ON GEODESICS OF A MODIFIED RIEMANNIAN MANIFOLD

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Introduction. In Riemannian geometry the autoparallels associated with the affine connexion coincide with the geodesics which arise from the metric. This is not the case in a modification of Riemannian geometry suggested by Lyra. A sufficient condition that the two classes of curves coincide is obtained.

The differential geometrical structure of a manifold is determined by

(i) an affine connexion characterized by its components $\Gamma^{\mu}_{\alpha\beta}$, which are defined by the infinitesimal parallel transfer of a vector ξ^{μ}. If we let $\delta\xi^{\mu}$ denote the quantity which must be subtracted from the ordinary differential $d\xi^{\mu}$ in order to obtain a tensorial differential, we have

$$(1) \qquad \delta\xi^{\mu} = - \Gamma^{\mu}_{\alpha\beta}\, \xi^{\alpha} dx^{\beta},$$

and (ii) a metrical connexion characterized by the metric fundamental tensor $g_{\mu\lambda}$ which is defined by the measure of length l of a vector ξ^{μ}:

$$(2) \qquad l^2 = g_{\mu\lambda}\, \xi^{\mu} \xi^{\lambda} \quad .$$

Riemannian geometry is characterized by the following assumptions.

$$\text{(a)} \qquad \Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha},$$

$$(3) \quad \text{(b)} \qquad \delta\, l^2 = \delta\,(g_{\mu\lambda}\, \xi^{\mu} \xi^{\lambda}) = 0,$$

$$\text{(c)} \qquad dg_{\mu\lambda} = \delta\, g_{\mu\lambda}.$$

From these it follows that

$$(4) \qquad \Gamma^{\mu}_{\alpha\beta} = \left\{ {\mu \atop \alpha\beta} \right\} ,$$

where the latter quantities are Christoffel symbols of the second kind.

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An autoparallel of an affine connexion is defined by a curve $x^{\mu} = x^{\mu}(s)$ (with s representing arc-length), whose tangential vector $\xi^{\mu} = dx^{\mu}/ds$ is transferred parallel to itself. Its equation is therefore

$$\frac{d^2 x^{\mu}}{ds^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dx^{\alpha}}{ds} \frac{dx^{\beta}}{ds} = 0. \tag{5}$$

A geodesic of a metrical connexion, on the other hand, is defined by the extremal curves of the problem in the calculus of variations:

$$\delta\left(\int ds\right) = \delta\left(\int \sqrt{g_{\mu\lambda} \frac{dx^{\mu}}{dt} \cdot \frac{dx^{\lambda}}{dt}}\, dt\right) = 0, \tag{6}$$

where s is arc-length and t is an arbitrary parameter. This yields

$$\frac{d^2 x^{\mu}}{ds^2} + \left\{ {\mu \atop \alpha\beta} \right\} \frac{dx^{\alpha}}{ds} \cdot \frac{dx^{\beta}}{ds} = 0. \tag{7}$$

In view of (4), the two classes of curves are the same.

A modified Riemannian geometry. Lyra [1] suggested a modification of Riemannian geometry, which may also be considered as a modification of Weyl's geometry [3]. Weyl introduced the concept of non-integrability of length transfer, thereby modifying (3b) to

$$\delta l^2 = -l^2 \phi_{\alpha} dx^{\alpha}.$$

As a result

$$\Gamma^{\mu}_{\alpha\beta} = \left\{ {\mu \atop \alpha\beta} \right\} + \tfrac{1}{2}\left(\delta^{\mu}_{\alpha} \phi_{\beta} + \delta^{\mu}_{\beta} \phi_{\alpha} - g_{\alpha\beta} \phi^{\mu} \right), \tag{8}$$

where

$$\phi^{\mu} = g^{\mu\nu} \phi_{\nu}.$$

A Weyl manifold is therefore characterized not only by $g_{\mu\lambda}$ but also by ϕ_{α}. The non-integrability of length transfer leads to the concept of gauge-transformation

$$l^2 \to \bar{l}^2 = \lambda(x^{\mu}) l^2,$$

under which

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$$(9)\quad \begin{aligned} &(a) \quad g_{\mu\lambda} \longrightarrow \bar{g}_{\mu\lambda} = \lambda g_{\mu\lambda}, \\ &(b) \quad \phi_\alpha \longrightarrow \bar{\phi}_\alpha = \phi_\alpha - \lambda^{-1}\frac{\partial\lambda}{\partial x^\alpha}. \end{aligned}$$

In Weyl's geometry the autoparallels and the geodesics are different.

