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CONNECTEDNESS OF THE INVERTIBLES IN CERTAIN NEST ALGEBRAS

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ABSTRACT. We show that if $\mathcal{N}$ is a nest with no isolated atoms of finite multiplicity, then the invertibles in $\mathcal{T}(\mathcal{N})$ are connected. The key technical ingredient is that in such nest algebras, every operator with zero atomic diagonal part factors through the non-atomic part of $\mathcal{N}$. In particular, these results apply for the Cantor nest.

In [3], the first two authors showed that the invertible elements are connected in every nest algebra of infinite multiplicity, which means that every atom is infinite rank. In this paper, we extend these results to include nests with finite rank atoms, as long as they are not isolated points in the order structure of $\mathcal{N}$. In particular, this applies to the Cantor nest [2, pp. 23, 27] which is atomic with all finite atoms, but is order equivalent to the Cantor set which is perfect.

Our result reduces the connectedness of the invertibles question for arbitrary nests to the case of the upper triangular operators with respect to a fixed orthonormal basis e_n for $n \geq 1$.

A nest will be a complete chain $\mathcal{N}$ of closed subspaces of a separable Hilbert space. The corresponding nest algebra $\mathcal{T}(\mathcal{N})$ consists of all operators leaving each element of $\mathcal{N}$ invariant. An interval of $\mathcal{N}$ is a projection onto the difference of two subspaces in $\mathcal{N}$.

Minimal intervals of $\mathcal{N}$ are called *atoms*. The expectations Δ onto the atomic part and Δ_f onto the finite atoms of $\mathcal{N}$ are given by

$$\Delta(T) := \sum E_k T E_k + \sum F_\ell T F_\ell \quad \text{and} \quad \Delta_f(T) := \sum E_k T E_k$$

as E_k runs over the set of all finite rank atoms, and F_ℓ runs over all infinite rank atoms. The projection $P_f = \Delta_f(I)$ projects onto the sum of these finite rank atoms, and $P_\infty = I - P_f$ is the projection onto the infinite multiplicity part of the nest.

Likewise $P_a = \Delta(I)$ is the projection onto the atomic part of $\mathcal{N}$ and $P_c = I - P_a$ is the projection onto the continuous part.

Every nest is order equivalent to a compact subset Ω of the unit interval. Each atom corresponds to a component of the complement of Ω in $[0, 1]$. Every nest induces a spectral measure on Ω. The continuous part corresponds to the non-atomic part of this measure, and may be supported only on the maximal perfect subset Ω_0 of Ω, as the isolated points of Ω do not support a non-atomic measure. It is a consequence of the

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Similarity Theorem for nests [1] that each nest $\mathcal{N}$ is similar to another nest which has a continuous part with full support Ω_0. In this case, we say that $\mathcal{N}$ has *maximal non-atomic part*. See [2] for an overview of these ideas.

In [3], some matrix manipulation techniques are combined with the factorization results of [5] to show that the invertibles are connected in every nest that has no finite rank atoms. That is the starting point for this paper. The key new ingredient is a factorization theorem which shows that if $\Delta_f(T) = 0$, then T can be factored through P_∞. It is easy to see why finite rank atoms cannot be isolated for this result. Then the off diagonal part of invertible operators can be shunted over to the infinite part, and eliminated by a continuous path of invertibles.

1. Factorization.

THEOREM 1.1. *Suppose that $\mathcal{N}$ is a nest with no isolated atoms of finite multiplicity and with maximal non-atomic part. Let $T \in \mathcal{T}(\mathcal{N})$ be such that $\Delta_f(T) = 0$. Then there are operators A and B in $\mathcal{T}(\mathcal{N})$ such that $T = AP_\infty B$.*

PROOF. Notice that P_∞ meets every interval which is not a finite rank atom. In fact one can partition P_∞ into pairwise orthogonal projections P_0 and $P_{i,k}$ for $i, k \geq 1$ with this same property.

For each finite rank atom E_k of $\mathcal{N}$, let G_k be the upper end point of E_k. Since $E_kT = E_kTG_k^\perp$ is compact, for each $k \geq 1$ we can pick a decreasing sequence $N_{i,k}$ in $\mathcal{N}$ with $G_k = \bigwedge_{i\geq 1} N_{i,k}$ so that

$$\sum_{i,k\geq 1} i\|E_kTN_{i,k}\| < \infty. \tag{1}$$

Set $N_{0,k} = I$ for each k, and define intervals $J_{i,k} = N_{i-1,k} - N_{i,k}$ for $i, k \geq 1$.

