

ON VARIANCES OF PARTIAL VOLUMES OF THE TYPICAL CELL OF A POISSON–VORONOI TESSELLATION AND LARGE-DIMENSIONAL VOLUME DEGENERACY

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Abstract

For a typical cell of a homogeneous Poisson–Voronoi tessellation in \mathbb{R}^d , it is shown that the variance of the volume of the intersection of the typical cell with any measurable subset of \mathbb{R}^d is bounded by the variance of the volume of the typical cell. It is also shown that the variance of the volume of the intersection of the typical cell with a translation of itself is bounded by four times the variance of the volume of the typical cell. These bounds are applied to show large-dimensional volume degeneracy as d tends to ∞ . An extension to the k th nearest-point Poisson–Voronoi tessellation for $k \geq 2$ is also considered.

Keywords: Poisson process; Voronoi tessellation; typical cell; geometric covariogram; nearest neighbor

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1. Introduction

Associated with a homogeneous Poisson point process $\Pi = \Pi_d$ in \mathbb{R}^d with intensity $\lambda > 0$, the Voronoi tessellation is defined as $\{C_X(\Pi) : X \in \Pi\}$, where $C_X(\Pi)$, the Voronoi cell of Π centered at X , is given by $C_X(\Pi) = \{x \in \mathbb{R}^d : \|x - X\| \leq \|x - X'\| \text{ for all } X' \in \Pi\}$, $\|\cdot\| = \|\cdot\|_d$ being the d -dimensional Euclidean norm. A ‘typical cell’ refers to a uniform random member (translated so as to be centered at the origin) of the Voronoi tessellation, and can be explicitly described (cf. [5]) by $C_t = C_{t,d} := C_{\mathbf{0}}(\Pi \cup \{\mathbf{0}\}) = \{x \in \mathbb{R}^d : \|x\| \leq \|x - X\| \text{ for all } X \in \Pi\}$, the Voronoi cell centered at the origin $\mathbf{0}$ after adding $\mathbf{0}$ to Π . A subscript d is attached to Π , C_t , $\|\cdot\|$, and μ (the Lebesgue measure on \mathbb{R}^d) in order to emphasize dependence on d , which is suppressed in the proofs for ease of notation when no danger of confusion arises.

In the literature on the moments and distributions of geometric characteristics of the typical cell most of the analytic and simulation studies have been concerned with the two important low-dimensional cases $d = 2$ and $d = 3$ (see [9] and [12] for a comprehensive review). At the other extreme, Newman *et al.* [11] first studied the large-dimensional limit of $\mu_d(C_{t,d})$, the d -dimensional Lebesgue measure (d -volume) of $C_{t,d}$, and showed that, as $d \rightarrow \infty$, $\mu_d(C_{t,d})$ converges in distribution to $1/\lambda$. (In fact, their Theorem 10 implies that, as $d \rightarrow \infty$, $\mu_d(C_{t,d})$ converges in L^p to $1/\lambda$ for all $p > 0$. Here a sequence of real-valued random variables $\{Y_n\}$, not necessarily defined on the same probability space, is said to converge in L^p to a constant γ as $n \rightarrow \infty$ if $\lim_{n \rightarrow \infty} E[|Y_n - \gamma|^p] = 0$.) Recently, Alishahi and Sharifitabar [1] established

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the following sharp bound on the variance of $\mu_d(C_{t,d})$:

$$\text{var}(\mu_d(C_{t,d})) = E\left[\left(\mu_d(C_{t,d}) - \frac{1}{\lambda}\right)^2\right] < \frac{c}{\lambda^2\sqrt{d}}\left(\frac{4}{3\sqrt{3}}\right)^d \quad \text{for all } d, \tag{1}$$

where c is a constant. By [1, Remark 3.1], $c = 5$ suffices for (1) to hold. Note that, to establish (1), they used the Blaschke–Petkantschin formula (see, e.g. [6, Chapter 4]) to obtain an integral formula for $E[\{\mu_d(C_{t,d})\}^2]$, while a similar integral representation was derived earlier by Gilbert [4, Formula (14)]. Alishahi and Sharifitabar further showed that, for each fixed $u > 0$, as $d \rightarrow \infty$, the d -volume of the intersection of $C_{t,d}$ with a ball of volume u centered at the origin converges in L^2 to $\lambda^{-1}(1 - e^{-\lambda u})$. These results indicate the phenomenon of large-dimensional volume degeneracy of the typical cell.

