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STATISTICAL CAUSALITY AND MARTINGALE REPRESENTATION PROPERTY WITH APPLICATION TO STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract

The paper considers a statistical concept of causality in continuous time between filtered probability spaces, based on Granger's definition of causality. This causality concept is connected with the preservation of the martingale representation property when the filtration is getting smaller. We also give conditions, in terms of causality, for every martingale to be a continuous semimartingale, and we consider the equivalence between the concept of causality and the preservation of the martingale representation property under change of measure. In addition, we apply these results to weak solutions of stochastic differential equations. The results can be applied to the economics of securities trading.

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1. Introduction

In this paper we consider a martingale representation. The representation property says that every martingale of an underlying filtration can be written uniquely as a stochastic integral with respect to a local martingale, for a suitable predictable process.

In Section 2, we give some definitions and basic properties of the causality concept (see [4, 12]) and the martingale representation property (see [15]), which will be used later.

The given causality concept is shown to be equivalent to a generalisation of the notion of weak uniqueness for weak solutions of stochastic differential equations (see [11]). In [12], it is shown that the causality concept is closely connected to the extremality of measures and the martingale problem. It is equivalent to orthogonality of local martingales (see [18]) and to stable subspaces of H^p which contain right-continuous modifications of martingales (see [13]).

Some new results are given in Section 3. We prove equivalence between the given concept of causality and the preservation of the martingale representation property

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when $(\mathcal{G}_t)$ is a subfiltration of $(\mathcal{F}_t)$, which is not true in general. Also, we give a generalisation of [1, Proposition 9], concerned with the characterisation of martingales which have the representation property. Further, in this section we give a generalisation of results introduced in [15], where the decomposition of a $(\mathcal{G}_t)$-martingale as a continuous $(\mathcal{F}_t)$-semimartingale is given. An extension of the Girsanov theorem, concerning the representation property for the continuous local martingale X and the Girsanov transform of X, is given in [17]. We give necessary and sufficient conditions, in terms of causality, for the preservation of the representation property under change of measure.

We also investigate a connection between weak solutions of stochastic differential equations and the martingale representation property. We consider an equation of the form

$$\begin{cases} dX_t = u_t(X)\,dZ_t, \\ X_0 = 0 \end{cases} \tag{1.1}$$

(introduced in [5]), where Z is a semimartingale (with $Z_0 = 0$) and $u_t(X)$ is an $(\mathcal{F}_t^{Z,X})$-predictable process. For a weak solution of (1.1), the concept of causality is closely connected with the martingale representation. As the most important example, we consider the stochastic differential equation driven by the process of Brownian motion and its martingale representation.

In the last section we give some examples and implications of our new results in finance (for details, see [2, 7, 16]).

2. Preliminaries and notation

The study of Granger causality has been mostly concerned with time series. But many of the systems to which it is natural to apply tests of causality take place in continuous time, so we will consider continuous-time processes.

A probabilistic model for a time-dependent system is described by $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, where $(\Omega, \mathcal{F}, P)$ is a probability space and $\{\mathcal{F}_t,\ t \in I\}$ is a 'framework' filtration, that is, $(\mathcal{F}_t)$ are all events in the model up to and including time t and $(\mathcal{F}_t)$ is a subset of $\mathcal{F}$. We suppose that the filtration $(\mathcal{F}_t)$ satisfies the usual conditions, which means that $\{\mathcal{F}_t, t \in I\}$ is right continuous and each is such that $(\mathcal{F}_t)$ is complete.

An analogous notation will be used for filtrations $\mathbf{H} = \{\mathcal{H}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$ and $\mathbf{F} = \{\mathcal{F}_t\}$.

A family of σ-algebras induced by a stochastic process $X = \{X_t, t \in I\}$ is given by $\mathbf{F}^X = \{\mathcal{F}_t^X, t \in I\}$, where $(\mathcal{F}_t^X) = \sigma\{X_u, u \in I, u \le t\}$, being the smallest σ-algebra with respect to which the random variables X_u, $u \le t$, are measurable. The process X_t is $(\mathcal{F}_t)$-adapted if $(\mathcal{F}_t^X) \subseteq (\mathcal{F}_t)$ for each t.