Choose isometries $V_{i,k}$ mapping E_k into $(N_{i,k} - G_k)P_{i,k}$. Notice that $V_{i,k}^*$ belongs to $\mathcal{T}(\mathcal{N})$. Then we can define operators in $\mathcal{T}(\mathcal{N})$ by

$$A = \sum_{i,k\geq 1} \frac{1}{i} V_{i,k}^* \quad \text{and} \quad B = \sum_{i,k\geq 1} iV_{i,k}E_kTJ_{i,k}.$$

Note that the factors of $1/i$ in the first sum ensure that A is bounded. To see that B is bounded, notice that

$$B_0 := \sum_{i\geq 2, k\geq 1} iV_{i,k}E_kTJ_{i,k}$$

is compact by (1). On the other hand,

$$B - B_0 = \Big(\sum_{k\geq 1} V_{1,k}\Big)\Big(P_fT - \sum_{k\geq 1} E_kTN_{1,k}\Big)$$

is bounded as $\sum_{k\geq 1} E_kTN_{1,k}$ is compact by (1) again. We obtain

$$AB = AP_1B = P_fT,$$

where $P_1 = \sum_{i,k\geq 1} P_{i,k}$. By [4], there are operators $A' = A'P_0$, and $B' = P_0B'$ in $\mathcal{T}(\mathcal{N})$ such that $A'P_0B' = P_\infty$. Hence

$$\begin{aligned}(A + A')P_\infty(B + B'P_\infty T) &= AB + A'B'P_\infty T\\ &= P_fT + P_\infty T = T.\end{aligned}$$

This is the desired factorization. ∎

COROLLARY 1.2. *Suppose that $\mathcal{N}$ is a nest with no isolated atoms of finite multiplicity. Let $T \in \mathcal{T}(\mathcal{N})$ be such that $\Delta_f(T) = 0$. Then there are operators A, B and P in $\mathcal{T}(\mathcal{N})$ such that P is an idempotent with $\Delta_f(P) = 0$ so that T factors as $T = APB$.*

PROOF. By the Similarity Theorem [1], there is an invertible operator S such that $S\mathcal{N}$ has maximal non-atomic part. Use Theorem 1.1 to factor STS^{-1} as $AP_\infty B$. Then

$$T = (S^{-1}AS)(S^{-1}P_\infty S)(S^{-1}BS)$$

is the desired factorization. ∎

2. **Connectedness.**

THEOREM 2.1. *Suppose that $\mathcal{N}$ is a nest with no isolated atoms of finite multiplicity. Then the set of invertible elements of $\mathcal{T}(\mathcal{N})$ is connected.*

PROOF. First we replace $\mathcal{N}$ by a similar nest with a maximal non-atomic part of uniform infinite multiplicity, so that the infinite multiplicity part splits as two unitarily equivalent parts. Thus we may decompose $\mathcal{H}$ as

$$P_f\mathcal{H} \oplus P_\infty\mathcal{H} = P_f\mathcal{H} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are identified with $P_\infty\mathcal{H}$. Let P_i be the orthogonal projections onto $\mathcal{H}_i$ for $i = 1, 2$.

Let T be an invertible element of $\mathcal{T}(\mathcal{N})$. With respect to the decomposition $\mathcal{H} = P_f\mathcal{H} \oplus P_\infty\mathcal{H}$, the operator T will have the matrix form

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Moreover, with respect to the decomposition $\mathcal{P}_\infty\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we may think of D as a 2 by 2 operator matrix:

(2) $$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}.$$

By Proposition 2.2 of [3], we may approximate D_{22} by an element D'_{22} so that there are elements X and Y in $\mathcal{T}(P_2\mathcal{N})$ satisfying $XD'_{22}Y = I$. Take this approximation to be within $\varepsilon/2$ where $\varepsilon = \|T^{-1}\|^{-1}$. Then T may be connected by a straight line to the matrix replacing D_{22} by D'_{22}. Thus we may (and do) assume for convenience of notation that D_{22} has this property.