In Lyra's geometry Weyl's concept of gauge, which is essentially a metrical concept, is modified by introducing a gauge function in the structureless manifold.

According to Lyra, the displacement vector $\overrightarrow{PP'}$ between two neighbouring points $P(x^\mu)$ and $P'(x^\mu + dx^\mu)$, has the components $\xi^\mu = x^0\, dx^\mu$, where $x^0(x^\mu)$ is a gauge function. The coordinate system (x^μ) together with the gauge x^0 form a reference system $(x^0;\ x^\mu)$. The transformation formula for a tensor under the general transformation of reference systems

$$(10)\qquad x^\mu \to x^{\mu'} = x^{\mu'}(x^\lambda)\ ;\ x^0 \to x^{0'} = x^{0'}(x^0,\ x^\mu),$$

with

$$(10')\qquad A^{\mu'}_{\mu} \equiv \frac{\partial x^{\mu'}}{\partial x^\mu}\ ; \qquad A^{\mu}_{\mu'} \equiv \frac{\partial x^\mu}{\partial x^{\mu'}},$$

is then

$$\xi^{\rho'_1 \ldots \rho'_s}_{\sigma'_1 \ldots \sigma'_r} = \lambda^{s-r} A^{\rho'_1}_{\rho_1} \ldots A^{\rho'_s}_{\rho_s} A^{\sigma_1}_{\sigma'_1} \ldots A^{\sigma_r}_{\sigma'_r}\ \xi^{\rho_1 \ldots \rho_s}_{\sigma_1 \ldots \sigma_r}$$

Thus the factor λ^{s-r}, where $\lambda = x^{0'}/x^0$, arises as a consequence of the introduction of the gauge function.

In a Riemannian manifold the components of the affine connexion $\Gamma^\mu_{\alpha\beta}$ can be considered to arise as a consequence of general coordinate transformations in the following manner (cf. [4]). Let us suppose that, in a coordinate system (x^μ), a vector ξ^μ is constant, i.e. $\partial\xi^\mu/\partial x^\lambda = 0$. Then, in another coordinate system $(x^{\mu'})$, we have

$$(11)\qquad \frac{\partial \xi^{\mu'}}{\partial x^{\lambda'}} + \Gamma^{\mu'}_{\nu'\lambda'}\ \xi^{\nu'} = 0,$$

where

$$\Gamma^{\mu'}_{\nu'\lambda'} = -A^{\mu}_{\nu'}\ A^{\mu'}_{\mu,\lambda'}\ ,\ A^{\mu'}_{\mu,\lambda'} \equiv \frac{\partial}{\partial x^{\lambda'}}(A^{\mu'}_{\mu}).$$

Another way of expressing the fact that ξ^μ = constant would be to say that equation (11) is valid in all coordinate systems, but $\Gamma^\mu_{\nu\lambda} = 0$ in the particular system (x^μ). $\Gamma^\mu_{\nu\lambda}$ vanishes also in all other coordinate systems obtained by an affine transformation from this one.

We shall show that a similar analysis of a constant vector in Lyra's geometry leads to the concept of a generalized affine connexion characterized not only by $\Gamma^\mu_{\nu\lambda}$ but also by a function ϕ_α, which arises through gauge transformation.

A vector ξ^μ in Lyra's geometry transforms as

$$\xi^{\mu'} = \lambda A^{\mu'}_{\mu} \xi^\mu .$$

If $\partial\xi^\mu / \partial x^\lambda = 0$ in the reference system $(x^0; x^\mu)$, then, in the reference system $(x^{0'}; x^{\mu'})$, we have

$$\frac{1}{x^{0'}} \frac{\partial \xi^{\mu'}}{\partial x^{\lambda'}} - \frac{1}{x^{0'}} A^{\mu'}_{\mu,\lambda'} A^{\mu}_{\nu'} \xi^{\nu'} - \frac{1}{x^{0'}} \frac{\partial \log \lambda}{\partial x^{\lambda'}} \xi^{\mu'} = 0$$

or

$$\frac{1}{x^{0'}} \frac{\partial \xi^{\mu'}}{\partial x^{\lambda'}} + \Gamma^{\mu'}_{\nu'\lambda'} \xi^{\nu'} - \tfrac{1}{2} \phi_{\lambda'} \xi^{\mu'} = 0, \tag{12}$$

where

$$\Gamma^{\mu'}_{\nu'\lambda'} = \frac{-1}{x^{0'}} A^{\mu'}_{\mu,\lambda'} A^{\mu}_{\nu'} ; \quad \phi_{\lambda'} = \frac{1}{x^{0'}} \frac{\partial \log \lambda^2}{\partial x^{\lambda'}} \tag{12'}$$

Note that $A^{\mu'}_{\mu} A^{\mu}_{\nu'} = \delta^{\mu'}_{\nu'}$, by (10') and hence, by partial differentiation with respect to $x^{\lambda'}$, $A^{\mu'}_{\mu,\lambda'} A^{\mu}_{\nu'} = -A^{\mu'}_{\mu} A^{\mu}_{\nu',\lambda'}$. Accordingly $\Gamma^{\mu'}_{\nu'\lambda'}$ is symmetrical in ν' and λ'.

