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NORMALITY IN ELEMENTARY SUBGROUPS OF CHEVALLEY GROUPS OVER RINGS

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1. Introduction. In [**6**] we have constructed certain normal subgroups G_I of the elementary subgroup G_R of the Chevalley group $G(L, R)$ over R corresponding to a finite dimensional simple Lie algebra L over the complex field, where R is a commutative ring with identity. The method employed was to augment somewhat the generators of the elementary subgroup E_I of G corresponding to an ideal I of the underlying Chevalley algebra L_R; E_I is thus the group generated by all $x_r(t)$ in G having the property that $te_r \in I$. In [**6**, § 5] we noted that in general E_I actually had to be enlarged for a normal subgroup of G_R to be obtained. In the present paper, we note that G_I is in fact the minimal normal subgroup of G_R which contains the $x_r(t)$ with r positive and $te_r \in I$; i.e., G_I is the normal closure of U_I in G. In his review of [**6**], I. Stewart has asked to what extent I is recoverable from G_I. This question is answered in Theorem 3.2 and its corollary. There then follows a study of normal closures in G_R of root elements $x_r(t)$ which correspond to transvections considered by Klingenberg [**7**; **8**], and we obtain analogues for G_R of a result of Klingenberg for $GL(n, R)$. As in [**6**] it is assumed that 2 and 3 are not zero divisors (or 0) in R.

In [**7**; **8**; **9**], Klingenberg and Mennicke have shown that if R is the ring of integers or a local ring, and $n \geqq 3$, then any normal subgroup N of $GL(n, R)$ satisfies $K_I \subseteq N \subseteq Z_I$ for I an ideal of R. Here $K_I = SL(n, R) \cap \operatorname{Ker} f_I$ where f_I is the natural map of $GL(n, R)$ onto $GL(n, R/I)$, and $Z_I = f_I^{-1}(C_I)$ where C_I is the center of $GL(n, R/I)$. Apart from some exceptions in low dimension, Abe [**1**] obtained this result for the Chevalley group $G(L, R)$ corresponding to a simple, simply connected Chevalley-Demazure group scheme over a local ring R of characteristic 0 or a prime $p \neq (l, l)/(s, s)$ where l is a long root and s is a short root. He did this by first showing that $G(L, R) = G$, the elementary subgroup considered in [**6**]. Thus G is a natural group in which to study congruence subgroups (in the sense of [**1**]). The negative solution of the congruence subgroup problem by Bass-Milnor-Serre and Matsumoto [**2**; **9**] warns us in advance that the results of Klingenberg and Abe should not generalize to commutative rings R with identity. Recently however, Wilson [**15**] has obtained similar (but of course weaker) sandwiching results for $GL(n, R)$ for such general rings R if $n \geqq 4$. He has, in fact, shown that if N is normal in $GL(n, R)$, then $P_I \subseteq N \subseteq Z_I$, where P_I is the normal closure of the elementary subgroup E_I in $GL(n, R)$, I an ideal of R.

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In the present paper, we identify the groups G_I of [**6**] with groups considered by Abe, and obtain analogues for G of Proposition 2 of [**7**], one of the two key results Klingenberg needed to obtain his sandwiching results mentioned above. These results relate the normal closures of root elements in G_R to ideals in the Chevalley algebra L_R corresponding to the principal ideals generated by the coefficients t in R.

2. Chevalley algebras and groups; elementary subgroups. For a detailed discussion of the construction of Chevalley groups over fields see [**4**; **13**]. Details regarding the construction of Chevalley algebras over rings can be found in [**5**], and in [**1**; **6**; **12**] can be found fairly complete discussions setting forth the constructions of the elementary subgroups of Chevalley groups over rings. For notational convenience we set down here an outline.

Let L be a finite dimensional simple Lie algebra over the complex field, H an n-dimensional Cartan subalgebra, with S the set of nonzero roots of L relative to H, ordered consistently with heights, and P the positive roots. Let $B = \{e_r | r \in S\} \cup \{h_1, h_2, \ldots, h_n\}$ be a Chevalley basis of L, and $L_{\mathbf{Z}}$ the free abelian group on B. Then $L_R = R \otimes_{\mathbf{Z}} L_{\mathbf{Z}}$ is the *(adjoint) Chevalley algebra* of L over R. The *elementary subgroup* G of the (adjoint) Chevalley group of L over R is the group generated by all $x_r(t) = \exp(\mathrm{ad}\, te_r)$ for $t \in R$ and $r \in S$. We use U_R and V_R to represent the subgroups generated by those $x_r(t)$ where r is in P or, respectively, $-r \in P$.