The geometric covariogram of a bounded measurable set $K \subset \mathbb{R}^d$ is the function $g_K(v) = \mu_d(K \cap (K + v))$, $v \in \mathbb{R}^d$, which was introduced by Matheron [8], who also asked the question of whether the geometric covariogram uniquely determines a convex set up to translations and reflections. While Bianchi [3] found counterexamples for $d \geq 4$, g_K clearly contains much information about the geometric characteristics of K and can uniquely determine K for some classes of convex sets. See Averkov and Bianchi [2] and the references therein. Alishahi and Sharifitabar [1] considered the (random) geometric covariogram of the typical cell, $g_{C_{t,d}}(v) = \mu_d(C_{t,d} \cap (C_{t,d} + v))$, $v \in \mathbb{R}^d$, which is a random function. (Note that Lantuéjoul [7, p. 23] defined the geometric covariogram of a random set K by taking the expectation $E[\mu_d(K \cap (K + v))]$.) For $r > 0$ and $v_d \in \mathbb{R}^d$ with $\|v_d\|_d = 1$, they proved that $\lim_{d \rightarrow \infty} E[g_{C_{t,d}}(rv_d)] = c(r, \lambda)$, where

$$c(r, \lambda) = \frac{1}{\lambda\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dt}{\exp(t^2/2)\Phi(t) + \exp((t - r\sqrt{2\pi e})^2/2)(1 - \Phi(t - r\sqrt{2\pi e}))}, \tag{2}$$

$\Phi(\cdot)$ being the standard normal cumulative distribution function. An interesting question was raised in [1, p. 934] as to whether $g_{C_{t,d}}(v_d)$ becomes deterministic for large d . Indeed, a more general question is to determine those partial volumes of the typical cell which become deterministic for large d .

In the next section we show that $\mu_d(C_{t,d} \cap S_1)$ and $\mu_d(C_{t,d} \cap S_2)$ have a nonnegative covariance for any measurable subsets S_1 and S_2 of \mathbb{R}^d , implying that $\text{var}(\mu_d(C_{t,d} \cap S_1)) \leq \text{var}(\mu_d(C_{t,d} \cap S_2)) \leq \text{var}(\mu_d(C_{t,d}))$ if $S_1 \subset S_2 \subset \mathbb{R}^d$. It then follows that, as $d \rightarrow \infty$, $\mu_d(C_{t,d} \cap A_d) - E[\mu_d(C_{t,d} \cap A_d)]$ converges in L^2 to 0 for any sequence of measurable sets $A_d \subset \mathbb{R}^d$, $d = 1, 2, \dots$. We further strengthen the L^2 convergence to L^p convergence for all $p > 0$. We also discuss conditions under which $\mu_d(C_{t,d} \cap A_d)/E[\mu_d(C_{t,d} \cap A_d)]$ converges in L^2 to 1 as $d \rightarrow \infty$.

In Section 3 we consider the geometric covariogram of the typical cell and show that $\text{var}(g_{C_{t,d}}(v)) \leq 4 \text{var}(\mu_d(C_{t,d}))$ for all $v \in \mathbb{R}^d$. Together with (1), this inequality implies that, as $d \rightarrow \infty$, $g_{C_{t,d}}(v_d) - E[g_{C_{t,d}}(v_d)]$ converges in L^2 to 0 for any sequence of $v_d \in \mathbb{R}^d$, $d = 1, 2, \dots$, which answers in the affirmative the aforementioned question raised in [1, p. 934]. (The L^2 convergence can in fact be strengthened to L^p convergence for all $p > 0$.)

In Section 4 we consider the typical cell (centered at the origin) of the k th nearest-point Poisson–Voronoi tessellation, $k = 1, 2, \dots$, defined by

$$C_{t,d}^{(k)} := \{x \in \mathbb{R}^d : \#\{X \in \Pi : \|x - X\| < \|x\|\} = k - 1\}. \tag{3}$$

(See [12, Section 3.3.2] for a brief discussion.) Note that $C_{t,d}^{(1)} = C_{t,d}$. We show that, for $k = 1, 2, \dots$ and $p > 0$, $\mu_d(C_{t,d}^{(k)})$ converges in L^p to $1/\lambda$ as $d \rightarrow \infty$, from which follows a

large-dimensional independent and identically distributed property for nearest-neighbor counts in two-ensemble Poisson point processes (see Remark 11).

