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Restrictions on the prime-to-p fundamental group of a smooth projective variety

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Abstract

The goal of this paper is to obtain restrictions on the prime-to-p quotient of the étale fundamental group of a smooth projective variety in characteristic $p \geqslant 0$. The results are analogues of some theorems from the study of Kähler groups. Our first main result is that such groups are indecomposable under coproduct. The second result gives a classification of the pro-ℓ parts of one-relator groups in this class.

Introduction

Our goal in this work is to obtain analogues, over fields of characteristic $p \geqslant 0$, of a few of the known restrictions on the class of Kähler groups. Recall that a group is Kähler if it can be realized as the fundamental group of a compact Kähler manifold. In this paper, we replace the usual fundamental group by the maximal prime-to-p quotient of the étale fundamental group. We focus on this group because it behaves most like its topological namesake. Let $\mathcal{P}(p)$ denote the class of profinite groups that arise as prime-to-p fundamental groups of smooth projective varieties defined over an algebraically closed field of characteristic p. Our first main result implies, among other things, the indecomposability of groups in $\mathcal{P}(p)$ under coproduct. This is an analogue of Gromov's theorem [Gro89] in the Kähler setting. In our second main result, we show that if $G \in \mathcal{P}(p)$ is the completion of a one-relator group, then for almost every ℓ the pro-ℓ quotient G_ℓ of G is isomorphic to the pro-ℓ fundamental group of a smooth projective curve. This was inspired by the recent classification of one-relator Kähler groups by Biswas and Mahan [BM12] and Kotschick [Kot12], although the argument used here is completely different. We deduce from the hard Lefschetz theorem that G_ℓ is a Demushkin group for almost all ℓ, and then the result follows from the classification of such groups.

1. Preliminaries

From the beginning, we fix an integer p which is either a prime number or 0. By a p'-group we mean a finite group of order prime to p (or of arbitrary order when $p = 0$), and by a pro-p' group we mean an inverse limit of p'-groups. The symbol ℓ will always stand for a fixed or variable prime different from p. Given a profinite group G, let $G_{p'}$ (respectively, G_ℓ) denote the maximal pro-p' (respectively, pro-ℓ) quotient of G. Given a discrete group G, we let

$$\hat{G} = \varprojlim_{G/N \text{ is a } p'\text{-group}} G/N$$

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be the pro-p' completion; so $\hat{\mathbb{Z}} = \prod_{\ell \neq p} \mathbb{Z}_\ell$. Then $\hat{G}_\ell$ can be identified with the pro-ℓ completion of G. Given a connected scheme X, let $\pi_1^{\mathrm{et}}(X)$ denote Grothendieck's étale fundamental group [SGA1], where we ignore the base point. This is the profinite group for which the category of finite sets with continuous action is equivalent to the category of étale covers of X. Let us write $\pi_1^{p'}(X)$ and $\pi_1^{\ell}(X)$ instead of $\pi_1^{\mathrm{et}}(X)_{p'}$ and $\pi_1^{\mathrm{et}}(X)_\ell$. Given an algebraically closed field k of characteristic p, let $\mathcal{P}(k)$ denote the class of pro-p' groups which are isomorphic to $\pi_1^{p'}(X)$, where X is a smooth projective k-variety; $\mathcal{P}(\mathbb{C})$ is the class of profinite completions of topological fundamental groups of complex smooth projective varieties. Set $\mathcal{P}(p) = \mathcal{P}(\overline{\mathbb{F}}_p)$, where $\overline{\mathbb{F}}_p$ is the algebraic closure of the prime field of characteristic p (so that $\overline{\mathbb{F}}_0 = \bar{\mathbb{Q}}$). There is no loss in focusing on this case because of the following fact.

Proposition 1.1. *If k is an algebraically closed field of characteristic p, then $\mathcal{P}(k) = \mathcal{P}(p)$.*

Proof. Clearly $\mathcal{P}(p) \subseteq \mathcal{P}(k)$ because extension of scalars from $\overline{\mathbb{F}}_p$ to k will not change the fundamental group [SGA1, Exposé X, Corollaire 1.8]. Suppose that X is a smooth projective k-variety. It is defined over a finitely generated extension K of $\overline{\mathbb{F}}_p$, i.e. there exists a K-scheme X_K such that $X = X_K \times_{\operatorname{Spec} K} \operatorname{Spec} k$. Let S be a variety defined over $\overline{\mathbb{F}}_p$ with function field K. After shrinking S if necessary, we can assume that there is a smooth projective morphism $\mathcal{X} \to S$ with geometric generic fiber X_K. Choose an $\overline{\mathbb{F}}_p$ rational point $y_0 \in S$, and let η denote the geometric generic point. Then $\pi_1^{p'}(\mathcal{X}_{y_0}) \cong \pi_1^{p'}(\mathcal{X}_\eta) \cong \pi_1^{p'}(X)$ by [SGA1, Exposé X, Corollaire 3.9]. □

Lemma 1.2. *If $G \in \mathcal{P}(p)$ and $H \subset G$ is open, then $H \in \mathcal{P}(p)$.*

Proof. If $G = \pi_1^{p'}(X)$, then $H = \pi_1^{p'}(Y)$ for some étale cover $Y \to X$. □

Given a finitely generated $\mathbb{Z}_\ell$-module V with a continuous action by a profinite group G, we define

$$H^i(G, V) := \varprojlim_n H^i(G, V/\ell^n V),$$
$$H^i(G, V \otimes \mathbb{Q}_\ell) = H^i(G, V) \otimes \mathbb{Q}_\ell.$$

This naive definition will suffice for our purposes, although there is one place where we would be better off with the more subtle definition of Jannsen [Jan88]. We summarize what we need about this in the following lemma.

