But this gives infinitely many elements in $\mathrm{cl}(b')$ which contradicts the fact that $\mathrm{cl}(b')$ is finite. $\dashv$

Corollary 4.17. *Let T be the theory of a non-principal ultraproduct of $\mathbb{Q}_p$'s. Then $\mathbb{T}_{\frac{1}{2}}$ is a strong theory with* $\mathrm{bdn}(\mathbb{T}_{\frac{1}{2}}) = (\aleph_0)_-$.

4.2. ***NSOP*$_1$ and simplicity of $\mathbb{T}_\alpha$.** In this subsection we examine whether the $NSOP_1$ and simplicity of T can be transferred to $\mathbb{T}_\alpha$. We first start by recalling the definition of $NSOP_1$. Let Γ be a complete theory and $\mathfrak{M} \models \Gamma$ be a monster model of Γ.

Definition 4.18 [15]. The formula $\varphi(x, y)$ has SOP_1 if there is a collection of tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ so that:

1. for all $\lambda \in 2^\omega$, $\{\varphi(x, a_{\lambda|_\alpha}) : \alpha < \omega\}$ is consistent,
2. for all $\eta \in 2^{<\omega}$, if $\nu \prec \eta \frown 0$, then $\{\varphi(x, a_\nu), \varphi(x, a_{\eta\frown 1})\}$ is inconsistent.

The theory Γ is called SOP_1 if there is a formula with SOP_1. Otherwise Γ is $NSOP_1$.

The following fact provides a technique for showing theory Γ is $NSOP_1$, ([15, Theorem 9.1]).

Fact 4.19. *Assume there is an $Aut(\mathfrak{M})$-invariant ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ on small subsets of the monster $\mathfrak{M} \models \Gamma$ which satisfies the following properties. For an arbitrary $M \models \Gamma$ and arbitrary tuples and subsets from $\mathfrak{M}$,*

1. *Strong finite character: If $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_M B$, then there is a formula $\varphi(\bar{x}, \bar{b}, \bar{m}) \in tp(\bar{a}/BM)$ such that for any $\bar{a}' \models \varphi(\bar{x}, \bar{b}, \bar{m})$, $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!|}_M B$.*
2. *Existence over models: $M \models \Gamma$ implies $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M M$ for any $\bar{a} \in \mathfrak{M}$.*
3. *Monotonicity: If $\bar{a}\bar{a}' \mathop{\smile\!\!\!\!\!\!|}_M BB'$ then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M B$.*
4. *Symmetry: $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M \bar{b}$ if and only if $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_M \bar{a}$.*
5. *The independence theorem: $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M B$, $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}_M C$, $B \mathop{\smile\!\!\!\!\!\!|}_M C$ and $\bar{a} \equiv_M \bar{a}'$ implies there is $\bar{a}''$ with $\bar{a}'' \equiv_{MB} \bar{a}$, $\bar{a}'' \equiv_{MC} \bar{a}'$ and $\bar{a}'' \mathop{\smile\!\!\!\!\!\!|}_M BC$.*

Then Γ is $NSOP_1$.

Now we want to show that the $NSOP_1$ can be transferred from T to $\mathbb{T}_\alpha$. Notice that since T is $NSOP_1$, there is an $Aut(\mathfrak{C})$-invariant ternary relation $\mathop{\smile\!\!\!\!\!\!|}^{\dagger}$ that fulfils the conditions of the above fact.

Remark 4.20. By taking $\mathop{\smile\!\!\!\!\!\!|}^{\dagger}$ to be the Kim-independence, [15], we may assume that:

1. $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\dagger}_M \bar{b}$ implies that $\bar{a} \cap \bar{b} \subseteq M$,
2. if $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\dagger}_M \bar{b}$ then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\dagger}_M \mathrm{acl}(M\bar{b})$,
3. $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\dagger}_M \bar{b}$ implies that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{dim}}_M \bar{b}$.