2. Nonnegative covariance of partial volumes of the typical cell and its implications

Theorem 1. *For any measurable subsets S_1 and S_2 of \mathbb{R}^d , we have*

$$\text{cov}(\mu_d(C_{t,d} \cap S_1), \mu_d(C_{t,d} \cap S_2)) \geq 0.$$

If $S_1 \subset S_2 \subset \mathbb{R}^d$ then

$$\text{var}(\mu_d(C_{t,d} \cap S_1)) \leq \text{var}(\mu_d(C_{t,d} \cap S_2)) \leq \text{var}(\mu_d(C_{t,d})).$$

Proof. Without loss of generality, assume that $\lambda = 1$. For ease of notation, the subscript d in $C_{t,d}$, μ_d , and $\|\cdot\|_d$ is suppressed. Letting $\mathbf{1}_S$ denote the indicator function of the set S and B_v denote the ball centered at v with radius $\|v\|$ (i.e. $B_v = \{x \in \mathbb{R}^d : \|x - v\| < \|v\|\}$), we have, for $x, y \in \mathbb{R}^d$,

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{C_t}(x) - e^{-\mu(B_x)}][\mathbf{1}_{C_t}(y) - e^{-\mu(B_y)}] &= \mathbb{E}[\mathbf{1}_{C_t}(x)\mathbf{1}_{C_t}(y)] - e^{-\mu(B_x) - \mu(B_y)} \\ &= e^{-\mu(B_x \cup B_y)} - e^{-\mu(B_x) - \mu(B_y)} \\ &\geq 0. \end{aligned} \tag{4}$$

Since, for any measurable set $S \subset \mathbb{R}^d$,

$$\mu(C_t \cap S) - \mathbb{E}[\mu(C_t \cap S)] = \int_S [\mathbf{1}_{C_t}(x) - e^{-\mu(B_x)}] dx,$$

it follows by Fubini’s theorem and (4) that

$$\begin{aligned} \text{cov}(\mu(C_t \cap S_1), \mu(C_t \cap S_2)) &= \mathbb{E} \int_{S_1} [\mathbf{1}_{C_t}(x) - e^{-\mu(B_x)}] dx \int_{S_2} [\mathbf{1}_{C_t}(y) - e^{-\mu(B_y)}] dy \\ &= \int_{S_1 \times S_2} \mathbb{E}[\mathbf{1}_{C_t}(x) - e^{-\mu(B_x)}][\mathbf{1}_{C_t}(y) - e^{-\mu(B_y)}] dx dy \\ &\geq 0. \end{aligned}$$

If $S_1 \subset S_2$ then $\mu(C_t \cap S_2) = \mu(C_t \cap S_1) + \mu(C_t \cap (S_2 \setminus S_1))$, so that

$$\begin{aligned} \text{var}(\mu(C_t \cap S_2)) &= \text{var}(\mu(C_t \cap S_1)) + \text{var}(\mu(C_t \cap (S_2 \setminus S_1))) + 2 \text{cov}(\mu(C_t \cap S_1), \mu(C_t \cap (S_2 \setminus S_1))) \\ &\geq \text{var}(\mu(C_t \cap S_1)). \end{aligned}$$

This completes the proof.

Remark 1. Note that the inequality in (4) is strict unless the origin $\mathbf{0}$ lies in the line segment connecting x and y . It follows that, for measurable subsets S_1 and S_2 of \mathbb{R}^d with $d \geq 2$, $\text{cov}(\mu_d(C_{t,d} \cap S_1), \mu_d(C_{t,d} \cap S_2)) > 0$ if $\mu_d(S_1) > 0$ and $\mu_d(S_2) > 0$. (For $d = 1$, $\mu_d(C_{t,d} \cap S_1)$ and $\mu_d(C_{t,d} \cap S_2)$ are independent if $S_1 \subset (-\infty, 0)$ and $S_2 \subset (0, \infty)$.)