In analogy to the Riemannian case then the parallel transfer of a vector ξ^μ in Lyra's geometry is given by

$$\delta\xi^\mu = -(\Gamma^\mu_{\alpha\beta} - \tfrac{1}{2}\delta^\mu_\alpha \phi_\beta)\, \xi^\alpha x^0 dx^\beta . \tag{13}$$

The transformation formulae for $\Gamma^\mu_{\alpha\beta}$ and ϕ_α are :

(i) Under coordinate transformation $x^\mu \longrightarrow x^{\mu'}$,

$$\begin{aligned} \Gamma^\mu_{\alpha\beta} &= A^\mu_{\mu'} A^{\alpha'}_{\alpha} A^{\beta'}_{\beta} \Gamma^{\mu'}_{\alpha'\beta'} + \frac{1}{x^0} A^\mu_{\nu'} A^{\nu'}_{\alpha,\beta} , \\ \phi_\alpha &= A^{\alpha'}_{\alpha} \phi_{\alpha'} . \end{aligned} \tag{14}$$

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(ii) Under gauge transformation $x^0 \longrightarrow x^{0'}$,

$$(14') \quad \begin{aligned} \Gamma^{\mu'}_{\alpha'\beta'} &= \lambda^{-1}\Gamma^{\mu}_{\alpha\beta} \quad (\mu' = \mu, \ \alpha' = \alpha, \ \beta' = \beta) \ , \\ \phi_{\alpha'} &= \lambda^{-1}\left(\phi_\alpha + \frac{1}{x^0} \frac{\partial \log \lambda^2}{\partial x^\alpha}\right) \quad (\alpha' = \alpha) \ . \end{aligned}$$

Autoparallels of the modified manifold. An autoparallel of the generalized affine connexion is defined by a curve $x^\tau = x^\tau(s)$, whose tangential vector $\xi^\tau = x^0(dx^\tau/ds)$ is transferred parallel to itself. Its equation is therefore

$$(15) \quad x^0 \frac{d^2x^\tau}{ds^2} + \Gamma^\tau_{\lambda\mu}\frac{dx^\lambda}{ds}\frac{dx^\mu}{ds}(x^0)^2 - \tfrac{1}{2}(\phi_\alpha - \mathring{\phi}_\alpha)\frac{dx^\alpha}{ds}\frac{dx^\tau}{ds}(x^0)^2 = 0$$

where

$$(15') \qquad \mathring{\phi}_\alpha = \frac{1}{x^0}\frac{\partial \log (x^0)^2}{\partial x^\alpha} \ .$$

A metrical connexion can be introduced in Lyra's geometry by means of a symmetric metric tensor $g_{\mu\lambda}$:

$$(16) \qquad ds^2 = g_{\mu\lambda}\,(x^0 dx^\mu)(x^0 dx^\lambda)$$

with the assumption that

$$(17) \qquad \delta\,(g_{\mu\lambda}\,\xi^\mu\,\xi^\lambda) = 0$$

for arbitrary vectors ξ^μ.

Assuming, as usual, that the process δ satisfies the product rule of differentiation and that $\delta g_{ij} = dg_{ij}$, we find, from (13) and (17), that

$$(18) \qquad \Gamma^\mu_{\alpha\beta} = \frac{1}{x^0}\left\{ {\mu \atop \alpha\beta} \right\} + \tfrac{1}{2}\left(\delta^\mu_\alpha\,\phi_\beta + \delta^\mu_\beta\,\phi_\alpha - g_{\alpha\beta}\,\phi^\mu\right),$$

where $\phi^\mu \equiv g^{\mu\lambda}\phi_\lambda$. A geodesic of the metrical connexion is therefore given by a solution of

$$\delta\left(\int ds\right) = \delta\left(\int \sqrt{(x^0)^2\, g_{\mu\lambda}\frac{dx^\mu}{dt}\frac{dx^\lambda}{dt}}\; dt\right) = 0 \ ,$$

i.e. by

$$(19) \qquad \delta\left(\int L dt\right) = 0 \ ,$$

where

$$L = \sqrt{(x^o)^2 \, g_{\mu\lambda} \, \frac{dx^\mu}{dt} \, \frac{dx^\lambda}{dt}} \quad .$$