Lemma 1.3. *Suppose that $1 \to K \to G \to H \to 1$ is an exact sequence of profinite groups where G is topologically finitely generated and V is a finitely generated $\mathbb{Z}_\ell$-module with continuous H action. Then there is the usual five-term exact sequence of Hochschild and Serre,*

$$0 \to H^1(H, V) \to H^1(G, V) \to H^0(H, H^1(K, V)) \to H^2(H, V) \to H^2(G, V).$$

Proof. Following Jannsen, we define $H^i_{\mathrm{cont}}(G, V)$ as the ith derived functor of

$$V \mapsto \varprojlim_n H^0(V/\ell^n V).$$

The Hochschild–Serre spectral sequence and the resulting five-term sequence for H^*_{cont} can be constructed in the usual way. By [Jan88, (2.1)], we have an exact sequence

$$0 \to \varprojlim{}^1 H^{i-1}(G, V/\ell^n V) \to H^i_{\mathrm{cont}}(G, V) \to H^i(G, V) \to 0. \quad (1)$$

When $i \leqslant 2$, we claim that $H^{i-1}(G, V/\ell^n V)$ is finite. For $i = 1$ this is clear, because V is finitely generated. So we have to show this for $i = 2$. We can find an open normal subgroup G_1 which acts trivially on $V/\ell^n V$. Then, by Hochschild–Serre, we have an exact sequence

$$H^1(G/G_1, V/\ell^n V) \to H^1(G, V/\ell^n V) \to H^1(G_1, V/\ell^n V).$$

The finiteness of the middle group is a consequence of the finiteness of the outer groups. The first group is finite because both G/G_1 and $V/\ell^n V$ are finite; the last group $H^1(G_1, V/\ell^n V)$ is isomorphic to $\mathrm{Hom}_{\mathrm{cont}}(G_1, V/\ell^n V)$. By assumption, G contains a finitely generated dense subgroup Γ. The group $\Gamma \cap G_1$ is easily seen to be finitely generated and dense in G_1. Therefore $\mathrm{Hom}_{\mathrm{cont}}(G_1, V/\ell^n V)$ is finite and the claim is proved.

The Mittag-Leffler condition holds for $H^{i-1}(G, V/\ell^n V)$ and $i \leqslant 2$ by the previous claim. Therefore the $\varprojlim^1$ in (1) vanishes, and so $H^i(G, V) \cong H^i_{\mathrm{cont}}(G, V)$ for $i \leqslant 2$. By the same argument, $H^i(H, V) \cong H^i_{\mathrm{cont}}(H, V)$. So the five-term sequence for H^*_{cont} can be identified with the one given in the statement of the lemma. □

LEMMA 1.4. *If G is a profinite group and V an abelian pro-ℓ group, then*

$$(G/[G,G])_\ell \cong G_\ell/[G_\ell, G_\ell],$$
$$\mathrm{Hom}(G, V) \cong \mathrm{Hom}(G_\ell, V),$$

where Hom *is the group of continuous homomorphisms.*

Proof. It is enough to prove the first isomorphism, because the second is a consequence of it. Since $(G/[G,G])_\ell$ is an abelian pro-ℓ group, the homomorphism $G \to (G/[G,G])_\ell$ factors through the abelianization of the maximal pro-ℓ quotient $G_\ell/[G_\ell, G_\ell]$. So we have a homomorphism $G_\ell/[G_\ell, G_\ell] \to (G/[G,G])_\ell$. On the other hand, the map $G \to G_\ell/[G_\ell, G_\ell]$ must factor through the maximal pro-ℓ quotient of the abelianization $(G/[G,G])_\ell$. This gives the inverse. □

PROPOSITION 1.5. *Suppose that X is a connected scheme of finite type over k. Let G be a quotient of $\pi_1^{\mathrm{et}}(X)$ by a closed normal subgroup, such that G dominates $\pi_1^\ell(X)$. Given a finitely generated $\mathbb{Z}_\ell$-module V with continuous G-action, there exists a homomorphism to ℓ-adic cohomology*

$$H^i(G, V) \to H^i(X, V). \tag{2}$$

This is compatible with the cup products

$$H^i(G, V) \otimes H^j(G, V') \to H^{i+j}(G, V \otimes V')$$

and

$$H^i(X, V) \otimes H^j(X, V') \to H^{i+j}(X, V \otimes V').$$

The map (2) is an isomorphism when $i \leqslant 1$.