The intuitively plausible notion of causality is given in [3] and generalised in [10] for families of Hilbert spaces. We now consider causality between arbitrary filtrations $\mathbf{H}$, $\mathbf{G}$ and $\mathbf{F}$. We can say that '$\mathbf{G}$ causes $\mathbf{H}$ within $\mathbf{F}$' if

$$\mathcal{H}_\infty \perp \mathcal{F}_t | \mathcal{G}_t \tag{2.1}$$

because the essence of (2.1) is that all information about $(\mathcal{H}_\infty)$ that gives $(\mathcal{F}_t)$ comes via $(\mathcal{G}_t)$ for arbitrary t; equivalently, $(\mathcal{G}_t)$ contains all information from $(\mathcal{F}_t)$ needed for predicting $(\mathcal{H}_\infty)$. Thus, it is natural to introduce the following definition of causality between filtrations.

Definition 2.1 (compare with [10]). We say that $\mathbf{G}$ causes $\mathbf{H}$ within $\mathbf{F}$ relative to P (and we write $\mathbf{H} \models \mathbf{G}; \mathbf{F}; P$) if $(\mathcal{H}_\infty) \subseteq (\mathcal{F}_\infty)$ and $\mathbf{G} \subseteq \mathbf{F}$, and if $(\mathcal{H}_\infty)$ is conditionally independent of $(\mathcal{F}_t)$ given $(\mathcal{G}_t)$ for each t. If there is no doubt about P, we omit 'relative to P'.

If $\mathbf{G}$ and $\mathbf{H}$ are such that $\mathbf{G} \models \mathbf{G}; \mathbf{H}$, we shall say that $\mathbf{G}$ is its own cause within $\mathbf{H}$ (compare with [9]).

This definition can be applied to a stochastic process: it will be said that stochastic processes are in a certain relationship if and only if the corresponding induced filtrations are in this relationship. For example, an $(\mathcal{F}_t)$-adapted stochastic process X_t is its own cause if $\mathbf{F}^X = (\mathcal{F}_t^X)$ is its own cause within $\mathbf{F} = (\mathcal{F}_t)$, that is, if

$$\mathbf{F}^X \models \mathbf{F}^X; \mathbf{F}; P.$$

The process X which is its own cause is completely described by its behaviour relative to $\mathbf{F}^X$.

Proposition 2.2 [12]. *Brownian motion $W = (W_t, t \in I)$ on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is its own cause within $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$ relative to probability P.*

The assertion $\mathbf{G} \models \mathbf{G}; \mathbf{F}; P$ implies that $\mathcal{G}_t = \mathcal{F}_t \cap \mathcal{G}_\infty$ for every $t \geq 0$. Also, $(\mathcal{G}_t)$ is a filtration generated by continuous martingales of the form $M_t = P(A \mid \mathcal{F}_t), A \in \mathcal{G}_\infty$ (see [1]). Let us mention that $E(Y \mid \mathcal{F}_t)$ admits a right-continuous progressively measurable version for all $Y \in L_1(P)$ (see [17]).

It is easy to see that the following result holds.

Proposition 2.3. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space and let $(\mathcal{G}_t)$ be generated by continuous martingales. Then from $\mathbf{G} \models \mathbf{G}; \mathbf{F}; P$ it follows that $E(M_t \mid \mathcal{F}_t) = E(M_t \mid \mathcal{G}_\infty)$.*

The question of martingale representation is surprisingly important in applications such as stochastic control and filtering theory, and it is particularly interesting in finance, for example.

Definition 2.4 [15]. The continuous local martingale X_t has the representation property if for any $(\mathcal{F}_t^X)$-local martingale M_t there is an $(\mathcal{F}_t^X)$-predictable process H such that

$$M_t = M_0 + \int_0^t H_s \, dX_s. \tag{2.2}$$

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Let $\mathcal{H}$ be the set of probability measures on $(\Omega, \mathcal{F})$ such that X is a local martingale. If $P \in \mathcal{H}$, $(\mathcal{F}_t, P)$ is the smallest right-continuous filtration complete for measure P and such that $(\mathcal{F}_t^X) \subset (\mathcal{F}_t)$. The representation property is a property of measure P: any $(\mathcal{F}_t, P)$-local martingale M may be written as $M = H \cdot X$, where H is $(\mathcal{F}_t, P)$-predictable and the stochastic integration is taken with respect to P. Suppose that $\mathcal{K}$ is the subset of $\mathcal{H}$ of those probability measures for which X is a martingale.

Definition 2.5 [15]. A probability measure P of $\mathcal{K}$ (respectively, $\mathcal{H}$) is called extremal if, whenever $P = \alpha P_1 + (1 - \alpha)P_2$ with $0 < \alpha < 1$ and $P_1, P_2 \in \mathcal{K}$ (respectively, $\mathcal{H}$), then $P = P_1 = P_2$.