The principal result of [**6**] is the following. Let $I \not\subseteq H_R$ $(= R \otimes_{\mathbf{Z}} H_{\mathbf{Z}}$, where $H_{\mathbf{Z}}$ is the free abelian group on $\{h_1, h_2, \ldots, h_n\})$ be an ideal of L_R. Let E_I be the subgroup of G generated by all $x_r(t)$ for which $te_r \in I$. We call E_I the *elementary subgroup corresponding to* I. We use U_I to represent the subgroup of G generated by all $x_r(t)$ for which $r \in P$ and $te_r \in I$. Let C_i be the inner automorphism of G given by conjugation by $x_{-r}(u_i)$ for i odd and given by conjugation by $x_r(u_i)$ for i even, where $u_i \in R$. Then G_I is the subgroup of G generated by E_I and all iterated conjugates $C_k \circ C_{k-1} \circ \ldots \circ C_1(x_r(t))$ as r runs over S and $t \in R$ satisfies $te_r \in I$. Then G_I is normal in G. We call G_I *the normal subgroup in* G *corresponding to* I.

3. Minimality of G_I as a normal subgroup containing U_I. A natural question is whether E_I needs to be enlarged as much as it was in constructing G_I, in order to obtain a normal subgroup of G. This question is answered affirmatively by our first result. Before stating it, we introduce for t a unit in R the "Weyl element" $\omega_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t)$. For $\omega_r(t)$ we have the corresponding Weyl reflection w_r, where $\omega_r(1)x_s(y)\omega_r(1)^{-1} = x_{w_r(s)}(\pm y)$.

3.1 THEOREM. *G_I is the normal closure of U_I in G.*

Proof. Let N_I be the normal closure of U_I in G. Then $N_I \subseteq G_I$ since G_I is normal in G and $U_I \subseteq G_I$. Moreover, N_I is the subgroup of G generated by all conjugates of the generators $x_r(t)$ $(r \in P,\ te_r \in I)$ of U_I by elements of G

[**11**, p. 53]. Among these elements are all the $x_r(t)$ themselves with $r \in P$ and also those $x_r(t)$ with r negative since $\omega_r(1)x_r(t)\omega_r(1)^{-1} = x_{-r}(\pm t)$. Also among these elements are all conjugates of elements $x_r(t)$ by elements of the form $x_{\pm r}(u_k) \ldots x_{-r}(u_3)x_r(u_2)x_{-r}(u_1)$, the first factor having a plus sign associated with r if and only if k is even. The conjugates by these latter elements are themselves generators of G_I denoted by $C_k \circ C_{k-1} \circ \ldots \circ C_1(x_r(t))$. Thus among the generators for N_I as a group are all those for G_I. Thus $G_I \subseteq N_I$, so $G_I = N_I$ as desired. (This result has also been obtained by R. Swan [**14**, Theorem 4.2].)

We note in passing that Theorem 3.1 (or [**6**] itself) shows that G_I is also the normal closure of E_I in G. We make use of this in § 4.

The question we now consider concerns the relationship between I and G_I. Given I, G_I is uniquely defined, but the correspondence is not bijective, for notice that in defining G_I, we are concerned only with $I \cap E_R$, which (see [**5**]) may coincide with $I' \cap E_R$ even when $I \neq I'$. Our next result says that $I \cap E_R$ is uniquely determined by G_I in all but one case.

3.2 THEOREM. *For all non-symplectic algebras L, $G_I = G_{I'}$ if and only if $I \cap E_R = I' \cap E_R$.*

Proof. As we just observed, if $I \cap E_R = I' \cap E_R$, then clearly $G_I = G_{I'}$. For the converse we distinguish the cases in which L has one and two root lengths, and we make use of the following rules giving the action of an element $x_r(t)$ on L_R.