Now by the proof of Theorem 1.4 of [3], there are invertible matrices in $\mathcal{T}(P_\infty\mathcal{N})$ which, with respect to the decomposition $P_\infty\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, have the form

$$\mathbf{X} = \begin{bmatrix} * & * \\ 0 & X \end{bmatrix} \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} * & 0 \\ * & Y \end{bmatrix}.$$

Also, by that same theorem, these matrices are connected to the identity. So T is connected to the identity exactly when

$$T' = \begin{bmatrix} I & 0 \\ 0 & \mathbf{X} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{Y} \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \tag{3}$$

is connected to the identity. Moreover

$$D' = \begin{bmatrix} * & * \\ 0 & X \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} * & 0 \\ * & Y \end{bmatrix} = \begin{bmatrix} * & * \\ * & I \end{bmatrix}.$$

Now by Lemma 1.2 of [3], we may use Gaussian elimination to connect T' to an element of the form

$$T'' = \begin{bmatrix} A'' & B'' & 0 \\ C'' & D'' & 0 \\ 0 & 0 & I \end{bmatrix} =: \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \tag{4}$$

with respect to the decompositions $P_f\mathcal{H} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ and $\mathcal{H}' \oplus \mathcal{H}_2$ respectively, where $\mathcal{H}' = P_f\mathcal{H} \oplus \mathcal{H}_1$. Let

$$E_0 = E - \Delta_f(E) - P_1 = \begin{bmatrix} A'' - \Delta_f(A'') & B'' \\ C'' & D'' - I \end{bmatrix}$$

so that $\Delta_f(E_0) = 0$.

Apply Theorem 1.1 to E_0 to factor it as

$$E_0 = FP_1G = \begin{bmatrix} 0 & F_1 \\ 0 & F_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ G_1 & G_2 \end{bmatrix}$$

in $\mathcal{T}(P_{\mathcal{H}'}\mathcal{N})$. Then notice that

$$\begin{bmatrix} \Delta_f(A'') & 0 & F_1 \\ 0 & I & F_2 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ G_1 & G_2 & I \end{bmatrix} = \begin{bmatrix} A'' & B'' & F_1 \\ C'' & D'' & F_2 \\ G_1 & G_2 & I \end{bmatrix} =: \begin{bmatrix} E & \tilde{F} \\ \tilde{G} & I \end{bmatrix} \tag{5}$$

is connected to the identity.

However, as E is invertible, Gaussian elimination now connects this to

$$\begin{bmatrix} I & 0 \\ -\tilde{G}E^{-1} & I \end{bmatrix} \begin{bmatrix} E & \tilde{F} \\ \tilde{G} & I \end{bmatrix} \begin{bmatrix} I & -E^{-1}\tilde{F} \\ 0 & I \end{bmatrix} = \begin{bmatrix} E & 0 \\ 0 & I - \tilde{G}E^{-1}\tilde{F} \end{bmatrix}. \tag{6}$$

Since the $2,2$ entry is connected to the identity by [3], we have connected the identity to T'' which completes the desired path. ∎

The condition that $\mathcal{N}$ have no isolated atoms of finite multiplicity can be slightly weakened.

COROLLARY 2.2. *Let $\mathcal{N}$ be a nest for which there is a finite bound N on the number of consecutive finite rank atoms in the nest. Then the invertibles are connected in $\mathcal{T}(\mathcal{N})$.*

PROOF. Let J_n be the maximal intervals of $\mathcal{N}$ consisting of finite rank atoms. Given an invertible element T of $\mathcal{T}(\mathcal{N})$, factor it as the product $T = DAS$ where

$$D = \Delta(T) + P_c \quad \text{and} \quad A = \sum_n J_n D^{-1} T J_n + P_\infty \quad \text{and} \quad S = A^{-1} D^{-1} T.$$

The first factor connects to the identity in the diagonal, and the second connects to the identity by a straight line. To see this latter claim, note that A is the direct sum of operators on nests with at most N elements. Hence A has the form $I + Q$ where Q is strictly upper triangular and is nilpotent of order at most N. So $I + tQ$ has inverse $\sum_{k=0}^{N}(-tQ)^k$. To connect the third factor to the identity, replace each interval J_n by a single (finite rank) atom and each infinite rank atom F_ℓ with a continuous nest on $F_\ell \mathcal{H}$. This new nest $\mathcal{M}$ satisfies the conditions of the last theorem, and S belongs to $\mathcal{T}(\mathcal{M})$. Connect S to the identity in $\mathcal{T}(\mathcal{M})$ by a path S_t using Theorem 2.1. Then $S_t' := \Delta_{\mathcal{M}}(S_t)^{-1} S_t$ is another such path with diagonal part (in $\mathcal{M}$) equal to the identity. This path then lies in $\mathcal{T}(\mathcal{N})$ as well. ■