So, the following theorem holds.

Theorem 2.6 [15]. *The probability measure P is extremal in $\mathcal{K}$ if and only if P has the representation property and $(\mathcal{F}_0, P)$ is trivial.*

This theorem can be extended from $\mathcal{K}$ to $\mathcal{H}$ (see [15, Theorem 4.7]).

We now introduce the generalisation of this notion, introduced in [6, 14]. Suppose that on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ there is defined a semimartingale $Z = \{Z_t \mid t \in I\}$ with characteristics (A, C, ν), relative to some truncation function h, Z^c is a continuous part of Z and $\mu = \mu^Z$ is defined in [6].

Definition 2.7 [6]. We say that a local martingale M has the representation property relative to Z if it has the form

$$M = M_0 + H \cdot Z^c + W * (\mu - \nu), \tag{2.3}$$

where H is a predictable process and the right integral is a stochastic integral with respect to random measure (defined in [6]).

3. Causality and representation property

In this section we study the preservation of the representation property when the filtration is getting smaller.

On a given probability space $(\Omega, \mathcal{F}, P)$, let $(\mathcal{G}_t)$ and $(\mathcal{F}_t)$ be two (different) filtrations, such that, for all t, $(\mathcal{G}_t) \subset (\mathcal{F}_t) \subset (\mathcal{F})$, which satisfies the usual conditions and M is a continuous $(\mathcal{F}_t)$-local martingale adapted to $(\mathcal{G}_t)$.

In this paper we are mainly concerned with the characterisation of martingales which have a martingale representation. If X_t has a representation property with respect to $(\mathcal{F}_t)$, then there is an $(\mathcal{F}_t)$-local martingale M_t which can be represented as

$$M_t = a + \int_0^t H_u \, dX_u,$$

where $a \in \mathbb{R}$ and H is an $(\mathcal{F}_t)$-predictable process. Similarly, if X_t has a representation property with respect to $(\mathcal{G}_t)$, then the $(\mathcal{G}_t)$-local martingale N_t can be represented as

$$N_t = b + \int_0^t K_u \, dX_u,$$

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where $b \in \mathbb{R}$ and K is a $(\mathcal{G}_t)$-predictable process. It is important here to know whether this representation property is relative to filtration $(\mathcal{F}_t)$ or $(\mathcal{G}_t)$. There are plenty of examples where X_t has the representation property with respect to $(\mathcal{G}_t)$, but not with respect to $(\mathcal{F}_t)$. Thus, it is natural to ask about the converse statement: if X_t has the representation property with respect to $(\mathcal{F}_t)$, is it also true with respect to $(\mathcal{G}_t)$? The general answer is negative. The next theorem gives conditions when this statement holds and establishes equivalence between the concept of causality and the representation property.

THEOREM 3.1. *Suppose that an $(\mathcal{F}_t)$-local martingale X_t has a representation property with respect to $(\mathcal{F}_t)$. Then X_t has a representation property with respect to $(\mathcal{G}_t)$ if and only if* $\mathbf{G}$ *is its own cause within* $\mathbf{F}$, *that is,* $\mathbf{G} \Bbbk \mathbf{G}; \mathbf{F}; P$.

PROOF. Suppose that X_t has a representation property with respect to filtrations $(\mathcal{G}_t)$ and $(\mathcal{F}_t)$. By [1, Proposition 9], we have that all $(\mathcal{G}_t)$-local martingales are $(\mathcal{F}_t)$-local martingales. (The martingale property is not preserved when the filtration is getting larger, in general.) Since X_t has a representation property, there is a $(\mathcal{G}_t)$-local martingale N_t of the form

$$N_t = E(N_\infty \mid \mathcal{G}_t) = b + \int_0^t K_u \, dX_u.$$

Then there exists a sequence $\{T_n\}$ of stopping times relative to $\mathbf{G}$ for which $\{N_{t\wedge T_n}\}$ is a sequence of $(\mathcal{G}_t)$-martingales. The sequence $\{T_n\}$ of stopping times relative to $\mathbf{G}$ is a sequence of stopping times relative to $\mathbf{F}$, too. Because X_t has a representation property with respect to $(\mathcal{F}_t)$, $\{N_{t\wedge T_n}\}$ is a sequence of $(\mathcal{F}_t)$-martingales, too. By [11, Theorem 4.1],

$$\mathbf{G} \Bbbk \mathbf{G}; \mathbf{F}; P$$

holds.