Remark 2. By direct calculations, Alishahi and Sharifitabar [1] showed that, as $d \rightarrow \infty$, $\text{var}(\mu_d(C_{t,d} \cap S))$ decays to 0 at a geometric rate when S is a ball centered at $\mathbf{0}$. By Theorem 1, this is true for general S . While a small value of $\text{var}(\mu_d(C_{t,d} \cap S))$ implies that $\mu_d(C_{t,d} \cap S)$ is close to its mean with high probability, it does not say much about volume degeneracy when $E[\mu_d(C_{t,d} \cap S)]$ is small, in which case a more relevant measure of volume degeneracy is the coefficient of variation

$$\text{CV}(\mu_d(C_{t,d} \cap S)) := \frac{\sqrt{\text{var}(\mu_d(C_{t,d} \cap S))}}{E[\mu_d(C_{t,d} \cap S)]},$$

since a small value of $\text{CV}(\mu_d(C_{t,d} \cap S))$ implies that $\mu_d(C_{t,d} \cap S)/E[\mu_d(C_{t,d} \cap S)]$ is close to 1 with high probability. With $r_d = d^{-1/4}(4/3\sqrt{3})^{d/2}$, it follows from (1) that $\lim_{d \rightarrow \infty} \text{CV}(\mu_d(C_{t,d} \cap A_d)) = 0$ whenever measurable sets $A_d \subset \mathbb{R}^d$ satisfy

$$\lim_{d \rightarrow \infty} \frac{E[\mu_d(C_{t,d} \cap A_d)]}{r_d} = \infty.$$

It would be of interest to find the optimal rate of r_d satisfying the above property.

Remark 3. As an example, consider $A_d = (0, \infty)^{n_d} \times \mathbb{R}^{d-n_d} = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i > 0, i = 1, \dots, n_d\}$, where $1 \leq n_d \leq d$ is an integer. Note that the total volume of the typical cell $\mu_d(C_{t,d})$ can be divided into 2^{n_d} parts, one of which is $\mu_d(C_{t,d} \cap A_d)$ and all of which have the same distribution by symmetry. It follows that

$$\begin{aligned} \lambda^{-1} &= E[\mu_d(C_{t,d})] = 2^{n_d} E[\mu_d(C_{t,d} \cap A_d)], \\ \text{var}(\mu_d(C_{t,d})) &\geq 2^{n_d} \text{var}(\mu_d(C_{t,d} \cap A_d)), \end{aligned} \tag{5}$$

since the covariance of each pair of the 2^{n_d} parts is nonnegative. We have, by (1),

$$\text{var}(\mu_d(C_{t,d} \cap A_d)) \leq \frac{c}{\lambda^2 \sqrt{d}} 2^{-n_d} \left(\frac{4}{3\sqrt{3}}\right)^d. \tag{6}$$

While the bound in (6) is crude, (5) and (6) imply that $\lim_{d \rightarrow \infty} \text{CV}(\mu_d(C_{t,d} \cap A_d)) = 0$ if $\limsup_{d \rightarrow \infty} n_d/d < \log 27 / \log 4 - 2$.

Remark 4. By [11, Theorem 10], $\lim_{d \rightarrow \infty} E[\mu_d(C_{t,d})]^n = \lambda^{-n}$, $n = 1, 2, \dots$, which implies that $\mu_d(C_{t,d})$ converges in L^p to λ^{-1} for all $p > 0$. We now show that, for any sequence of measurable sets $A_d \subset \mathbb{R}^d$, $\mu_d(C_{t,d} \cap A_d) - E[\mu_d(C_{t,d} \cap A_d)]$ converges in L^p to 0 as $d \rightarrow \infty$ for all $p > 0$. Letting $U_d := \mu_d(C_{t,d} \cap A_d)$, $m_d := E(U_d)$, and $V_d := \mu_d(C_{t,d})$, and noting that $m_d \leq E(V_d) = 1/\lambda$, we have, for $p > 2$,

$$\begin{aligned} E|U_d - m_d|^p &= E|U_d - m_d|^p \mathbf{1}_{\{|U_d - m_d| \leq 2/\lambda\}} + E|U_d - m_d|^p \mathbf{1}_{\{|U_d - m_d| > 2/\lambda\}}, \\ E|U_d - m_d|^p \mathbf{1}_{\{|U_d - m_d| \leq 2/\lambda\}} &\leq \left(\frac{2}{\lambda}\right)^{p-2} E|U_d - m_d|^2 \mathbf{1}_{\{|U_d - m_d| \leq 2/\lambda\}} \\ &\leq \left(\frac{2}{\lambda}\right)^{p-2} \text{var}(U_d) \\ &\rightarrow 0 \quad \text{as } d \rightarrow \infty, \end{aligned}$$