(1) $x_r(t)e_r = e_r$

(2) $x_r(t)e_{-r} = e_{-r} + th_r - t^2e_r$

(3) $x_r(t)h_r = h_r - 2te_r$

(4) $x_r(t)h_{r_i} = h_{r_i} - tc(r, r_i)e_r$

(5) $x_r(t)e_s = e_s + \sum_{i=1}^{q} \pm \binom{p+i}{i} t^i e_{s+ir}$

(Here $s - pr$ and $s + qr$ are the extremes of the r-string of roots through s.) In the single root length cases, these formulas show that if $I \cap E_R = JE_R$ (3.4 of [**5**]), then any generating element $x_r(t)$ for G_I acts as the identity on L_R/I. If $I \cap E_R = JE_R$ differs from $I' \cap E_R = J'E_R$, then we can find, say $k \in J$, such that $k \notin J'$, so $ke_r \in I$, $ke_r \notin I'$. Then clearly $x_r(k) \in G_I$, but if we choose s so that $r + s$ is a root, (5) then gives $x_r(k)e_s = e_s \pm ke_{s+r} \not\equiv e_s$ modulo I'. From this we conclude $x_r(k) \notin G_{I'}$, since if it were a product of conjugates of generating elements $x_u(t)$ for $G_{I'}$, each of which acts as the identity on L_R/I', then $x_r(k)$ would also act as the identity on L_R/I'. Thus $G_I \neq G_{I'}$.

In case L is of type B_n ($n \geqq 3$), F_4, or G_2, then $I \cap E_R = JE_L \oplus J_1E_S$ where $J \subseteq J_1 \subseteq m^{-1}J$ by 3.5 of [**5**]. Here m is the ratio of the squares of lengths of long and short roots. Suppose first that $I \cap E_S = J_1E_S = I' \cap E_S$

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but $I \cap E_L = JE_L$ differs from $I' \cap E_L = J'E_L$. Then we claim that any generating element $x_r(k)$, r long, for G_I acts as the identity on L_R/I. For if r is a long root and s is arbitrary, then $x_r(k)e_s = e_s$ or $e_s \pm ke_{r+s} \equiv e_s \bmod I$. Next, if r is short and s is long, then $x_r(k)e_s = e_s \pm ke_{r+s} \pm k^2e_{2r+s}$ ($e_s \pm ke_{s+r} \pm k^2e_{s+2r} \pm k^3e_{s+3r}$ in type G_2). Thus $x_r(k)e_s \equiv e_s \pm k^2e_{2r+s}$ ($e_s \pm k^3e_{s+3r}$ in type G_2) modulo I. Finally, if r and s are short, then in type B_n if $r + s$ is a root, it must be long. In this case or type F_4 when $r + s$ is long, we have $x_r(k)e_s = e_s \pm 2ke_{r+s}$. In type F_4 when $r + s$ is short, we have $x_r(k)e_s = e_s \pm ke_{r+s}$. In type G_2, $x_r(k)e_s = e_s \pm 2ke_{r+s} \pm 3k^2e_{2r+s}$ or $e_s \pm 3ke_{r+s}$ if $r + s$ is a root (2.10 of [**5**]). So in all cases if $te_r \in I$, then $x_r(t)e_s \equiv e_s \bmod I$ since $mJ \subseteq J \subseteq J_1$ (and in type G_2, $2J \subseteq J_1$). Now if r is long and we choose say $k \in J$, $k \notin J'$, then we claim $x_r(k) \notin G_{I'}$. For we can find a long root s such that $r + s$ is long, so that $x_r(k)e_s = e_s \pm ke_{r+s} \not\equiv e_s \bmod I'$. Then as before $x_r(k)$ can't be a product of conjugates of generators for $G_{I'}$ since all such conjugates would either fix $e_s \bmod I'$ or would send e_s to $e_s \pm \sum c_ie_{s+mv(i)} \bmod I'$, where $s + mv(i)$ is a long root. In no event then could a product of such elements send e_s to $e_s \pm ke_{r+s} \bmod I'$. Thus $x_r(k) \notin G_{I'}$. Since $x_r(k) \in G_I$, $G_I \neq G_{I'}$.