Conversely, suppose that the local martingale X_t has the $(\mathcal{F}_t)$ representation property and $\mathbf{G} \Bbbk \mathbf{G}; \mathbf{F}; P$ holds, that is,

$$\forall A \in \mathcal{G}_\infty, \quad P(A \mid \mathcal{G}_t) = P(A \mid \mathcal{F}_t).$$

Then there exists an $(\mathcal{F}_t)$-local martingale N_t of the form

$$N_t = b + \int_0^t K_u \, dX_u,$$

where K_t is an $(\mathcal{F}_t)$-predictable process. For a sequence $\{T_n\}$ of stopping times relative to $\mathbf{F}$, $\{N_{t\wedge T_n}\}$ is a sequence of $(\mathcal{F}_t)$-martingales. From $\mathbf{G} \Bbbk \mathbf{G}; \mathbf{F}; P$, it follows that $\mathcal{G}_t = \mathcal{F}_t \cap \mathcal{G}_\infty$ and $\mathcal{G}_T = \mathcal{F}_T \cap \mathcal{G}_\infty$ for the $(\mathcal{F}_t)$-stopping time T, so, by consequence (c.1) of [1, Theorem 3], $\{T_n\}$ is a sequence of $(\mathcal{G}_t)$-stopping times, too. By [11, Theorem 4.1], $\{N_{t\wedge T_n}\}$ is a sequence of $(\mathcal{G}_t)$-martingales. So, N_t is a $(\mathcal{G}_t)$-local martingale and can be represented as

$$N_t = b + \int K_u \, dX_u. \tag{3.1}$$

Also, by consequence (c.2) of [1, Theorem 3], K_t is a $(\mathcal{G}_t)$-predictable process if and only if it is an $(\mathcal{F}_t)$-predictable process, such that, for all t, K_t is $(\mathcal{G}_\infty)$-measurable and if $\mathbf{G} \ltimes \mathbf{G}; \mathbf{F}; P$ holds. Moreover, if K_t is bounded then it is $(\mathcal{F}_t)$-predictable and a $(\mathcal{G}_t)$-predictable projection cannot be distinguished from its 'predictable' projection on the constant filtration $(\mathcal{G}_\infty)$. So, the assertion is proved, because N_t in representation (3.1) is a $(\mathcal{G}_t)$-local martingale for a $(\mathcal{G}_t)$-predictable process K_t. □

The following equivalence may be useful.

Theorem 3.2. *Suppose that X_t has the representation property with respect to $(\mathcal{F}_t)$. Then every $(\mathcal{G}_t)$-martingale is a continuous $(\mathcal{F}_t)$-semimartingale if and only if $(\mathcal{G}_t)$ is its own cause within $(\mathcal{F}_t)$, that is, $\mathbf{G} \ltimes \mathbf{G}; \mathbf{F}; P$ holds.*

Proof. Let N_t be a $(\mathcal{G}_t)$-martingale and a continuous $(\mathcal{F}_t)$-semimartingale. By assumption, N_t may be written as

$$N_t = c + \int_0^t \varphi_s \, dX_s + A_t,$$

where $c \in \mathbb{R}$, φ is an $(\mathcal{F}_t)$-predictable process and A an $(\mathcal{F}_t)$-adapted, continuous process with finite variation.

Then, as X is continuous, the process $[N, X]_t = \int_0^t \varphi_s \, d\langle X, X\rangle_s$ is continuous and $(\mathcal{G}_t)$-adapted. Thus, $\varphi = d[N, X]/d\langle X, X\rangle$ may be chosen to be $(\mathcal{G}_t)$-predictable. So, $N_t - (c + \int_0^t \varphi_s \, dX_s)$ is a continuous $(\mathcal{G}_t)$-martingale, equal to A_t, a process with finite variation. This implies that $A \equiv 0$. So, X_t has the representation property with respect to $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$. Then, by Theorem 3.1, it follows that $(\mathcal{G}_t)$ is its own cause, that is,

$$\mathbf{G} \ltimes \mathbf{G}; \mathbf{F}; P.$$

Conversely, suppose that X_t has the representation property with respect to $(\mathcal{F}_t)$ and $\mathbf{G} \ltimes \mathbf{G}; \mathbf{F}; P$ holds. Because $(\mathcal{G}_t)$ is its own cause within $(\mathcal{F}_t)$,

$$\forall A \in \mathcal{G}_\infty, \quad P(A \mid \mathcal{G}_t) = P(A \mid \mathcal{F}_t). \tag{3.2}$$