Proof. We start by proving the analogous statements over $\Lambda_n = \mathbb{Z}/\ell^n\mathbb{Z}$, and then we take the limit. We indicate two different constructions of the map; the first is simpler, but the second gives more, so it is the one that we shall use. First of all, both $H^i(G, -)$ and $H^i(X, -)$ are δ-functors from the category of discrete $\Lambda_n[G]$-modules to abelian groups, with the first being universal in the sense of [Gro57]. By the connectedness assumption, $H^0(G, V) \cong H^0(X, V)$. Thus we get a map

$$H^*(G, -) \to H^*(X, -)$$

of δ-functors. Compatibility with cup products can be proved in principle by dimension shifting and induction, but it seems simpler to give an alternate interpretation. Suppose that $Y \to X$ is a Galois étale cover with Galois group H being a quotient of G by an open normal subgroup. Then we have an isomorphism of simplicial schemes

$$\mathrm{cosk}(Y \to X)_\bullet \cong (Y \times EH_\bullet)/G,$$

where $\mathrm{cosk}(Y \to X)_\bullet$ is the simplicial scheme

$$\dots Y \times_X Y \rightrightarrows Y$$

and $EH_\bullet \to BH_\bullet$ is a simplicial model for the universal H-bundle over the classifying space (cf. [Del74, §§ 5.1 and 6.1] or [Mil80, pp. 99–100]). The projection $\mathrm{cosk}(Y \to X)_\bullet \to EH_\bullet/G = BH_\bullet$ induces a map from the bar complex $C^\bullet(H, V)$ with coefficients in an H-module V to the Čech complex $\check{C}^\bullet(Y \to X, V)$. Thus we obtain maps

$$H^*(G, V) \to H^*(H, V) \to H^*(X, V).$$

The compatibility with cup products now follows easily from the standard simplicial formulas for them (see [Mil80, p. 172]).

We have already seen that the map (2) is an isomorphism when $i = 0$. We next prove that it is an isomorphism when $i = 1$. First, suppose that V is a finite Λ_n-module with trivial G-action. Then we have an isomorphism

$$H^1(G, V) \cong \mathrm{Hom}(G, V).$$

On the other hand,

$$H^1(X, V) \cong \mathrm{Hom}(\pi_1^{\mathrm{et}}(X), V)$$

because both groups classify V-torsors [Mil80, pp. 121–123]. By Lemma 1.4, we can also make the identification

$$\mathrm{Hom}(\pi_1^{\mathrm{et}}(X), V) \cong \mathrm{Hom}(G, V).$$

Now suppose that V is a nontrivial finite $\Lambda_n[G]$-module. Let $\pi : Y \to X$ be an étale cover such that π^*V is trivial. We can assume that π is a Galois cover, with Galois group H being a quotient of G. Set $K = \pi_1^{\mathrm{et}}(Y)$. Then K acts trivially on π^*V. It follows that

$$H^i(K, V) \cong H^i(Y, \pi^*V), \quad i = 0, 1, \tag{3}$$

and this isomorphism is compatible with the H-action. Then Hochschild–Serre gives a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(H, H^0(K, V)) & \longrightarrow & H^1(G, V) & \longrightarrow & H^0(H, H^1(K, V)) & \longrightarrow & H^2(H, H^0(K, V)) \\
 & & \downarrow \cong & & \downarrow f & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & H^1(H, H^0(Y, V)) & \longrightarrow & H^1(X, V) & \longrightarrow & H^0(H, H^1(Y, V)) & \longrightarrow & H^2(H, H^0(X, V))
\end{array}$$

with exact rows. The maps labeled with $\cong$ are isomorphisms by (3). Thus f is an isomorphism by the five lemma.

To summarize, we have canonical multiplicative homomorphisms

$$H^i(G, V) \to H^i(X, V)$$

for $\Lambda_n[G]$-modules, which are isomorphisms for $i \leqslant 1$. The proposition follows by taking the inverse limit over n. $\square$

We will apply the preceding proposition in the two cases $G = \pi_1^{p'}(X)$ and $G = \pi_1^{\ell}(X)$. It is worth remarking that when $V = \mathbb{Q}_\ell$, we have $H^1(\pi_1^{p'}(X), V) \cong H^1(\pi_1^{\ell}(X), V)$, but there is no reason to expect this for higher cohomology.

We have the following basic finiteness property.

Theorem 1.6 (Raynaud). *Any element of $P(p)$ is topologically finitely presented.*

Proof. This follows from [SGA7, Théorème 2.3.1 and Remarque 2.3.2]. □

The analogous statement for Kähler groups is a well-known consequence of the finite triangulability of compact manifolds. We wish to point out that topological finite presentability does not preclude some fairly wild examples such as $\prod_{\ell \neq p} \mathbb{Z}\ell\mathbb{Z}$. However, such examples cannot lie in $\mathcal{P}(p)$.

Proposition 1.7. *If $G \in P(p)$, then $G/[G,G]$ is the product of a finite abelian group and $\hat{\mathbb{Z}}^b = \prod_{\ell \neq p} \mathbb{Z}_\ell^b$ where b is an even integer.*

Proof. We can decompose $G/[G,G]$ as $\prod_{\ell \neq p} \mathbb{Z}_\ell^{b_\ell} \times A_\ell$, where A_ℓ is a finite abelian ℓ-group. We have to show that b_ℓ is constant and that $A_\ell = 0$ for $\ell \gg 0$. The Kummer sequence [Mil80, p. 66] gives an isomorphism

$$\mathrm{Hom}\left(\prod \mathbb{Z}_\ell^{b_\ell} \times A_\ell, \mathbb{Z}_\ell\right) \cong T_\ell \mathrm{Pic}(X) = T_\ell \mathrm{Pic}^0(X)_{\mathrm{red}},$$

where we make the identification $\mathbb{Z}_\ell \cong \mathbb{Z}_\ell(1)$. For the last equality, we use the exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0$$

and the fact that the Neron–Severi group $\mathrm{NS}(X)$ is finitely generated. Since $\mathrm{Pic}^0(X)_{\mathrm{red}}$ is an abelian variety, it follows that $b_\ell = b = 2 \dim \mathrm{Pic}^0(X)_{\mathrm{red}}$ (see, for example, [Mil86, Theorem 15.1]).