REMARK 2.3. The *discrete intervals* of a nest are the maximal open intervals without limit points or infinite rank atoms. These intervals consist of finite rank atoms with order type ω, ω^*, $\omega^* + \omega$, or $\mathbf{n} = \{1, \ldots, n\}$. Every invertible operator in $\mathcal{T}(\mathcal{N})$ factors as a product of two operators. The first has the identity operator as its compression to each discrete interval; and the second is the direct sum of invertible operators on each discrete interval plus the projection onto the complement of these intervals.

The first of these factors can be connected to the identity in the same manner as in the corollary above. So the problem reduces to dealing with the invertible elements for a nest of order type $\omega^* + \omega$. However, this case is easily reduced to the analysis of the order ω nest. Also, since every finite rank matrix has an upper triangular form, it is easy to replace finite rank atoms with a set of rank one atoms. This reduces the connectedness question for invertibles in arbitrary nest algebras to the standard algebra of upper triangular operators with respect to an orthonormal basis e_n for $n \geq 1$.

3. **Products of symmetries.** A somewhat stronger result can be obtained from the proof of Theorem 2.1. Call a square root of the identity in a unital Banach algebra a *symmetry*. It is easy to see that a symmetry in any Banach algebra can be connected to the identity. Hence if an invertible operator can be factored as a finite product of symmetries, it can be connected to the identity. Of course, the converse is not always true: the group of invertible operators in $C[0,1]$ is connected, but $C[0,1]$ does not contain any symmetries which are not scalar multiples of the identity. However, for the nest algebras we consider, we have Proposition 3.5 and Corollary 3.6 below which show that each invertible element is the product of a diagonal operator and a finite product of symmetries. The following lemma is known; we include it for completeness.

LEMMA 3.1. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be a unital weakly closed operator algebra.*

(a) If $A \in \mathcal{A}$, the elementary operator $E = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$ is a product of two symmetries in $M_2(\mathcal{A})$.

(b) Let $U, V \in \mathcal{A}$ satisfy $UV = I$, $VU \neq I$. The operators

$$X = \begin{bmatrix} U & 0 \\ I - VU & V \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} V & I - VU \\ 0 & U \end{bmatrix}$$

are each products of six symmetries.

(c) Suppose in addition, that $\mathcal{H}$ is infinite dimensional, A is an invertible element of $\mathcal{A}$, and $\mathcal{A}$ is isomorphic to $\mathcal{A} \otimes \mathcal{B}(\mathcal{H})$. Then the operator $Z = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ is a product of four symmetries.

PROOF. For (a), note that

$$E = \begin{bmatrix} I & -A \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

For part (b), observe that X and Y are inverses of each other, hence if one is the product of symmetries, so is the other. We show that X is the product of six symmetries. Indeed, a computation shows

$$\begin{bmatrix} V & I - VU \\ 0 & U \end{bmatrix} = \begin{bmatrix} I & 0 \\ U & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ V & I \end{bmatrix} \begin{bmatrix} I & U \\ 0 & I - 2VU \end{bmatrix}. \tag{7}$$

The second and fourth terms in this factorization are symmetries, while the other two terms are products of two symmetries. Part (b) follows.

Before proving (c), we pause to introduce some notation. For each i, suppose $A_i \in \mathcal{B}(\mathcal{H}_i)$ and let $\operatorname{diag}(A_1, A_2, \ldots) = \bigoplus_{i=1}^{\infty} A_i$ acting on $\bigoplus_{i=1}^{\infty} \mathcal{H}_i$.

To prove (c) we resort to a modification of a standard trick: use the isomorphism of $\mathcal{A}$ with $\mathcal{A} \otimes \mathcal{B}(\mathcal{H})$ to view Z as $\operatorname{diag}(A, I, I, \ldots)$. Let

$$\begin{aligned}
S_1 &= \operatorname{diag}\left(\begin{bmatrix} 0 & A \\ A^{-1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & A \\ A^{-1} & 0 \end{bmatrix}, \begin{bmatrix} 0 & A \\ A^{-1} & 0 \end{bmatrix}, \ldots \right), \\
S_2 &= \operatorname{diag}\left(\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \ldots \right), \\
S_3 &= \operatorname{diag}\left(I, \begin{bmatrix} 0 & A^{-1} \\ A & 0 \end{bmatrix}, \begin{bmatrix} 0 & A^{-1} \\ A & 0 \end{bmatrix}, \ldots \right), \\
S_4 &= \operatorname{diag}\left(I, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \ldots \right).
\end{aligned}$$

Then $Z = S_1 S_2 S_3 S_4$ is a product of 4 symmetries. ∎

We remark that part (a) of Lemma 3.1 shows that when performing Gaussian elimination, two symmetries are needed for each elementary row or column operation.