Suppose that N_t is a $(\mathcal{G}_t)$-martingale. Then, by [11, Theorem 4.1], N_t is an $(\mathcal{F}_t)$-martingale. By Theorem 3.1, X_t has the representation property with respect to $(\mathcal{G}_t)$. Then every $(\mathcal{G}_t)$-martingale N_t may be written as

$$N_t = E(N_\infty \mid \mathcal{G}_t) = c + \int_0^t \varphi_u \, dX_u,$$

where $c \in \mathbb{R}$ and φ is predictable with respect to $(\mathcal{G}_t)$. Obviously, N_t may be written as a stochastic integral with respect to X, so it is a continuous $(\mathcal{F}_t)$-martingale. Therefore, it is a continuous $(\mathcal{F}_t)$-semimartingale. □

Let Q be any probability measure on $(\Omega, \mathcal{F})$, absolutely continuous with respect to P. The P-martingale $L_t = dQ/dP$ denotes the Radon–Nikodym derivative. The next theorem gives necessary and sufficient conditions for the preservation of the martingale representation property under change of measure.

THEOREM 3.3. *Suppose that* $(\mathcal{G}_t)$ *and* $(\mathcal{F}_t)$ *are filtrations in the measurable space* $(\Omega, \mathcal{F})$ *and* P *and* Q *are probability measures on* $(\mathcal{F}_t)$ *satisfying* $Q \ll P$ *with* dQ/dP *as* $(\mathcal{G}_\infty)$*-measurable. Let* $\mathbf{G} \not\models \mathbf{G}; \mathbf{F}; P$ *hold; then* X *has a representation property with respect to* $(\mathcal{F}_t, P)$ *if and only if* X *has a representation property with respect to* $(\mathcal{F}_t, Q)$.

PROOF. Suppose that $\mathbf{G} \not\models \mathbf{G}; \mathbf{F}; P$ holds and X_t has the representation property with respect to $(\mathcal{F}_t, P)$. According to Theorem 3.1,

$$M_t = E_P(M_\infty \mid \mathcal{F}_t) = x_0 + \int_0^t H_u \, dP(X_u).$$

Because Q is absolutely continuous relative to P ($Q \ll P$), the Radon–Nikodym derivative $L_\infty = dQ/dP$, which is $(\mathcal{G}_\infty)$-measurable, can be defined. Define L_t as a right-continuous version of $E_P(L_\infty \mid \mathcal{F}_t)$, $t \geq 0$. Then

$$M_t L_t = E_P(M_\infty \mid \mathcal{F}_t) E_P(L_\infty \mid \mathcal{F}_t) = E_P(M_\infty L_\infty \mid \mathcal{F}_t).$$

So, for $L_t \neq 0$,

$$M_t = \frac{1}{L_t} E_P\Big(M_\infty \frac{dQ}{dP} \,\Big|\, \mathcal{F}_t\Big) = E_Q(M_\infty \mid \mathcal{F}_t) = x_0 + \int_0^t H_u \, dQ(X_u).$$

Based on the last assertion, X_t has the representation property relative to $(\mathcal{F}_t, Q)$.

Conversely, suppose that $\mathbf{G} \not\models \mathbf{G}; \mathbf{F}; P$ holds and X_t has the representation property relative to $(\mathcal{F}_t, Q)$. From $\mathbf{G} \not\models \mathbf{G}; \mathbf{F}; P$,

$$\forall A \in \mathcal{G}_\infty, \quad P(A \mid \mathcal{G}_t) = P(A \mid \mathcal{F}_t).$$

Also, X_t has the representation property with respect to $(\mathcal{F}_t, Q)$, or

$$M_t = E_Q(M_\infty \mid \mathcal{F}_t) = x_0 + \int_0^t H_u \, dQ(X_u).$$

Then, for $L_t \neq 0$,

$$M_t = E_Q(M_\infty \mid \mathcal{F}_t) = \frac{1}{L_t} E\Big(M_\infty \frac{dQ}{dP} \,\Big|\, \mathcal{F}_t\Big) = \frac{1}{L_t} E\Big(M_\infty \frac{dQ}{dP} \,\Big|\, \mathcal{G}_t\Big).$$