Again by Kummer, we have an isomorphism

$$\mathrm{Hom}\left(\prod \mathbb{Z}_\ell^{b} \times A_\ell, \mathbb{Z}/\ell\mathbb{Z}\right) \cong \ell\text{-torsion subgroup of } \mathrm{Pic}(X).$$

Since $\mathrm{NS}(X)$ is finitely generated, the ℓ-torsion subgroups of $\mathrm{Pic}(X)$ and $\mathrm{Pic}^0(X)$ coincide for all $\ell \gg 0$. The ℓ-torsion subgroup of $\mathrm{Pic}^0(X)$ is isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^b$. Therefore, for $\ell \gg 0$, we must have $\mathrm{Hom}(A_\ell, \mathbb{Z}/\ell\mathbb{Z}) = 0$, which implies that $A_\ell = 0$. □

2. Consequences of the hard Lefschetz theorem

By far the simplest restriction on Kähler groups is what we will refer to as the parity test: a finitely generated Γ cannot be Kähler unless $\mathrm{rank}(\Gamma/[\Gamma,\Gamma])$ is even. This is a consequence of the Hodge decomposition. Proposition 1.7 gives an analogue in our setting. It is convenient to record the relevant part of it as a corollary.

Corollary 2.1. *If $G \in \mathcal{P}(p)$, $\mathrm{rank}\, G_\ell/[G_\ell, G_\ell]$ is a fixed even integer for each $\ell \neq p$.*

The following fact, which refines the previous result, was first observed by Johnson and Rees [JR87] in the Kähler group setting.

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Theorem 2.2. *Let $G \in \mathcal{P}(p)$, and let H be a quotient of G by a closed normal subgroup such that H dominates G_ℓ. Suppose that $\rho : H \to O_n(\mathbb{Q}_\ell)$ is an orthogonal representation such that $\rho(H)$ is finite, and let V be the corresponding H-module with quadratic form $q : V \otimes V \to \mathbb{Q}_\ell$. Then there exists a linear map $\lambda : H^2(H, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell$ such that $\lambda(q(\alpha \cup \beta))$ defines a symplectic pairing on $H^1(H, V)$.*

Proof. Suppose that $G = \pi_1^{p'}(X)$, where X is an n-dimensional smooth projective variety. Fix an ample line bundle $\mathcal{O}_X(1)$, and let L denote the corresponding Lefschetz operator. We claim that

$$H^1(X, V) \times H^1(X, V) \xrightarrow{q \circ \cup} H^2(X, \mathbb{Q}_\ell) \xrightarrow{L^{n-1}} H^{2n}(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell \tag{4}$$

gives a nondegenerate symplectic pairing. When ρ is trivial, (V, q) is a sum of n copies of $\mathbb{Q}_\ell$ with the standard pairing, and the claim follows from the hard Lefschetz theorem for étale cohomology [Del80, 4.1.1]. In general, let $\pi : Y \to X$ be a Galois étale cover with Galois group K such that π^*V is trivial. We can decompose $\pi_*\pi^*V$ into $V \oplus V'$ under the K-action. Let $p : \pi_*\pi^*V \to V$ denote the projection. We equip Y with the Lefschetz operator corresponding to $\pi^*\mathcal{O}_X(1)$. Then the pairing (4) is obtained by applying p to the nondegenerate pairing

$$H^1(Y, \pi^*V) \times H^1(Y, \pi^*V) \longrightarrow H^2(Y, \mathbb{Q}_\ell) \xrightarrow{L^{n-1}} H^{2n}(Y, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell,$$

and the claim follows for general V with finite monodromy.

By Proposition 1.5, we have a commutative diagram

$$\begin{array}{ccccc} H^1(H, V) \times H^1(H, V) & \longrightarrow & H^2(H, \mathbb{Q}_\ell) & \overset{\lambda}{\searrow} & \\ \downarrow \cong & & \downarrow \iota & & \\ H^1(X, V) \times H^1(X, V) & \longrightarrow & H^2(X, \mathbb{Q}_\ell) & \xrightarrow{L^{n-1}} & \mathbb{Q}_\ell \end{array}$$

where λ is defined as $L^{n-1} \circ \iota$. □

Second proof of Corollary 2.1. $H^1(G, \mathbb{Q}_\ell)$ carries a symplectic pairing, so it must be even-dimensional. □

The theorem itself gives more subtle information than the parity test. For example, we have the following consequence.

Proposition 2.3. *Suppose that*

$$1 \to K \to G \to H \to 1$$

is an extension of pro-p' groups such that $H^1(H, \mathbb{Q}_\ell) \neq 0$ and the trangression $\mathrm{Hom}_H(K, \mathbb{Q}_\ell) \to H^2(H, \mathbb{Q}_\ell)$ *is an isomorphism. Then $G \notin \mathcal{P}(p)$.*

Proof. From the Hochschild–Serre sequence

$$0 \to H^1(H, \mathbb{Q}_\ell) \xrightarrow{\alpha} H^1(G, \mathbb{Q}_\ell) \to \mathrm{Hom}_H(K, \mathbb{Q}_\ell) \xrightarrow{\sim} H^2(H, \mathbb{Q}_\ell) \xrightarrow{\beta} H^2(G, \mathbb{Q}_\ell)$$

(Lemma 1.3), we conclude that α is an isomorphism and $\beta = 0$. Therefore

$$\cup : \wedge^2 H^1(G, \mathbb{Q}_\ell) \to H^2(G, \mathbb{Q}_\ell)$$

is zero, because it factors through β. Thus $G \notin \mathcal{P}(p)$ by Theorem 2.2. □

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The conditions of the proposition are easy to check for the following example.