In case L is of type B_n, $n \geqq 3$, F_4, or G_2 and $I \cap E_S = J_1E_S \neq J_1'E_S = I' \cap E_S$, pick $k \in J_1$ such that $k \notin J_1'$ say. Given any short root r and long root s such that r and s form a system of type B_2 (respectively, G_2) we have $x_r(k)e_s = e_s \pm ke_{s+r} \pm k^2e_{s+2r} \equiv e_s \pm k^2e_{s+2r} \bmod I$ (respectively, $= e_s \pm ke_{s+r} \pm k^2e_{s+2r} \pm k^3e_{s+3r} \equiv e_s \pm k^3e_{s+3r} \bmod I$). Then we claim $x_r(k) \notin G_{I'}$. For if $x_r(k)$ were a product of conjugates of generating elements for $G_{I'}$, then those involving long root generators $x_u(t)$ would fix $e_s \bmod I'$ (since the long roots form a system of type D_n or A_2) and those involving short root generators $x_v(t)$ would map e_s to $e_s \pm \sum c_ie_{s+mw(i)}$ where $s + mw(i)$ is a long root. In no event then would a product of such elements send $e_s + I'$ to $e_s \pm ke_{r+s} \pm k^2e_{s+2r} + I'$ (respectively, $e_s \pm ke_{r+s} \pm k^2e_{s+2r} \pm k^3e_{s+3r} + I'$). But the latter is precisely the action of $x_r(k)$ on $e_s + I'$. So $x_r(k) \notin G_{I'}$ and $G_I \neq G_{I'}$ then. This completes the proof.

Turning now to case C_n, $n \geqq 3$, we have $I \cap E_S = JE_S$. If $I \cap E_R$ and $I' \cap E_R$ differ in their intersections with E_S, then use of 2.5 and 2.6 of [**5**] in conjunction with (5) above gives $G_I \neq G_{I'}$. If $I \cap E_R$ and $I' \cap E_R$ differ in their intersections with $E_L + H_R$ (cf. 5.4 of [**5**]), then the result of the theorem may fail. Consider for example the ring $R = \mathbf{Z}$ of integers. Suppose $I = \langle 2 \rangle L_R$. Let I' be the ideal generated by $\langle 2 \rangle E_S$, H_R, and the element $e = \sum_{l \text{ long}} e_l$. Then I' is an ideal in L_R, $I' \neq I$, but $\{t \in R | te_r \in I\} = \{t \in R | te_r \in I'\} = \langle 2 \rangle$, and so $G_I = G_{I'}$ even though I and I' differ in their intersections with E_R. (Note $e \in I' \cap E_R$, $e \notin I \cap E_R$.)

If L is not symplectic, then let $\mathscr{I}(L_R)$ be the set of all ideals of L_R. We introduce the equivalence relation $\sim$ on $\mathscr{I}(L_R)$ by $I \sim I'$ if and only if $I \cap E_R = I' \cap E_R$. Theorem 3.2 now takes the following simple form.

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3.3 COROLLARY. *For non-symplectic algebras L, the set of normal sub-groups G_I is in one-to-one correspondence with the set $\mathscr{I}(L_R)/\sim$ of equivalence classes of ideals of $\mathscr{I}(L_R)$.*

We close this section with the remark that the groups G_I occur in [**1**] with the notation $E(R, J)$, J an ideal of R. For the restrictions on R in [**1**] (described in § 1 above) assure that save for type A_n the only ideals in L_R have the form $JL_R = L_J$ (by 3.3 of [**5**]). Also G_I is an analogue of the group P_J of [**14**], and an exact analogue if m and the determinant of the Cartan matrix are invertible in R.

4. Normal closures of root elements. In [**7**; **8**] a key fact used to obtain the sandwiching relation quoted in § 1 above is Proposition 2 of [**7**] (Satz 3 of [**8**]). This is as follows. The order of an element $\sigma \in GL(n, R)$ is defined as the smallest ideal J of R such that $f_J(\sigma) \in C_J$. Then a transvection τ of order J has normal closure K_I, provided in dimension 2 that R/I does not have characteristic 2. What corresponds to a transvection in the present setting? A natural candidate is a root element $x_r(t)$ [**6**, pp. 1067–1068]. Corresponding to the order of a transvection as just defined we have the ideal $J = \langle t \rangle$ of R. In this section we show that a result analogous to that just described for $GL(n, R)$ holds for G in several cases, namely, $x_r(t)$ has normal closure G_I where I is an ideal of the Chevalley algebra L_R which arises in a natural way from the ideal $J = \langle t \rangle$ in R. The theorems of this section make these remarks precise.