The last step holds because of supposed causality, where M_∞ and dQ/dP are $(\mathcal{G}_\infty)$-measurable. So,

$$\begin{aligned} M_t &= \frac{1}{L_t} E(L_\infty \mid \mathcal{G}_t) E(M_\infty \mid \mathcal{G}_t) = \frac{1}{L_t} E(L_\infty \mid \mathcal{F}_t) E(M_\infty \mid \mathcal{F}_t) \\ &= E(M_\infty \mid \mathcal{F}_t) = x_0 + \int_0^t H_u \, dP(X_u). \end{aligned}$$

Obviously, X_t has the representation property with respect to $(\mathcal{F}_t, P)$ and the theorem holds. □

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The martingale representation property can also be connected with weak solutions of stochastic differential equations. First, we consider equations driven with a process of Brownian motion, the most important example of the martingale representation property.

Consider the stochastic differential equation

$$\begin{cases} dX_t = a_t(X)dt + b_t(X)\,dW_t, \\ X_0 = \eta, \end{cases} \tag{3.3}$$

where W is a Brownian motion and a_t, b_t are causal functionals.

Proposition 3.4. *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, W_t)$ be a weak solution of (3.3). Then a process X_t has a representation property if and only if X_t is its own cause, that is, $\mathbf{F}^X \Bbbk \mathbf{F}^X; \mathbf{F}; P$.*

Proof. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, W_t)$ be a weak solution of (3.3). If X_t has a representation property, then there exists an $(\mathcal{F}_t^X)$-martingale M_t, which can be represented as

$$M_t = \int_0^t H_s\,dX_s + M_0, \tag{3.4}$$

where H_s is an $(\mathcal{F}_t^X)$-predictable process. Then the induced measure P of the process X_t has a representation property. So, by Theorem 2.6, the measure P defined on $(\mathcal{F}_\infty^X)$ is extremal in the set M_X (the set of all probability measures on $(\Omega, \mathcal{F})$ such that the process X_t is a local martingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$). According to [12, Theorem 4.4], the weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, W_t)$ of (3.3) is weakly unique and, by [9, Proposition 4.1], X_t is its own cause, that is, $\mathbf{F}^X \Bbbk \mathbf{F}^X; \mathbf{F}; P$ holds.

Conversely, suppose that X_t is its own cause, that is, $\mathbf{F}^X \Bbbk \mathbf{F}^X; \mathbf{F}; P$ holds for (3.3). The weak solution of (3.3) is of the form $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, W_t)$ and, since X_t is its own cause, by [9, Proposition 4.1] it follows that this solution is weakly unique. So, by [12, Theorem 4.4], the measure P on $(\mathcal{F}_\infty^X)$ is extremal among all measures for which X_t is a solution of (3.3). By Theorem 2.6, the measure P has the representation property on $(\mathcal{F}_\infty^X)$ or the solution process X_t has the representation property and may be written $M = H \cdot X$, where H is an $(\mathcal{F}_t^X, P)$-predictable process and the stochastic integral is taken with respect to P, or

$$M_t = \int_0^t H_s\,dX_s + M_0,$$

where $M_0 = 0$. So, by Definition 2.4, X_t has the representation property. □

Probabilists have long been interested in problems of extremality of measure. The relation between extremality and martingale representation was discovered by Dellacherie (see [8]) and was taken further and given its definite form by Jacod and Yor (see [8]).

Now we consider a more general equation, a stochastic differential equation driven with semimartingales (see [5]),

$$\begin{cases} dX_t = u_t(X)\,dZ_t, \\ X_0 = 0, \end{cases} \tag{3.5}$$

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where Z is a continuous semimartingale (with $Z_0 = 0$), $u_t(X)$ is an $(\mathcal{F}_t^{Z,X})$-predictable process and X is a solution process. In [5, 9], a definition of weak solution for (3.3) is given.

The driving process Z_t of (3.5) is a semimartingale and the solution process X_t is also a semimartingale (see [5]) relative to the filtration $(\mathcal{F}_t^{Z,X})$. So, the process X_t can be represented as

$$X_t = N_t + B_t, \tag{3.6}$$

where N_t is a local martingale and B_t is the process of finite variation.

Proposition 3.5. *Let* $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t)$ *be a weak solution of (3.5). Then the process* N_t *from the decomposition (3.6) has a representation property relative to the filtration* $(\mathcal{F}_t^{Z,X})$ *if and only if* (Z_t, X_t) *is its own cause, that is,* $\mathbf{F}^{Z,X} \Bumpeq \mathbf{F}^{Z,X}; \mathbf{F}; P$.