In the light of the above fact a notion of independence is introduced to prove that $\mathbb{T}_\alpha$ is $NSOP_1$.

Definition 4.21. Suppose $\bar{a}, B, C$ are given. We say $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{*}_C B$ if and only if $\mathrm{cl}(\bar{a}C) \mathop{\smile\!\!\!\!\!\!|}^{D}_{\mathrm{cl}(C)} \mathrm{cl}(BC)$ and $\mathrm{cl}(\bar{a}C) \mathop{\smile\!\!\!\!\!\!|}^{\dagger}_{\mathrm{cl}(C)} \mathrm{cl}(BC)$.

Theorem 4.22. *If T is $NSOP_1$ then so is $\mathbb{T}_\alpha$.*

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PROOF. It is enough to prove that $\mathop{\smile\!\!\!\!\!\mid}^*$ satisfies the conditions of Fact 4.19. It is easily seen that $\mathop{\smile\!\!\!\!\!\mid}^*$ is an automorphism invariant and satisfies the properties *existence over models*, *monotonicity,* and *symmetry* by the definition of $\mathop{\smile\!\!\!\!\!\mid}^*$.

So we only prove the other properties. Notice that without loss of generality B, B' and C can be considered to be closed in $\mathfrak{C}$.

- *Strong finite character:* Assume that $\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}^*_M B$. We prove that there is a formula $\varphi(\bar{x}, \bar{b}, \bar{m}) \in \text{tp}(\bar{a}/BM)$ such that for any $\bar{a}' \models \varphi(\bar{x}, \bar{b}, \bar{m})$, $\bar{a}' \not\mathop{\smile\!\!\!\!\!\mid}^*_M B$.

  $\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}^*_M B$ implies either $\text{cl}(\bar{a}M) \not\mathop{\smile\!\!\!\!\!\mid}^D_M \text{cl}(BM)$ or $\text{cl}(\bar{a}M) \not\mathop{\smile\!\!\!\!\!\mid}^\dagger_M \text{cl}(BM)$. Then we have two cases to consider:

  Case 1. $\text{cl}(\bar{a}M) \not\mathop{\smile\!\!\!\!\!\mid}^D_M \text{cl}(BM)$ but $\text{cl}(\bar{a}M) \cap \text{cl}(BM) = M$.

  So $D(\bar{a}/M) > D(\bar{a}/MB)$. Therefore there exist a finite set $M_0 \leqslant M$ and a finite tuple $\bar{b}$ of B such that

  $$D(\bar{a}/M) = D(\bar{a}/M_0) > D(\bar{a}/MB) = D(\bar{a}/M_0\bar{b}).$$

  By D-symmetry this is equivalent to

  $$D(\bar{b}/M_0) > D(\bar{b}/M_0\bar{a}). \quad (*)$$

  Let $\gamma = D(\bar{b}/M_0)$. Now in the light of Lemma 4.5 there exists a partial type $\Sigma_\gamma(\bar{x}, \bar{y})$ over M expressing that $D(\bar{y}/M\bar{x}) \geq \gamma$. Hence by $(*)$ one can find a formula $\theta(\bar{x}, \bar{m}_0, \bar{b})$ which is satisfied by $\bar{a}$ and

  $$D(\bar{b}/M) = D(\bar{b}/M_0) > D(\bar{b}/M\bar{a}'),$$

  for every $\bar{a}'$ with $\theta(\bar{a}', \bar{m}_0, \bar{b})$. Hence by D-symmetry

  $$D(\bar{a}'/M) > D(\bar{a}'/M\bar{b}).$$

  Thus $\text{cl}(\bar{a}'M) \not\mathop{\smile\!\!\!\!\!\mid}^D_M \text{cl}(BM)$.

  Case 2. $\text{cl}(\bar{a}M) \cap \text{cl}(BM) \neq M$ or $\text{cl}(\bar{a}M) \not\mathop{\smile\!\!\!\!\!\mid}^\dagger_M \text{cl}(BM)$.