4.1 THEOREM. *If L has rank at least two and a single root length, then the normal closure N of $x_r(k)$ is G_I, $I = JL_R$, $J = \langle k \rangle$.*

Proof. Choose a root s so that $r + s$ is a root. Then $(x_s(1), x_r(k)) = x_{r+s}(\pm k) \in N$ [**13**, p. 24]. Then $(x_{r+s}(\pm k), x_{-r}(1)) = x_s(\pm k) \in N$. Now given any root $u \neq r$, find a sequence $s_0 = r, s_1, \ldots, s_m = u$ of roots such that $s_{i+1} - s_i$ is a root for $0 \leqq i \leqq m - 1$ [**5**, 4.1]. Using the successive s_i in place of s just considered, we obtain $x_u(\pm k) \in N$. For any $y \in R$, $(x_s(y), x_u(\pm k)) = x_{s+u}(\pm ky)$, so $x_u(ky) \in N$ for any $y \in R$. Thus every generating element of the elementary subgroup E_I belongs to N. Hence the normal closure G_I of E_I is included in N. But G_I is a normal subgroup of G_R which contains $x_r(k)$, so $G_I \supseteq N$. Thus $N = G_I$ as desired.

4.2 THEOREM. *Suppose L is of type B_n, $n \geqq 3$, or F_4. If r is a long root, then the normal closure N of $x_r(k)$ is G_I, $I = JL_R$, $J = \langle k \rangle$. If L is of type G_2, then $G_{2I} \subseteq N \subseteq G_I$.*

Proof. Recall that in type B_n or F_4 the long roots form a system of type D_n [**5**, 2.3] and in type G_2 form a system of type A_2. Then as in 4.1, every $x_u(\pm ky) \in N$ for all long roots u and all $y \in R$. We now distinguish the cases B_n and F_4 from the case G_2. Supposing the first case, let any short root v be given. We want to obtain $x_v(\pm ky) \in N$ for arbitrary $y \in R$. To this end, find a long root u so that u and v form a system of type B_2. From $x_u(\pm k) \in N$ we get $(x_v(y), x_u(\pm k)) = x_{u+v}(\pm ky)x_{u+2v}(\pm ky^2) \in N$ by the Commutator Lemma of [**6**]. For $c_{1,1,u,v} = n_{u,v} = \pm 1$ since $u - v$ is not a root, and

$$c_{1,2,u,v} = \frac{1}{2!} n_{v,u} n_{v,u+v} = \left(\frac{1}{2}\right)(\pm 1)(\pm 2) = \pm 1$$

since $u + v - v$ is a root, but $u + v - 2v$ is not a root. Now since $u + 2v$ is a long root, we know $x_{u+2v}(\pm ky^2) \in N$. Thus $x_{u+v}(\pm ky) \in N$. Let w be an element of the Weyl group such that $v = w(u + v)$. Such a w exists since v and $u + v$ are short [**3**, p. 151, Prop. 11]. Write $w = \prod_{i=1}^{q} w_i$ where w_i is the Weyl reflection corresponding to the simple root r_i [**13**, p. 269, Theorem 16], and let $\omega = \prod_{i=1}^{q} \omega_i(1)$. Then $\omega x_{u+v}(\pm ky)\omega^{-1} = x_v(\pm ky) \in N$. Thus all generators of E_I are in N, so $G_I \subseteq N$. Hence as in 4.1, $G_I = N$.

Supposing next that we are in case G_2, again let v be any short root. Find a long root u so that u and v form a system of type G_2. Then we have $(x_{u+v}(1), x_{-u}(\pm k)) = x_v(\pm k)x_{u+2v}(\pm k)x_{2u+3v}(\pm k)x_{u+3v}(qk^2)$ where $q = \pm 1$ or ± 2, since

$$c_{1,1,u+v,-u} = n_{u+v,-u} = \pm 1,$$

$$c_{2,1,u+v,-u} = \frac{1}{2!} n_{u+v,-u} n_{u+v,v} = \pm 1,$$

$$c_{3,1,u+v,-u} = \frac{1}{3!} n_{u+v,-u} n_{u+v,v} n_{u+v,u+2v} = \pm 1,$$

$$c_{3,2,u+v,-u} = \pm q c_{3,1,u+v,-u} n_{-u,2u+3v} = q.$$

Since $u + 3v$ and $2u + 3v$ are long roots, $x_{2u+3v}(\pm k) \in N$ and $x_{u+3v}(qk^2) \in N$ by the beginning of the proof. So our calculation yields $x_v(\pm k)x_{u+2v}(\pm k) \in N$. Now