Proof. Suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t)$ is a weak solution of (3.5) and the process X_t is a solution process. Also, by assumption, the process N_t has the representation property relative to $(\mathcal{F}_t^{Z,X})$:

$$M_t = M_0 + \int_0^t H_s\, dN_s,$$

where M_t is an $(\mathcal{F}_t^{Z,X})$-local martingale and H_t a suitable $(\mathcal{F}_t^{Z,X})$-predictable process. Then the measure P defined on $(\mathcal{F}_\infty^{Z,X})$, relative to which the stochastic integration is taken, has the representation property, too. By Theorem 2.6, P is an extremal measure on the set of measures which defines the martingales of the form $M_t = P(A \mid \mathcal{F}_t^{Z,X})$. Such a martingale allows a right-continuous modification. Let us denote such a set by $\mathcal{K}$. The filtration generated by the continuous processes M_t is $(\mathcal{F}_t^{Z,X})$ and $(\mathcal{F}_\infty^{Z,X})$ is the minimal filtration on which the solution of (3.5) is defined. Namely, for $M_t \in (\mathcal{F}_t^{Z,X})$,

$$\begin{aligned} E(M_\infty \mid \mathcal{F}_t^{Z,X}) &= E(P(A \mid \mathcal{F}_\infty^{Z,X}) \mid \mathcal{F}_t^{Z,X}) = E(E(1_A \mid \mathcal{F}_\infty^{Z,X}) \mid \mathcal{F}_t^{Z,X}) \\ &= E(1_A \mid \mathcal{F}_\infty^{Z,X} \cap \mathcal{F}_t^{Z,X}) = P(A \mid \mathcal{F}_t^{Z,X}) = M_t, \end{aligned}$$

so M_t is an $(\mathcal{F}_t^{Z,X})$-martingale. According to [12, Theorems 4.3 and 4.4], P is a weakly unique solution of (3.5) and (Z_t, X_t) is its own cause, that is,

$$\mathbf{F}^{Z,X} \Bumpeq \mathbf{F}^{Z,X}; \mathbf{F}; P.$$

Conversely, suppose that $\mathbf{F}^{Z,X} \Bumpeq \mathbf{F}^{Z,X}; \mathbf{F}; P$ holds. Then, by [12, Theorems 4.3 and 4.4], the weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t)$ of (3.5) is weakly unique on $(\mathcal{F}_\infty^{Z,X})$ and the measure P is extremal on every weak solution. From $\mathbf{F}^{Z,X} \Bumpeq \mathbf{F}^{Z,X}; \mathbf{F}; P$, that is,

$$\forall A \in \mathcal{F}_\infty^{Z,X}, \quad P(A \mid \mathcal{F}_t^{Z,X}) = P(A \mid \mathcal{F}_t),$$

and from $\mathcal{F}_t^{Z,X} = \mathcal{F}_t \cap \mathcal{F}_\infty^{Z,X}$, it follows that $(\mathcal{F}_t^{Z,X})$ is generated by processes of the form $M_t = P(A \mid \mathcal{F}_t)$, which are martingales. The measure P is extremal and defined on $(\mathcal{F}_\infty^{Z,X})$, so, according to Theorem, 2.6 P has the representation property on $(\mathcal{F}_\infty^{Z,X})$ or any $(\mathcal{F}_t^{Z,X}, P)$-local martingale M may be written as $M = H \cdot N$, where H is $(\mathcal{F}_t^{Z,X}, P)$-predictable and the stochastic integration is taken with respect to P. So, N_t has the representation property relative to filtration $(\mathcal{F}_t^{Z,X})$. □

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4. Some examples and their application to finance

The martingale representation says that every martingale of the underlying filtration can be written uniquely as a stochastic integral with respect to a local martingale, for a suitable predictable process. This property has found application in, for example, stochastic control, filtering and more recently in economics of securities trading. The interpretation in finance is that the risk-minimising hedging strategy of a contingent claim in an incomplete market has 'smooth' regular sample paths.

In Section 3, we have already seen the most important example of martingale representation, that of Brownian motion. Here, we give some examples which are local martingales with jumps, and which have the representation property.