COROLLARY 2.4. *The completion of the Heisenberg group*

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \hat{\mathbb{Z}} \right\}$$

is not in $\mathcal{P}(p)$.

A more general class of examples to which the proposition applies comes from generalized universal central extensions. Given a pro-p' group H, we have a (generally noncanonical) central extension

$$0 \to H_2(H, \hat{\mathbb{Z}}) \to G \to H \to 1, \tag{5}$$

with extension class lifting the identity under the surjection

$$H^2(H, H_2(H, \hat{\mathbb{Z}})) \to \mathrm{Hom}(H_2(H, \hat{\mathbb{Z}}), H_2(H, \hat{\mathbb{Z}})).$$

Here the (co)homologies are defined by taking inverse limits of the usual groups with coefficients in $\mathbb{Z}/n\mathbb{Z}$. Transgression gives an isomorphism, so that the following holds.

COROLLARY 2.5. *If $H^1(H, \mathbb{Q}_\ell)$ is nonzero, then the group G of the above extension (5) is not in $\mathcal{P}(p)$.*

The last statement should be compared with [Rez02, p. 717, Corollary].

3. Free products

Given two pro-p' groups G_1 and G_2, their coproduct in the category of pro-p' groups exists [RZ10, § 9.1]. We denote it by $G_1 \hat{*} G_2$. It is closely related to the usual free product $*$.

LEMMA 3.1. *Given discrete groups G_i, we have $\widehat{G_1 * G_2} \cong \hat{G}_1 \hat{*} \hat{G}_2$.*

Proof. See [RZ10, 9.1.1]. □

The completion $\hat{F}^r$ of the usual free group on r generators is a free pro-p' group. It can also be expressed as a coproduct

$$\hat{F}^r = \hat{\mathbb{Z}} \hat{*} \cdots \hat{*} \hat{\mathbb{Z}} \quad (r \text{ factors}).$$

In [ABR92], it is shown that a Kähler group cannot be an extension of a group with infinitely many ends by a finitely generated group. We observe that any nontrivial free product other than $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ has infinitely many ends. Since we do not (yet) have a theory of ends in the profinite setting, we give a slightly weaker statement involving the aforementioned class. On the other hand, the hypothesis on the kernel can be relaxed slightly.

THEOREM 3.2. *Let $p \neq 2$. Suppose that we have an extension of pro-p' groups*

$$1 \to K \to G \to H \to 1 \tag{6}$$

such that:

(a) *$(K/[K,K])_2/(\text{torsion})$ is a finitely generated $\mathbb{Z}_2$-module; and*
(b) *H is a nontrivial coproduct other than $(\mathbb{Z}/2\mathbb{Z}) \hat{*} (\mathbb{Z}/2\mathbb{Z})$.*

Then $G \notin \mathcal{P}(p)$.

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Proof. Suppose that G fits into the exact sequence (6) with $H = H_1 \hat{*} H_2$, where the H_i are nontrivial and not both of order 2. We will show that G cannot lie in $\mathcal{P}(p)$. We first reduce to the case where H is of the form $J \hat{*} \hat{F}^2$. Choose nontrivial finite quotients Q_i of H_i such that $|Q_i| > 2$ for some i. Let $L \subset H$ be the kernel of the projection $H \to Q_1 \times Q_2$. Then, by the profinite version of the Kurosh subgroup theorem [RZ10, 9.1.9], we see that $L \cong J \hat{*} \hat{F}^2$ for some group J. It suffices to prove that the preimage $\tilde{G}$ of L in G is not in $\mathcal{P}(p)$ by Lemma 1.2. Since it fits into an extension

$$1 \to K \to \tilde{G} \to L \to 1,$$

we may replace G by $\tilde{G}$ and H by L.

From the exact sequence (6), we get a continuous action of H on $K/[K,K]$. Therefore $M = (K/[K,K])_2 \otimes_{\mathbb{Z}_2} \mathbb{Q}_2$ is a finite-dimensional representation of H. With respect to the factor $\hat{F}^2 = \hat{\mathbb{Z}} \hat{*} \hat{\mathbb{Z}}$ of H, we get two actions of $\hat{\mathbb{Z}}$ on M, which we refer to as the first and second actions. Let $\{\xi_1, \ldots, \xi_n\}$ be the (possibly empty) set of one-dimensional characters of $\hat{F}^2$ corresponding to one-dimensional subquotients of M. We may suppose that $\xi_1, \ldots, \xi_m$ are the characters among these of finite order. Let $S \subset \hat{F}^2$ be the intersection of kernels of $\xi_1, \ldots, \xi_m$. The group S is necessarily of the form $\hat{F}^r$ with $r \geqslant 2$; see [RZ10, 3.6.2]. After replacing H by $J \hat{*} S = (J \hat{*} \hat{F}^{r-2}) \hat{*} \hat{F}^2$, J by $J \hat{*} \hat{F}^{r-2}$ and G by the preimage of the new H in the old G, we may assume that all the characters ξ_i are either trivial or of infinite order. Let ξ_i' denote the restrictions of the ξ_i to the first factor of $\hat{F}^2 = \hat{\mathbb{Z}} \hat{*} \hat{\mathbb{Z}}$. Then the sign character $\sigma : \hat{\mathbb{Z}} \to \mathbb{Z}_2 \to \mathbb{Q}_2^*$, defined by