$$\begin{aligned}(x_{u+v}(y),\ & x_v(\pm k)x_{u+2v}(\pm k) \\ &= (x_{u+v}(y), x_v(\pm k))x_v(\pm k)(x_{u+v}(y), x_{u+2v}(\pm k))x_v(\pm k)^{-1} \\ &= x_{u+2v}(\pm 2ky)x_{u+3v}(\pm 3k^2y)x_{2u+3v}(\pm 3ky^2) \\ &\qquad \times x_v(\pm k)x_{2u+3v}(\pm 3ky)x_v(\pm k)^{-1}\end{aligned}$$

since

$$c_{1,1,u+v,v} = n_{u+v,v} = \pm 2,$$

$$c_{1,2,u+v,v} = \frac{1}{2!} n_{v,u+v} n_{v,u+2v} = \pm 3,$$

$$c_{2,1,u+v,v} = \frac{1}{2!} n_{u+v,v} n_{u+v,u+2v} = \pm 3,$$

$$c_{1,1,u+v,u+2v} = n_{u+v,u+2v} = \pm 3.$$

Since $x_{2u+3v}(\pm 3ky^2)$ commutes with $x_v(\pm k)$ and since $x_{u+3v}(\pm 3k^2y)$,

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$x_{2u+3v}(\pm 3ky)$, and $x_{2u+3v}(\pm 3ky^2)$ are in N by the beginning of the proof, we thus obtain $x_{u+2v}(\pm 2ky) \in N$. Then as in the B_n and F_4 cases we can use the fact that there is an element w in the Weyl group such that $w(u + 2v) = v$ to produce an element $\omega \in G$ such that $\omega x_{u+2v}(\pm 2ky)\omega^{-1} = x_v(\pm 2ky) \in N$. Thus we have all the generators of E_{2I} in N, so $G_{2I} \subseteq N$. As $N \subseteq G_I$ is clear, the proof is complete.

4.3 THEOREM. *If r is a short root and L is of type B_n, $n \geqq 2$, C_n, $n \geqq 3$, or F_4, then the normal closure N of $x_r(k)$ in G satisfies $G_{2I} \subseteq N \subseteq G_I$, where $I = JL_R$, $J = \langle k \rangle$.*

Proof. Find a long root s so that r and s form a system of type B_2. Then $(x_{s+r}(y), x_r(k)) = x_{s+2r}(\pm 2ky) \in N$, for the only positive integers i and j such that $i(s + r) + jr$ is a root are $i = j = 1$, and $c_{1,1,s+r,r} = n_{s+r,r} = \pm 2$ ($s + r - 2r$ is not a root). As in 4.2 we can now obtain $x_u(\pm 2ky) \in N$ for any long root u and $y \in R$. Also $(x_{-r}(1), x_{s+2r}(\pm 2ky)) = x_{s+r}(\pm 2ky)x_s(\pm 2ky) \in N$ since $s + 3r$ is not a root and

$$c_{2,1,-r,s+r} = \frac{1}{2!} n_{-r,s+2r} n_{-r,s+r} = \pm 1.$$

Since s is a long root, $x_s(\pm 2ky) \in N$, hence $x_v(\pm 2ky) \in N$ for all short roots v. Thus $G_{2I} \subseteq N$. On the other hand, G_I is a normal subgroup of G containing $x_r(k)$, so $G_I \supseteq N$. This completes the proof.

In the case of C_n, $n \geqq 3$, we are in a position to describe N more exactly.

4.4 THEOREM. *Let L be of type C_n, $n \geqq 3$. If r is a short root, then the normal closure of $x_r(k)$ is G_I, where $I = JE_S \oplus J'E_L \oplus (J'H_L + JH_S)$, $J = \langle k \rangle$, $J' = \langle 2k \rangle$.*

Proof. Note by 3.6 of [**5**] I is the smallest ideal of L_R such that $I \cap E_S = JE_S$. Next note that if we consider just the short roots of L, then we have a system of type D_n. The reasoning of 4.1 thus shows that for any short root v, $x_v(\pm ky) \in N$. Given a long root u, find a short root v so that u and v form a system of type $B_2 = C_2$. We have $x_v(\pm ky)$ in N and then as in 4.3, $x_u(\pm 2ky) \in N$. With all the generators of E_I in N, we once again come to the desired conclusion.

If r is a long root of L of type C_n, $n \geqq 3$ or a short root of L of type G_2, then the normal closure N of $x_r(k)$ is certainly contained in G_I for $I = JL_R$, $J = \langle k \rangle$. But attempts to obtain a lower bound for N in the manner of 4.3 are obstructed by the nature of the root systems. An explicit description of N in these cases (as well as in those of 4.3) would be desirable.