Recall from [3] that an operator X in a unital algebra $\mathcal{A}$ is *interpolating* if there are elements $A, B \in \mathcal{A}$ such that $AXB = I$.

LEMMA 3.2. *Suppose $\mathcal{M}$ is a continuous nest. Then each invertible element of $\mathcal{T}(\mathcal{M})$ is the product of at most* 28 *symmetries.*

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PROOF. By the Similarity Theorem, $\mathcal{T}(\mathcal{M})$ and $M_2(\mathcal{T}(\mathcal{M}))$ are isomorphic. We will show that every invertible operator in $M_2(\mathcal{T}(\mathcal{M}))$ factors as a product of at most 28 symmetries.

Suppose that

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

is an invertible in $M_2(\mathcal{T}(\mathcal{M}))$. The proof of Theorem 1.4 of [3] together with parts (b) and (c) of Lemma 3.1 shows that if R_{11} is an interpolating element of $\mathcal{T}(\mathcal{M})$, then R is the product of 16 symmetries.

Now let T be an arbitrary invertible element of $M_2(\mathcal{T}(\mathcal{M}))$. By Proposition 2.2 of [3], we may find an invertible element R of $M_2(\mathcal{T}(\mathcal{M}))$ such that R_{11} is interpolating and $\|I - TR^{-1}\| < 1$. Write $TR^{-1} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then A is invertible in $\mathcal{T}(\mathcal{M})$ and performing Gaussian elimination, we obtain the factorization,

$$TR^{-1} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D - CA^{-1} \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}.$$

Applying Lemma 3.1 we find that TR^{-1} is the product of 12 symmetries, so $T = TR^{-1}R$ is the product of 28 symmetries. ∎

LEMMA 3.3. *Let $\mathcal{M}$ be a nest. If every atom of $\mathcal{M}$ has infinite dimension, then every invertible element of $\mathcal{T}(\mathcal{M})$ is the product of at most* 44 *symmetries.*

PROOF. First note that if $\mathcal{K}$ is an infinite dimensional Hilbert space, and $R \in \mathcal{B}(\mathcal{K})$ is invertible, then R is the product of at most 16 symmetries. To see this, consider the polar decomposition of R, $R = UP$. Then U is a normal operator, so by spectral theory, U may be factored as $U = U_1U_2$, where both U_1 and U_2 have an infinite dimensional eigenspace corresponding to the eigenvalue 1. Now by Lemma 3.1(c), each of U_1 and U_2 may be factored as the product of 4 symmetries, so U is the product of 8 symmetries. Similar considerations show that P is the product of 8 symmetries, so R is the product of no more than 16 symmetries. Moreover, note that the terms in the factorization of Lemma 3.1(c) have norms at most $\max\{\|Z\|, \|Z^{-1}\|\}$. Thus the 16 terms in the factorization of R are all bounded by $\max\{\|R\|, \|R^{-1}\|\}$. Consequently, an invertible operator which is the direct sum of bounded operators is also the product of 16 symmetries which are the direct sum of the corresponding factors of each summand.

Given the previous lemma, we may assume that $\mathcal{M}$ has some infinite atoms. Now let T be an invertible element of $\mathcal{T}(\mathcal{M})$ and factor T as $T = AB$, where $A = (P_c + \Delta(T))$ and $B = (P_c + \Delta(T))^{-1}T$. Now $\Delta(T)$ belongs to the direct sum of at most countably many copies of $\mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is an infinite dimensional Hilbert space. So by the preceding paragraph, A is the product of at most 16 symmetries in $\mathcal{T}(\mathcal{M}) \cap (\mathcal{T}(\mathcal{M}))^*$. As in [3], we may view B as an element of a continuous nest algebra, so by Lemma 3.2, we see that T is the product of at most $16 + 28 = 44$ symmetries. ∎

REMARK 3.4. The method used for factoring an invertible operator in $\mathcal{B}(\mathcal{K})$ into the product of symmetries above is simple, yet crude. A result of Radjavi [6] shows that it is possible to factor such operators into the product of 7 symmetries. For an interesting survey of factorization problems, see [7].