  By Remark 4.20 this case is equivalent to $\text{cl}(\bar{a}M) \not\mathop{\smile\!\!\!\!\!\mid}^\dagger_M \text{cl}(BM)$.

  Let $\text{cl}(\bar{a}M) \not\mathop{\smile\!\!\!\!\!\mid}^\dagger_M \text{cl}(BM)$. Since T is $NSOP_1$, there are $\bar{a}' \subseteq \text{cl}(\bar{a}M)$, disjoint from $\bar{a}$, and an $\mathcal{L}$-formula $\theta(\bar{x}, \bar{x}', \bar{b}, \bar{m}) \in \text{tp}_{\mathcal{L}}(\bar{a}\bar{a}'/\text{cl}(BM))$ witnessing the strong finite character property with respect to T. Let $\psi(\bar{x}, \bar{x}', \bar{m})$ be an $\mathcal{L}_p$-intrinsic formula satisfied by $\bar{a}\bar{a}'$. Set $\varphi(\bar{x}, \bar{b}, \bar{m}) := \exists \bar{x}'(\theta(\bar{x}, \bar{x}', \bar{b}, \bar{m}) \wedge \psi(\bar{x}, \bar{x}', \bar{m}))$. We claim that $\varphi(\bar{x}, \bar{b}, \bar{m})$ is the desired formula. Let $\bar{a}_1$ be a solution of $\varphi(\bar{x}, \bar{b}, \bar{m})$. Then there exits $\bar{a}_2 \subseteq \text{cl}(\bar{a}_1M)$ with $\theta(\bar{a}_1, \bar{a}_2, \bar{b}, \bar{m})$. Hence $\bar{a}_1\bar{a}_2 \not\mathop{\smile\!\!\!\!\!\mid}^\dagger_M \text{cl}(BM)$. Therefore $\text{cl}(\bar{a}_1M) \not\mathop{\smile\!\!\!\!\!\mid}^\dagger_M \text{cl}(BM)$.
- *The independence theorem:* Assume that $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^*_M B$, $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}^*_M C$, $B \mathop{\smile\!\!\!\!\!\mid}^*_M C$ and $\bar{a} \equiv_M \bar{a}'$. Further, suppose that $B = \langle B \rangle$, $C = \langle C \rangle$, both include M and they are closed in $\mathfrak{C}$. We prove that there is $\bar{a}''$ with $\bar{a}'' \equiv_B \bar{a}$, $\bar{a}'' \equiv_C \bar{a}'$ and $\bar{a}'' \mathop{\smile\!\!\!\!\!\mid}^*_M BC$. By definition of $\mathop{\smile\!\!\!\!\!\mid}^*_M$ we have $\text{cl}(\bar{a}M) \mathop{\smile\!\!\!\!\!\mid}^\dagger_M B$, $\text{cl}(\bar{a}'M) \mathop{\smile\!\!\!\!\!\mid}^\dagger_M C$, $B \mathop{\smile\!\!\!\!\!\mid}^\dagger_M C$ and $\bar{a} \equiv_M \bar{a}'$ (in the sense of $\mathcal{L}_p$). Since $\bar{a} \equiv_M \bar{a}'$, by Corollary 3.13 it follows that $\langle \text{cl}(\bar{a}M) \rangle \equiv_M \langle \text{cl}(\bar{a}'M) \rangle$.

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So by independence theorem for T there exists an $\mathcal{L}$-structure E such that $\mathrm{tp}_{\mathcal{L}}(E/B) = \mathrm{tp}_{\mathcal{L}}(\langle \mathrm{cl}(\bar{a}M)\rangle/B)$, $\mathrm{tp}_{\mathcal{L}}(E/C) = \mathrm{tp}_{\mathcal{L}}(\langle \mathrm{cl}(\bar{a}'M)\rangle/C)$ and $E \mathop{\smile\!\!\!\!\!\mid}^{\dagger}_{M} BC$. By Remark 4.20 we have $E \mathop{\smile\!\!\!\!\!\mid}^{\dim}_{M} BC$. Existence of an $\mathcal{L}$-isomorphism $f : \langle \mathrm{cl}(\bar{a}M)\rangle \to E$ allows us to color E the same as $\langle \mathrm{cl}(\bar{a}M)\rangle$. So $E \in \mathcal{K}^{+}_{\alpha}$.