The Euler-Lagrange equations for the problem (19) are then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\nu}\right) - \frac{\partial L}{\partial x^\nu} = 0 \quad \left(\dot{x}^\nu \equiv \frac{dx^\nu}{dt}\right) .$$

Now

$$\frac{\partial L}{\partial x^\nu} = \left(\frac{ds}{dt}\right)^{-1} \tfrac{1}{2}\, \dot{x}^\mu \, \dot{x}^\lambda \left\{ (x^o)^2 \, g_{\mu\lambda} \right\}_{,\nu}$$

and

$$\frac{\partial L}{\partial \dot{x}^\nu} = \left(\frac{ds}{dt}\right)^{-1} (x^o)^2 \, g_{\mu\nu} \, \dot{x}^\mu .$$

Substituting in the Euler-Lagrange equations, performing the differentiation and putting t=s, we find that the geodesics of the metrical connexion can therefore be written

$$\text{(20)} \quad \begin{aligned} &(x^o)^2 \, g_{\mu\nu} \, \frac{d^2 x^\mu}{ds^2} + (x^o)^2 \, [\nu,\lambda\,\mu] \frac{dx^\lambda}{ds} \, \frac{dx^\mu}{ds} \\ &+ (2 g_{\mu\nu} \, x^o_{,\lambda} - x^o_{,\nu} \, g_{\mu\lambda}) \; x^o \, \frac{dx^\mu}{ds} \, \frac{dx^\lambda}{ds} = 0, \end{aligned}$$

where $[\nu,\lambda\,\mu]$ is the Christoffel symbol of the first kind and $\overset{o}{x}_{,\lambda}$ denotes $\partial x^o/\partial x^\lambda$. Multiplying (20) by $g^{\tau\nu}$ and using (15'), we obtain

$$\text{(21)} \quad \begin{aligned} &\frac{d^2 x^\tau}{ds^2} + \left\{ {\tau \atop \lambda\mu} \right\} \frac{dx^\lambda}{ds} \, \frac{dx^\mu}{ds} \\ &+ \frac{x^o}{2} \left(\delta^\tau_\mu \, \overset{o}{\phi}_\lambda + \delta^\tau_\lambda \, \overset{o}{\phi}_\mu - \overset{o}{\phi}{}^\tau g_{\lambda\mu} \right) \frac{dx^\mu}{ds} \, \frac{dx^\lambda}{ds} = 0 , \end{aligned}$$

where $\overset{o}{\phi}{}^\tau \equiv g^{\tau\lambda} \, \overset{o}{\phi}_\lambda$.

On the other hand, in view of (18) and (15), the equation of an autoparallel of the generalized affine connexion becomes

$$x^0 \frac{d^2x^\tau}{ds^2} + \left\{ \frac{1}{x^0} \left\{ {\tau \atop \mu\lambda} \right\} + \frac{1}{2} (\delta^\tau_\mu \phi_\lambda + \delta^\tau_\lambda \phi_\mu - g_{\mu\lambda} \phi^\tau) \right\} \tag{22}$$
$$\cdot (x^0)^2 \frac{dx^\mu}{ds} \frac{dx^\lambda}{ds} - \frac{1}{2} (\phi_\alpha - \mathring{\phi}_\alpha) (x^0)^2 \frac{dx^\alpha}{ds} \frac{dx^\tau}{ds} = 0 .$$

A comparison of equations (21) and (22) shows that a sufficient condition that the two types of curves be the same is

$$\phi_\alpha = \mathring{\phi}_\alpha .$$

It can easily be seen that the above condition is invariant under gauge transformations because $\mathring{\phi}_\alpha$ transforms exactly as ϕ_α, when $x^0 \to x^{0\prime}$.

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REFERENCES

1. G. Lyra, Über eine Modifikation der riemannschen Geometrie, Math. Z. 54 (1951), 52-64.

2. E. Scheibe, Über einen Verallgemeinerten affinen Zusammenhang, Math. Z. 57(1952), 65-74.

3. H. Weyl, Gravitation und Elektrizität, S.-B. Preuss. Akad. Wiss, Berlin (1918), 465-480.

4. E. Schroedinger, Space-Time Structure (Cambridge, 1954), p. 27.

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