$$\begin{aligned}
 \mathbb{E} |U_d - m_d|^p \mathbf{1}_{\{|U_d - m_d| > 2/\lambda\}} &\leq \mathbb{E} |U_d - m_d|^p \mathbf{1}_{\{U_d > 2/\lambda\}} \\
 &\quad (\text{since } |U_d - m_d| > 2/\lambda \text{ implies that } U_d > 2/\lambda) \\
 &\leq \mathbb{E} |V_d - m_d|^p \mathbf{1}_{\{V_d > 2/\lambda\}} \quad (\text{since } V_d \geq U_d) \\
 &\leq 2^p \mathbb{E} |V_d - \lambda^{-1}|^p \mathbf{1}_{\{V_d > 2/\lambda\}} \\
 &\leq 2^p \mathbb{E} |V_d - \lambda^{-1}|^p \\
 &\rightarrow 0 \quad \text{as } d \rightarrow \infty,
 \end{aligned}$$

where the second-to-last inequality follows from the fact that $V_d > 2/\lambda$ implies that $2(V_d - 1/\lambda) > V_d - m_d$. This proves that $\mu_d(C_{t,d} \cap A_d) - \mathbb{E}[\mu_d(C_{t,d} \cap A_d)]$ converges in L^p to 0 as $d \rightarrow \infty$ for all $p > 0$.

Remark 5. The following simple fact may have something to do with the volume degeneracy property of the typical cell. Let Q_d denote a typical point of the Poisson point process Π_d , and let $X_{i,d} \in \Pi_d$ denote Q_d 's i th nearest neighbor. Then $X_{i,d} - Q_d, i = 1, 2, \dots$, are asymptotically orthogonal as $d \rightarrow \infty$, i.e. for $i \neq j$,

$$\frac{\langle X_{i,d} - Q_d, X_{j,d} - Q_d \rangle_d}{\|X_{i,d} - Q_d\|_d \|X_{j,d} - Q_d\|_d} \rightarrow 0 \quad \text{in probability as } d \rightarrow \infty,$$

where $\langle \cdot, \cdot \rangle_d$ is the inner product associated with the norm $\|\cdot\|_d$. This fact is related to a result of Newman *et al.* [11] which states that the number of points among $X_{i,d}, i = 1, 2, \dots$, that have Q_d as their nearest neighbor has a limiting Poisson distribution with mean 1 as $d \rightarrow \infty$. Further extensions to general k th nearest-neighbor counts with respect to the l^p distance ($1 \leq p \leq \infty$) can be found in [10] and [13], where $p = \infty$ refers to the supnorm distance.

3. Bound on the variance of the geometric covariogram of the typical cell

Theorem 2. For all $v \in \mathbb{R}^d$, we have

$$\text{var}(\mu_d(C_{t,d} \cap (C_{t,d} + v))) \leq 4 \text{var}(\mu_d(C_{t,d})).$$

Proof. Without loss of generality, assume that $\lambda = 1$. For ease of notation, we suppress the subscript d in C_t, μ_d , and $\|\cdot\|_d$. Recall that $B_v = \{x \in \mathbb{R}^d : \|x - v\| < \|v\|\}$. Since

$$\mu(C_t \cap (C_t + v)) = \int_{\mathbb{R}^d} \mathbf{1}_{C_t}(x) \mathbf{1}_{C_t}(x - v) \, dx$$

and

$$\mathbb{E}[\mu(C_t \cap (C_t + v))] = \int_{\mathbb{R}^d} e^{-\mu(B_x \cup B_{x-v})} \, dx,$$

we have, by Fubini's theorem,

$$\begin{aligned}
 &\text{var}(\mu(C_t \cap (C_t + v))) \\
 &= \mathbb{E} \int_{\mathbb{R}^d} [\mathbf{1}_{C_t}(x) \mathbf{1}_{C_t}(x - v) - e^{-\mu(B_x \cup B_{x-v})}] \, dx \int_{\mathbb{R}^d} [\mathbf{1}_{C_t}(y) \mathbf{1}_{C_t}(y - v) - e^{-\mu(B_y \cup B_{y-v})}] \, dy \\
 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} [\mathbb{E} \mathbf{1}_{C_t}(x) \mathbf{1}_{C_t}(x - v) \mathbf{1}_{C_t}(y) \mathbf{1}_{C_t}(y - v) - e^{-\mu(B_x \cup B_{x-v}) - \mu(B_y \cup B_{y-v})}] \, dx \, dy \\
 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} [e^{-\mu(B_x \cup B_{x-v} \cup B_y \cup B_{y-v})} - e^{-\mu(B_x \cup B_{x-v}) - \mu(B_y \cup B_{y-v})}] \, dx \, dy. \tag{7}
 \end{aligned}$$