Let $D = \langle BC\rangle$ and $F = E \oplus_M D$ be an $\mathcal{L}$-structure given according to Convention 2.7. We subsequently turn F to an $\mathcal{L}_p$-structure by taking $p(F) = p(E) \cup p(D)$. Then by Lemma 3.3 we have that $D \leqslant F \in \mathcal{K}^{+}_{\alpha}$. Therefore there is a strong embedding $g : F \to \mathfrak{C}$ fixing D pointwise. Put $g(f(\bar{a})) = \bar{a}''$. It can be easily seen that $\bar{a}'' \equiv_B \bar{a}$, $\bar{a}'' \equiv_C \bar{a}'$ (in the sense of $\mathcal{L}_p$)and $\bar{a}'' \mathop{\smile\!\!\!\!\!\mid}^{\dagger}_{M} BC$. Thus $\bar{a}'' \mathop{\smile\!\!\!\!\!\mid}^{\dim}_{M} BC$. Furthermore $\mathrm{cl}(\bar{a}''BC) = \mathrm{cl}(\bar{a}''M) \oplus_M BC$. Therefore in the light of Lemma 4.4 we have $\bar{a}'' \mathop{\smile\!\!\!\!\!\mid}^{D}_{M} BC$ and consequently $\bar{a}'' \mathop{\smile\!\!\!\!\!\mid}^{*}_{M} BC$. ⊣

The following corollary presents an important example of a theory with $NSOP_1$. Recall that by Theorem 5.28 of [12] the theory ACF_pG is an $NSOP_1$ geometric theory.

Corollary 4.23. *For rational α, $\mathbb{ACF}_p\mathbb{G}_\alpha$ is $NSOP_1$.*

Now we turn our discussion to verifying simplicity of the theory $\mathbb{T}_\alpha$. So from now on we assume that T is a simple theory. A fact similar to Fact 4.19 states that a theory Γ is simple if and only if it supports a notion of independence with certain properties (see [16, Theorem 4.2]).

Fact 4.24. *Let Γ be an arbitrary theory, and $\mathop{\smile\!\!\!\!\!\mid}$ be an $Aut(\mathfrak{M})$-invariant ternary relation on small subsets of the monster model $\mathfrak{M} \models \Gamma$. Suppose $\mathop{\smile\!\!\!\!\!\mid}$ has the following properties.*

1. *Local character: For any $\bar{a}$ and B there is $A \subseteq B$ such that the cardinality of A is at most the cardinality of Γ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A} B$.*
2. *Finite character: $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A} B$ if and only if for every finite tuple $\bar{b}$ from B, $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A} \bar{b}$.*
3. *Extension: For any $\bar{a}, A$ and $B \supseteq A$ there is $\bar{a}'$ such that $\mathrm{tp}(\bar{a}'/A) = \mathrm{tp}(\bar{a}/A)$ and $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{A} B$.*
4. *Symmetry: If $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A} \bar{b}$ then $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_{A} \bar{a}$.*
5. *Transitivity: Suppose $A \subseteq B \subseteq C$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{B} C$ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A} B$ if and only if $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{A} C$.*
6. *The independence theorem: Suppose M is a model of Γ, $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_{M} B$, $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_{M} C$, $B \mathop{\smile\!\!\!\!\!\mid}_{M} C$ and $\bar{a} \equiv_M \bar{a}'$. Then there is $\bar{a}''$ with $\bar{a}'' \equiv_{MB} \bar{a}$, $\bar{a}'' \equiv_{MC} \bar{a}'$, and $\bar{a}'' \mathop{\smile\!\!\!\!\!\mid}_{M} BC$.*

Then $\mathop{\smile\!\!\!\!\!\mid}$ is exactly the forking independence and Γ is simple.