Example 4.1. We consider Emery's structure equation

$$[X, X]_t - t = \int_0^t H_s \, dX_s \tag{4.1}$$

(see [14]), for which $X_0 = 0$ and the compensator of $[X, X]_t$ is $A_t = t$. Consider (4.1) given in a form resembling a differential equation:

$$d[X, X]_t = dt + \phi(X_{t-}) \, dX_t. \tag{4.2}$$

Since no probability space is specified, the only reasonable interpretation of (4.2) is that of a weak solution. This means that there exist a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, satisfying the usual hypothesis, and the local martingale X which satisfies (4.2). If the solution is weakly unique, which means that if X and Y are solutions of (4.2), then X and Y have the same distributions as the processes. Then, by [12, Theorem 4.3], the process X is its own cause.

From (4.2), if we set $\phi(x) = \alpha + \beta x, \beta = 0$,

$$d[X, X]_t = dt + \alpha \, dX_t \tag{4.3}$$

and, when $\alpha = 0$, the process X is a standard Brownian motion, it is its own cause and, by Proposition 3.4, it has a martingale representation.

For $\alpha = 1$ and $\beta = 0$, (4.2) is

$$[X, X]_t = t + (X_t - X_0) = t + x_t. \tag{4.4}$$

Hence, X is a finite variation martingale, $\Delta X_t = 1$, so X only jumps up, with jumps always of size 1. Let $X_t = N_t - t$ be the compensated Poisson process. Then X satisfies the equation, because of weak uniqueness, it is its own cause and a compensated standard Poisson process, by Proposition 3.4, has a representation property with respect to its natural filtration.

Example 4.2. Consider an economy that puts at our disposal a number of assets $\mathbf{S}_t = (S_t^{(1)}, \ldots, S_t^{(n)})$, so that $S_t^{(i)}$ is the price at time t of the ith asset. Thus, at any time t, an agent of the wealth at time t holds a portfolio $\mathbf{H}_t = (H_t^{(1)}, \ldots, H_t^{(n)})$, where $H_t^{(i)}$ is the number of i-assets held at time t.

The wealth of the portfolio V_t at time t is therefore

$$V_t = K_t B_t + \mathbf{H}_t \cdot \mathbf{S}_t, \tag{4.5}$$

where K_t is the number of the 0-assets held at time t, B_t is the value at time t of one unit of money invested in the deposit account at time zero and $K_t B_t$ is the wealth not invested in $\mathbf{S}$.

But, because any value kept in the deposit account grows at some positive rate, it is more useful to express asset prices in terms of B, writing $V_t^B = V_t/B_t, \mathbf{S}_t^B = \mathbf{S}_t/B_t$. The wealth equation (4.5) then becomes

$$V_t^B = K_t + H_t \cdot \mathbf{S}_t^B. \tag{4.6}$$

The consequence of ϕ being self-financing is then that $dV_t^B = H_t\, dS_t^B$, so

$$V_t^B = V_0 + \int_0^t H_u \cdot d\mathbf{S}_u^B, \tag{4.7}$$

so that the discounted wealth is the integral of the portfolio holdings against the discounted asset price process.

The fundamental theorem of asset pricing, formalised by Harrison and Pliska, states that arbitrage is excluded if and only if there is some equivalent martingale measure under which the discounted asset price processes are martingales. So, by [12, Theorem 4.3], arbitrage is excluded if and only if discounted asset price processes are their own cause within the market. This implies that the price of the contingent claim can be computed as the expectation in the martingale measure of the discounted payoff of that claim. If the market is also complete, so that all claims can be replicated perfectly, then the martingale measure (and hence the market price for any claim) is unique. So, by Proposition 3.5, there exists a martingale representation of the trading strategy $Y = (K_t, H_t)$.

Now, if P^B is a measure under which $\mathbf{S}^B$ is a martingale, and $h = f(\mathbf{S})$ is a contingent claim on $\mathbf{S}^B$, the discounted price at time t of h, π_t^B, being the price of the traded asset, is itself a P^B martingale. It follows that π has a representation as

$$\pi_t = B_t \pi_t^B = B_t E^B(h/B_T \mid \mathcal{F}_t).$$

In the absence of any other condition enforcing a unique price for the claim h, there will be potentially as many prices π for h as there are market agents. If the market is complete, there is a price-enforcing mechanism: the price of h will be the cost V_0^h of setting up a portfolio worth $V_0(\phi) = \pi_0(h)$ at time zero and $V^h(T) = h$ at time T.

In the simple Black–Scholes model, P^B as constructed above is the unique equivalent probability measure with respect to which the price processes at the market assets are martingales.

So, by [12, Theorem 4.3], the claim process and the price process are caused by themselves and, by Proposition 3.5, it follows that there exists a martingale representation of the wealth V_t^β relative to the trading strategy $\phi = (K_t, H_t)$.

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