$$\sigma(x) = \begin{cases} +1 & \text{if } x \in 2\hat{\mathbb{Z}}, \\ -1 & \text{otherwise,} \end{cases}$$

is not in $\{\xi_1', \ldots, \xi_n'\}$. Let $\chi_1 = \sigma$ and $\chi_2 \in \{1, \sigma\}$, where the precise choice will be determined below. Let $V = \mathbb{Q}_2$ denote the $H = J \hat{*} \hat{\mathbb{Z}} \hat{*} \hat{\mathbb{Z}}$ module where J acts trivially and the two $\hat{\mathbb{Z}}$ factors act through χ_1 and χ_2, respectively. We note that V is orthogonal, so that we can apply Theorem 2.2 when the time comes.

We now compute $\dim H^1(G, V)$. From the Hochschild–Serre sequence (Lemma 1.3), we obtain the exact sequence

$$0 \to H^1(H, V) \to H^1(G, V) \to H^0(H, H^1(K, V)). \tag{7}$$

We can identify

$$H^0(H, H^1(K, V)) \cong H^0(H, \mathrm{Hom}(K/[K,K], V)) \cong \mathrm{Hom}_H(M, V).$$

Since we chose $\chi_1 \notin \{\xi_1', \ldots, \xi_n'\}$, the latter space is zero. Therefore, by (7) we obtain an isomorphism

$$H^1(G, V) \cong H^1(H, V).$$

By an appropriate Mayer–Vietoris sequence [RZ10, Proposition 9.2.13], we see that

$$H^1(H, V) \cong H^1(J, V) \oplus H^1(\hat{\mathbb{Z}}, \mathbb{Q}_{2,\chi_1}) \oplus H^1(\hat{\mathbb{Z}}, \mathbb{Q}_{2,\chi_2}),$$

where the subscripts χ_i indicate the action. The middle group on the right-hand side vanishes because χ_1 was nontrivial. By choosing χ_2 to be trivial or not according to the parity of $\mathrm{rank}(J/[J, J])$, we see that the right-hand side can be made to have odd dimension. Therefore G cannot be the pro-p' fundamental group of a smooth projective variety, by Theorem 2.2. □

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COROLLARY 3.3. *A group in $\mathcal{P}(p)$ cannot decompose as a coproduct of nontrivial pro-p' groups; in particular, it cannot be free.*

Proof. The only case not covered by the last theorem is $(\mathbb{Z}/2\mathbb{Z}) \hat{*} (\mathbb{Z}/2\mathbb{Z})$, but since this contains $\hat{\mathbb{Z}}$ as an open subgroup, it is ruled out by Corollary 2.1. □

COROLLARY 3.4. *Suppose that G satisfies all the assumptions of the theorem but with* (a) *replaced by*

(a′) *K is topologically finitely generated.*

Then $G \notin \mathcal{P}(p)$.

Proof. The condition (a′) implies (a). □

COROLLARY 3.5. *Suppose that $1 \to K \to G \to H_1 * H_2 \to 1$ is an exact sequence of discrete groups, with K finitely generated and $\hat{H}_i$ nontrivial and not both of order 2. Then $\hat{G} \notin \mathcal{P}(p)$.*

Proof. By [RZ10, Proposition 3.2.5] and Lemma 3.1, we have an exact sequence

$$\hat{K} \to \hat{G} \xrightarrow{f} \hat{H}_1 \hat{*} \hat{H}_2 \to 1.$$

Therefore $\ker f$ is topologically finitely generated. □

As an illustration of the use of this theorem, we show that the pure braid group does not lie in this class. This is a direct translation of the argument in [Ara95] for showing that braid groups are not Kähler. Recall that B_n is given by generators $s_1, \ldots, s_{n-1}$ with relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ if $|i-j| > 1$. This maps to the symmetric group S_n via $s_i \mapsto (i\, i+1)$. The kernel is the pure braid group P_n. More geometrically, P_n is the fundamental group of the configuration space of n distinct ordered points in the plane.

PROPOSITION 3.6. *We have $\hat{P}_n \notin \mathcal{P}(p)$.*

Proof. We have $P_2 = \mathbb{Z}$, so $\hat{P}_2 \notin \mathcal{P}(p)$ by Corollary 2.1. The group B_3 is generated by $a = s_1 s_2 s_1$ and $b = s_1 s_2$ with the relation $a^2 = b^3$. There is a surjective homomorphism from $f : B_3 \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ which sends a and b to the generators of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, respectively. The kernel of f is the cyclic group generated by $a^2 \in P_3$. Thus we have an extension

$$0 \to \mathbb{Z} \to P_3 \to f(P_3) \to 1.$$

By Kurosh's subgroup theorem [Ser03, § 5.5], the image $f(P_3)$ is a free product of a nonabelian free group and some additional factors. Therefore $\hat{P}_3 \notin \mathcal{P}(p)$ by Corollary 3.5. When $n > 3$, projection of the configuration spaces gives a fibration resulting in a surjective homomorphism $P_n \to P_3$ with finitely generated kernel. It follows that $P_n \to f(P_3)$ is again surjective with finitely generated kernel. So once again Corollary 3.5 shows that $\hat{P}_n \notin \mathcal{P}(p)$. □