Theorem 4.25. *The theory $\mathbb{T}_\alpha$ is simple.*

Proof. As T is simple there is a notion of independence $\mathop{\smile\!\!\!\!\!\mid}^{\dagger}$ which satisfies the properties of Fact 4.24. Furthermore since $\mathop{\smile\!\!\!\!\!\mid}^{\dagger}$ is the T-forking independence, we may assume that $B \mathop{\smile\!\!\!\!\!\mid}^{\dagger}_{B} C$ and, furthermore $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\dagger}_{A} B$ implies $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{\dim}_{A} B$.

Let $\mathop{\smile\!\!\!\!\!\mid}^{*}$ be the notion of independence introduced in Definition 4.21. To simplify our discussion sometimes $\mathrm{cl}(\bar{a}C) \mathop{\smile\!\!\!\!\!\mid}^{\dagger,D}_{\mathrm{cl}(C)} \mathrm{cl}(BC)$ is used as an abbreviation of $\bar{a} \mathop{\smile\!\!\!\!\!\mid}^{*}_{C} B$. We prove that $\mathop{\smile\!\!\!\!\!\mid}^{*}$ satisfies the conditions 1–6 of Fact 4.24.

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One can easily prove *invariance*, *monotonicity*, *symmetry,* and *finite character*. Further the proof of independence theorem is the same as of Theorem 4.22. So we prove the other properties.

- *Local character.* Let $\bar{a}$ and a closed subset B be given. Take a finite closed subset B_0 of B such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{D}_{B_0} B$. So by D-closure preservation of Fact 4.3, we have $\mathrm{cl}(\bar{a}B_0) \mathop{\smile\!\!\!\!\!\!|}^{D}_{B_0} B$. Now by local character property of $\mathop{\smile\!\!\!\!\!\!|}^{\dagger}$ there exists a countable closed subset B_1 of B including B_0 such that $\mathrm{cl}(\bar{a}B_0) \mathop{\smile\!\!\!\!\!\!|}^{\dagger}_{B_1} B$ and $\mathrm{cl}(\bar{a}B_1) \cap B = B_1$. Therefore $\mathrm{cl}(\bar{a}B_1) \mathop{\smile\!\!\!\!\!\!|}^{D}_{B_1} B$. By iterating this process a sequence $B_0 \leqslant B_1 \leqslant B_2 \leqslant \dots$ is found such that $\mathrm{cl}(\bar{a}B_i) \mathop{\smile\!\!\!\!\!\!|}^{D}_{B_i} B$ and $\mathrm{cl}(\bar{a}B_i) \mathop{\smile\!\!\!\!\!\!|}^{\dagger}_{B_{i+1}} B$ for each $i \in \mathbb{N}$. Now put $A = \bigcup_{i\in\mathbb{N}} B_i$. Hence $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{*}_{A} B$ is obtained.
- *Extension.* Let $\bar{a}, A$ and $B \supseteq A$ be given. Without loss of generality we may assume that $A \leqslant B \leqslant \mathfrak{C}$. By simplicity of T and the extension property of $\mathop{\smile\!\!\!\!\!\!|}^{\dagger}$ there are B_0 and a partial $\mathcal{L}$-isomorphism $f : \mathrm{cl}(\bar{a}A) \to B_0$ fixing A such that $B_0 \mathop{\smile\!\!\!\!\!\!|}^{\dagger}_{A} B$. Now by taking $\bar{b} = f(\bar{a})$ (and coloring B_0 using f and leaving $B_0 - Im(f)$ non-colored) the function f can be considered as a partial $\mathcal{L}_p$-isomorphism between $\mathrm{cl}(\bar{a}A)$ and $\mathrm{cl}(\bar{b}A)$.