5. Normal closures of products of root elements. In [**6**] it was remarked that if $R = \mathbf{Z}$ and N is any normal subgroup of G, then $N \subseteq G_I$ where I is the ideal in $L_{\mathbf{Z}}$ generated by all $d_r e_r$ where d_r is the g.c.d. of

$$\{n \in \mathbf{Z} | x_r(n) \text{ is a factor of some element of } N\}.$$

(Here we write each element of N in one way as a reduced product of generators $x_r(n)$, i.e. any abutting terms $x_r(-n)x_r(n)$ are cancelled.) In fact for any ring R, if J is the ideal generated by all t such that $x_r(t)$ is a factor of some element of N, and $I = JL_R$, then we easily see that $N \subseteq G_I$. If $\bar{I}$ is an ideal of L_R maximal relative to the property that $G_{\bar{I}} \subseteq N$, then a natural question is how $\bar{I}$ and I are related and whether the sandwich relation $G_{\bar{I}} \subseteq N \subseteq G_I$ can be refined to obtain a result analogous to that of Wilson mentioned in § 1 above. First we remark, in the single root length case at least, if for each $\prod_{i=1}^m x_{r_i}(t_i) \in N$, we have $x_{r_i}(t_i) \in N$ for each i, then $N \supseteq G_I$ since for any $y \in R$ we obtain $x_{r_i}(t_i y) \in N$ as in the proof of 4.1. Thus in this circumstance the only normal subgroups of G would be of the form G_I, I an ideal of L_R. In this direction we have the following result for the case $m = 2$.

5.1 THEOREM. *Let L have a single root length and rank at least two. Then the normal closure N of $x_r(t_1)x_s(t_2)$, $r \neq s$, is G_I, $I = JL_R$, $J = \langle t_1, t_2 \rangle$.*

Proof. Find $q \neq -r$ so that $r + q$ is a root, but $s + q$ is not a root. (One verifies easily that this is always possible for L of type A_2 or A_3, so for all L considered here.) Then

$$\begin{aligned}(x_q(y), x_r(t_1)x_s(t_2)) &= (x_q(y), x_r(t_1))x_r(t_1)(x_q(y), x_s(t_2))x_r(t_1)^{-1} \\ &= x_{q+r}(\pm t_1 y) \in N.\end{aligned}$$

Also $x_r(t_1)^{-1}x_r(t_1)x_s(t_2)x_r(t_1) = x_s(t_2)x_r(t_1) \in N$. Now find $q' \neq -s$ so that $r + q'$ is not a root but $s + q'$ is. We have

$$\begin{aligned}(x_{q'}(y), x_s(t_2)x_r(t_1)) &= (x_{q'}(y), x_s(t_2))x_s(t_2)(x_{q'}(y), x_r(t_1))x_s(t_2)^{-1} \\ &= x_{q'+s}(\pm t_2 y) \in N.\end{aligned}$$

Now using the reasoning of 4.1, we get all $x_u(\pm t_1 y)$ and $x_u(\pm t_2 y)$ in N. Thus $G_I \subseteq N$ and so $G_I = N$ as desired.

A similar result holds for the normal closure of a product $x_{r_1}(t_1)x_{r_2}(t_2)x_{r_3}(t_3)$, but it is more complicated in this case to break off single factors as we have done in 5.1. Then is the normal closure of $\prod_{i=1}^q x_{r_i}(t_i) = G_I$ where $I = JL_R$, $J = \langle t_1, t_2, \ldots, t_q \rangle$? The method of 5.1 fails to generalize without some efficient tool for writing factors in a systematic way. In obtaining the result mentioned in § 1, Abe made heavy use (cf. § 3 of [**1**]) of a normal form (2.8 of [**1**]) for writing products in G as uhv, where $u \in U_R$, $v \in V_R$, and $h \in T_R'$, the subgroup of G generated by $h_r(t) = \omega_r(t)\omega_r(1)^{-1}$, t a unit on R. The development of this normal form in turn relied on R being a local ring. Some added hypotheses on R seem to be essential in order to obtain such a normal form. The difficulty is that in the absence of a tool like 2.8 of [**1**], repeated application of the scheme used in

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proving 5.1 yields longer and longer products of generating root elements, even after as much reduction as possible has been effected by conjugation.

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