4. One-relator groups

Recently, Biswas and Mahan [BM12] and Kotschick [Kot12] classified one-relator Kähler groups: they are all fundamental groups of one-dimensional compact orbifolds with at most one orbifold point. In more explicit terms, such a group is of the form

$$\Gamma_{g,m} = \begin{cases} \langle x_1, \ldots, x_{2g} \mid ([x_1, x_{g+1}] \ldots [x_g, x_{2g}])^m \rangle & \text{if } g > 0, \\ \mathbb{Z}/m\mathbb{Z} = \langle x \mid x^m \rangle & \text{if } g = 0. \end{cases}$$

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(Note that both [BM12] and [Kot12] classify infinite one-relator Kähler groups, but the statement as given above is an immediate consequence.) We prove a pro-ℓ version for large ℓ, assuming that the relation lies in the commutator subgroup. To reconcile the statement below with the one just given, observe that $(\hat{\Gamma}_{g,m})_\ell \cong (\hat{\Gamma}_{g,1})_\ell$ when ℓ is coprime to m.

Theorem 4.1. *Suppose that $G \in \mathcal{P}(p)$ is the pro-p' completion of a discrete one-relator group. Then there exists an explicit finite set S of primes such that if $\ell \notin S$, the maximal pro-ℓ quotient G_ℓ of G is isomorphic to the pro-ℓ completion of the genus-g surface group $\Gamma_{g,1}$ where $g = \frac{1}{2}\dim H^1(G, \mathbb{Q}_\ell)$.*

Before giving the proof, we need the following version of Stallings's theorem [Sta65].

Lemma 4.2. *If $f : G \to H$ is a continuous homomorphism of pro-ℓ groups such that the induced map $H^i(H, \mathbb{Z}/\ell\mathbb{Z}) \cong H^i(G, \mathbb{Z}/\ell\mathbb{Z})$ is an isomorphism for $i = 1$ and an injection for $i = 2$, then f is an isomorphism.*

Proof. The surjectivity of f follows from [Ser65, I, Proposition 23], so it remains to check injectivity. Define the ℓ-central series by $C^0(G) = G$ and $C^{n+1}(G) = [G, C^n(G)]C^n(G)^\ell$. We claim that f induces an isomorphism $G/C^n(G) \to H/C^n(H)$. The injectivity of f will follow from this claim because one has $\bigcap C^n(G) = \{1\}$. The proof of the claim is essentially identical to the argument in [Sta65] in dual form; nevertheless, we give it for completeness. This proof proceeds by induction. The initial case $n = 1$ follows from the isomorphism $H^1(H, \mathbb{Z}/\ell\mathbb{Z}) \cong H^1(G, \mathbb{Z}/\ell\mathbb{Z})$. For the induction step, we use the following commutative diagram.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C^nH/C^{n+1}H & \longrightarrow & H/C^{n+1}H & \longrightarrow & H/C^nH & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \\ 1 & \longrightarrow & C^nG/C^{n+1}G & \longrightarrow & G/C^{n+1}G & \longrightarrow & G/C^nG & \longrightarrow & 1 \end{array}$$

We have to show that f_{n+1} is an isomorphism, assuming this is true for f_n. It is enough to check that γ is an isomorphism. We have the following diagram coming from Hochschild–Serre.

$$\begin{array}{ccccccccc} H^1(H/C^n) & \longrightarrow & H^1(H) & \longrightarrow & \mathrm{Hom}(C^nH, \mathbb{Z}/\ell) & \longrightarrow & H^2(H/C^n) & \longrightarrow & H^2(H) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma^*} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\ H^1(G/C^n) & \longrightarrow & H^1(G) & \longrightarrow & \mathrm{Hom}(C^nG, \mathbb{Z}/\ell) & \longrightarrow & H^2(G/C^n) & \longrightarrow & H^2(G) \end{array}$$

The hypotheses, including the induction hypothesis, imply that α, β and δ are isomorphisms, while ϵ is injective. Therefore γ^* is an isomorphism by the five lemma. This implies that γ is an isomorphism. $\square$

Proof of Theorem 4.1. Let G be the completion of the quotient of the free group on d letters, $F = F^d$, by the normal subgroup R generated by a single element $r \in F$ with $r \neq 1$.

We have two cases. The first case is where $r \in [F, F]$. The associated graded algebra of F with respect to the lower central series

$$\mathrm{Gr}(F) = F/[F, F] \oplus [F, F]/[F, [F, F]] \oplus \cdots$$

is a graded Lie algebra over $\mathbb{Z}$ with Lie bracket induced by the commutator [Laz54]. Let $x_1, \ldots, x_d$ denote generators of F. The first summand $F/[F, F]$ is a free $\mathbb{Z}$-module freely generated by the classes $\bar{x}_i$ of x_i, and the next summand $[F, F]/[F, [F, F]]$ is freely generated by $[\bar{x}_i, \bar{x}_j]$ with $i < j$.