  Since $\mathop{\smile\!\!\!\!\!\!|}^{\dagger}$ implies $\mathop{\smile\!\!\!\!\!\!|}^{\dim}$ we have $B_0 \mathop{\smile\!\!\!\!\!\!|}^{\dim}_{A} B$. Let $C = B \oplus_A B_0$. Then $B_0 \leqslant C$. Take $g : C \to \mathfrak{C}$ to be a strong embedding and $\bar{a}' = g(\bar{b})$. Then we have $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}^{*}_{A} B$.
- *Transitivity.* Suppose $A \subseteq B \subseteq C$. We show that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{*}_{B} C$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{*}_{A} B$ if and only if $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{*}_{A} C$.Fix $A \leqslant B \leqslant C \leqslant \mathfrak{C}$. Let $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{*}_{B} C$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{*}_{A} B$. So by the definition of $\mathop{\smile\!\!\!\!\!\!|}^{*}$ we have $\mathrm{cl}(\bar{a}B) \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{B} C$ and $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{A} B$. We prove that $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{A} C$.

  Since $\mathrm{cl}(\bar{a}B) \mathop{\smile\!\!\!\!\!\!|}^{D}_{B} C$ and $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{D}_{A} B$, Lemma 4.4 implies $\mathrm{cl}(\bar{a}C) = \mathrm{cl}(\bar{a}B) \oplus_B C$ and $\mathrm{cl}(\bar{a}B) = \mathrm{cl}(\bar{a}A) \oplus_A B$. Hence $\mathrm{cl}(\bar{a}C) = \mathrm{cl}(\bar{a}A) \oplus_A C$ and $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{D}_{A} C$.

  Then $\mathrm{cl}(\bar{a}A) \subseteq \mathrm{cl}(\bar{a}B)$ yields $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{\dagger}_{B} C$. Now as we have $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{\dagger}_{A} B$, by transitivity of $\mathop{\smile\!\!\!\!\!\!|}^{\dagger}$, $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{\dagger}_{A} C$ is obtained

  Conversely assume that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^{*}_{A} C$. Then $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{A} C$. We prove that $\mathrm{cl}(\bar{a}B) \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{B} C$ and $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{A} B$.

  By transitivity of $\mathop{\smile\!\!\!\!\!\!|}^{\dagger}$ and $\mathop{\smile\!\!\!\!\!\!|}^{D}$ it is clear that $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{A} B$ and $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{B} C$. On the other hand since $\mathrm{cl}(\bar{a}A) \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{A} B$, by Lemma 4.4 we have $\mathrm{cl}(\bar{a}B) = \mathrm{cl}(\bar{a}A) \oplus_A B$. Hence $\mathrm{cl}(\bar{a}B) \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{B} C$ is equivalent to $\mathrm{cl}(\bar{a}A)B \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{B} C$. But by symmetry, transitivity of $\mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}$ and $B \mathop{\smile\!\!\!\!\!\!|}^{\dagger,D}_{B} C$ the latter holds. ⊣

Since any complete theory of a pseudo finite field is a simple geometric theory, the following corollary is immediate.

COROLLARY 4.26. *Let T be any complete theory of a pseudo finite field. Then for any α, the theory $\mathbb{T}_\alpha$ is simple.*

It is known that for simple theories the burden of a partial type p is the supremum of weights of its complete extensions (see [8, Fact 5.2]). Therefore in the light of Theorem 4.16 the following corollary is established.

COROLLARY 4.27. *Let T be any complete theory of a pseudo finite field. Then $\mathbb{T}_{\frac{1}{2}}$ is a simple theory of unbounded weight.*

PROOF. Since T is supersimple, by Proposition 5.5 of [8], it is strong. Hence by Theorem 4.16 the theory $\mathbb{T}_{\frac{1}{2}}$ is a strong simple theory of unbounded weight. ⊣

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