PROPOSITION 3.5. *Suppose that $\mathcal{N}$ is a nest with no isolated atoms of finite multiplicity. Then each invertible element T of $\mathcal{T}(\mathcal{N})$ may be written as $T = \big(\Delta_f(T) + P_\infty\big)S$, where S is the product of at most 80 symmetries belonging to $\mathcal{T}(\mathcal{N})$.*

PROOF. We shall examine the proof of Theorem 2.1 carefully. Let T be an invertible element of $\mathcal{T}(\mathcal{N})$. As in the proof of Theorem 2.1, we first assume that lower-right corner, D_{22}, of D (see line 2) is an interpolating operator.

Using Lemma 3.1 repeatedly, we see that twelve symmetries are needed for passing from T to T' in line 3, an additional four are required to pass from T to T'' (see line 4), four more are needed in line 5, and finally (using Lemma 3.3), $4 + 44 = 48$ are required for line 6. Hence we require a total of $12 + 4 + 4 + 48 = 68$ symmetries to connect T to the identity.

For a general invertible operator T, we use an argument similar to that used in the final paragraph of the proof of Lemma 3.2 to conclude that T will factor as a product of at most $68 + 12 = 80$ symmetries. ∎

Finally, we show that the group of invertibles in nest algebras which have no more than N atoms of finite rank in a row is generated by the invertibles in the diagonal and the symmetries belonging to the algebra.

COROLLARY 3.6. *Let $\mathcal{N}$ be a nest for which there is a finite bound N on the number of consecutive finite rank atoms in the nest. If $T \in \mathcal{T}(\mathcal{N})$ is an invertible operator, then T is the product of a diagonal operator and finitely many symmetries. The number of symmetries is bounded by $81 + \lceil \log_2(N) \rceil$.*

PROOF. As in the proof of Corollary 2.2, T factors as $T = DAS$, where D belongs to the diagonal, S is an operator to which Proposition 3.5 applies and $A = \bigoplus A_i$ is the direct sum of operators A_i, where each A_i belongs to a nest $\mathcal{N}_i$ with no more than N elements.

Let n be chosen so that $2^{n-1} < N \leq 2^n$. By taking $\mathcal{M}_i$ to be the ordinal sum of $\mathcal{N}_i$ with a nest of $2^n - N$ elements and adding a direct sum of the identity to A_i, we may assume without loss of generality that each A_i belongs to the algebra of a nest with 2^n elements.

Since $n < 1 + \log_2(N)$, it thus suffices to show that if $\mathcal{M}$ is a nest with 2^n elements, and $A \in \mathcal{T}(\mathcal{M})$ satisfies $EAE = \pm E$ for each atom of $\mathcal{M}$, then A is the product of at most $n+1$ symmetries each of whose norms do not exceed $\|A\|\,\|A^{-1}\|$.

To see this, let $P \in \mathcal{M}$ be the projection which is the sum of exactly 2^{n-1} atoms of $\mathcal{M}$. Then the matrix for A relative to the decomposition of $\mathcal{H} = P\mathcal{H} \oplus P^\perp\mathcal{H}$ has a 2×2 upper triangular block form $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, and factors as

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & -A_{22} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & -I \end{bmatrix}. \tag{8}$$

The second term in this factorization is a symmetry whose norm is no greater than $||A|| \, ||A^{-1}||$. The first term of the factorization is the direct sum of smaller operators whose dimension is half the size of the original block operator. Using a similar argument on each of the summands A_{11} and A_{22}, an easy induction shows that A may be factored into the product of at most $n+1$ symmetries whose norms are bounded by $||A|| \, ||A^{-1}||$. ∎

REMARK 3.7. Unfortunately, we have been unable to provide a bound on the number of symmetries required for the factorization in Corollary 3.6 which is independent of N while also maintaining control of the norms of the symmetries. We are able to show that if $\mathcal{N}$ is a nest with N elements, then each operator in $\mathcal{T}(\mathcal{N})$ of the form I + nilpotent is the product of four symmetries belonging to $\mathcal{T}(\mathcal{N})$, but we have been unable to show that the norms of the symmetries are bounded, even for N fixed.

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