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Thus we can expand the class $\bar{r} \in [F,F]/[F,[F,F]]$ of r as $\bar{r} = \sum a_{ij}[\bar{x}_i, \bar{x}_j]$ with $a_{ij} \in \mathbb{Z}$. We extend (a_{ij}) to a skew-symmetric matrix by setting $a_{ji} = -a_{ij}$ and $a_{ii} = 0$. By [RZ10, 3.2.5], we have an exact sequence

$$\hat{R}_\ell \to \hat{F}_\ell \to G_\ell \to 1.$$

Thus $\hat{G}_\ell$ is also a one-relator group in the topological sense, and therefore $\dim_{\mathbb{F}_\ell} H^2(G_\ell, \mathbb{Z}/\ell\mathbb{Z}) = 1$ by [Ser65, p. 31, Corollaire]. We can also conclude that

$$G_\ell/[G_\ell, G_\ell] \cong \hat{F}_\ell/[\hat{F}_\ell, \hat{F}_\ell] \cong \mathbb{Z}_\ell^d.$$

Thus $d = \dim_{\mathbb{F}_\ell} H^1(G_\ell, \mathbb{Z}/\ell\mathbb{Z})$ is the minimal number of generators of G_ℓ. From Theorem 2.2, it follows that $d = 2g$ for some integer g and $H^2(G_\ell, \mathbb{Q}_\ell) \neq 0$. Therefore $H^2(G_\ell, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell$ and the cup product pairing

$$H^1(G_\ell, \mathbb{Z}_\ell) \times H^1(G_\ell, \mathbb{Z}_\ell) \to H^2(G_\ell, \mathbb{Z}_\ell)$$

is nondegenerate. By an argument identical to the proof of [Lab67, Proposition 3], we see that this pairing is represented by the matrix (a_{ij}). Let S denote the union of $\{2, p\}$ and the set of all prime factors of the a_{ij}. Then we can reduce modulo $\ell \notin S$ to obtain a nondegenerate cup product pairing

$$H^1(G_\ell, \mathbb{Z}/\ell\mathbb{Z}) \times H^1(G_\ell, \mathbb{Z}/\ell\mathbb{Z}) \to H^2(G_\ell, \mathbb{Z}/\ell\mathbb{Z}).$$

It follows that G_ℓ is a so-called Demushkin group [Dem61, Lab67]; these groups are classified. Since ℓ is odd, the only possibility is

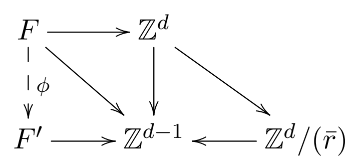
$$G_\ell \cong \langle y_1, \ldots, y_{2g} \mid y_1^{\ell^n}[y_1, y_{g+1}] \ldots [y_g, y_{2g}] \rangle$$

for some integer $n \geqslant 0$. When $n > 0$, $G_\ell/[G_\ell, G_\ell]$ has torsion contrary to what was shown above. Therefore $n = 0$ and the theorem is proved in this case.

Now we turn to the remaining case where $r \notin [F,F]$. Let $\bar{r} \in F/[F,F] \cong \mathbb{Z}^d$ be the image of r. Fix an isomorphism

$$(F/[F,F])/(\bar{r})/(\text{torsion}) \cong \mathbb{Z}^{d-1},$$

and lift the generators on the right to the free group $F' = F^{d-1}$. We thus have a commutative diagram



given by the solid arrows. We can choose a homomorphism $\phi : F \to F'$ which completes the commutative diagram as indicated. Let K be the quotient of F' by the normal subgroup generated by $r' = \phi(r)$, and let $H = \hat{K}$. The homomorphism ϕ induces a continuous homomorphism $f : G \to H$.

Let S_1 be the set of primes ℓ such that $\mathbb{Z}^d/(\bar{r})$ has ℓ-torsion. Equivalently, S_1 is the minimal set of primes such that $(\mathbb{Z}^d/(\bar{r}))_\ell$ is torsion-free whenever $\ell \notin S_1$. We assume $\ell \notin S_1$ for the remainder of this paragraph. We claim that f induces an isomorphism $G_\ell \cong H_\ell$. By construction, we have $H^1(H_\ell, \mathbb{Z}/\ell\mathbb{Z}) \cong H^1(G_\ell, \mathbb{Z}/\ell\mathbb{Z})$. If we can show that there is an injection on H^2, the claim will follow from Lemma 4.2. We split this into subcases. Suppose that $r' = 1$; then H_ℓ is a free pro-ℓ group. Therefore $H^2(H_\ell) = 0$ and so the claim follows in this case. But, in fact, this case is impossible because G_ℓ cannot be free. Thus $r' \neq 1$. Then

$$H^2(H_\ell, \mathbb{Z}/\ell\mathbb{Z}) \cong H^1(R', \mathbb{Z}/\ell\mathbb{Z})^{\hat{F}'_\ell} \cong \mathbb{Z}/\ell\mathbb{Z}$$

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where $R' \subset \hat{F}'_\ell$ is the closed normal subgroup generated by r' (cf. [Ser65, pp. 30–31]). We have a similar description for $H^2(G_\ell)$. From this it follows easily that the map $H^2(H_\ell) \to H^2(G_\ell)$ is nonzero and therefore an isomorphism.

With the claim now proven, we can work with H instead of G provided we choose $\ell \notin S_1$. By construction, $r' \in [F', F']$, so we are now in the same situation as the previous case. The arguments for that case show that there is a finite set of primes S_2, explicitly determined by r', such that when $\ell \notin S_2$ we have $H_\ell \cong \hat{\Gamma}_{g,1;\ell}$ for some g. In conclusion, the theorem holds in the second case when $S = S_1 \cup S_2$. □

Corollary 4.3. *With notation as in Theorem 4.1, the maximal pro-nilpotent prime-to-S quotients of G and $\hat{\Gamma}_{g,1}$ are isomorphic.*

Proof. This follows from the theorem and [LO10, Lemma 2.10]. □

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