Note that

$$\begin{aligned}
 &\mu(B_x \cup B_{x-v}) + \mu(B_y \cup B_{y-v}) - \mu(B_x \cup B_{x-v} \cup B_y \cup B_{y-v}) \\
 &= \mu((B_x \cup B_{x-v}) \cap (B_y \cup B_{y-v})) \\
 &= \mu((B_x \cap B_y) \cup (B_x \cap B_{y-v}) \cup (B_{x-v} \cap B_y) \cup (B_{x-v} \cap B_{y-v})) \\
 &\leq \mu(B_x \cap B_y) + \mu(B_x \cap B_{y-v}) + \mu(B_{x-v} \cap B_y) + \mu(B_{x-v} \cap B_{y-v}), \tag{8}
 \end{aligned}$$

so that, by (7),

$$\begin{aligned}
 &\text{var}(\mu(C_t \cap (C_t + v))) \\
 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\mu(B_x \cup B_{x-v} \cup B_y \cup B_{y-v})} \\
 &\quad \times [1 - e^{-\mu(B_x \cup B_{x-v}) - \mu(B_y \cup B_{y-v}) + \mu(B_x \cup B_{x-v} \cup B_y \cup B_{y-v})}] dx dy \\
 &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\mu(B_x \cup B_{x-v} \cup B_y \cup B_{y-v})} \\
 &\quad \times [1 - e^{-\mu(B_x \cap B_y) - \mu(B_x \cap B_{y-v}) - \mu(B_{x-v} \cap B_y) - \mu(B_{x-v} \cap B_{y-v})}] dx dy \\
 &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\mu(B_x \cup B_{x-v} \cup B_y \cup B_{y-v})} [1 - e^{-\mu(B_x \cap B_y)} + 1 - e^{-\mu(B_x \cap B_{y-v})} \\
 &\quad + 1 - e^{-\mu(B_{x-v} \cap B_y)} + 1 - e^{-\mu(B_{x-v} \cap B_{y-v})}] dx dy,
 \end{aligned}$$

where the last inequality follows from the simple fact that

$$1 - \exp\left(-\sum_{i=1}^k a_i\right) \leq \sum_{i=1}^k (1 - e^{-a_i}) \quad \text{for } a_i \geq 0, i = 1, \dots, k.$$

Then $\text{var}(\mu(C_t \cap (C_t + v))) \leq \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, where

$$\begin{aligned}
 \alpha_1 &:= \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\mu(B_x \cup B_{x-v} \cup B_y \cup B_{y-v})} [1 - e^{-\mu(B_x \cap B_y)}] dx dy, \\
 \alpha_2 &:= \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\mu(B_x \cup B_{x-v} \cup B_y \cup B_{y-v})} [1 - e^{-\mu(B_x \cap B_{y-v})}] dx dy, \\
 \alpha_3 &:= \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\mu(B_x \cup B_{x-v} \cup B_y \cup B_{y-v})} [1 - e^{-\mu(B_{x-v} \cap B_y)}] dx dy, \\
 \alpha_4 &:= \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\mu(B_x \cup B_{x-v} \cup B_y \cup B_{y-v})} [1 - e^{-\mu(B_{x-v} \cap B_{y-v})}] dx dy.
 \end{aligned}$$

We have

$$\begin{aligned}
 \alpha_1 &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\mu(B_x \cup B_y)} [1 - e^{-\mu(B_x \cap B_y)}] dx dy \\
 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} [e^{-\mu(B_x \cup B_y)} - e^{-\mu(B_x) - \mu(B_y)}] dx dy \\
 &= \text{var}(\mu(C_t)),
 \end{aligned}$$

and similarly $\alpha_i \leq \text{var}(\mu(C_t))$, $i = 2, 3, 4$. This completes the proof.

Remark 6. By Theorem 2 we have

$$1 \leq \sup_{v \in \mathbb{R}^d} \frac{\text{var}(\mu_d(C_{t,d} \cap (C_{t,d} + v)))}{\text{var}(\mu_d(C_{t,d}))} \leq 4.$$

In view of Theorem 1, it seems natural to ask whether

$$\text{var}(\mu_d(C_{t,d} \cap (C_{t,d} + v))) \leq \text{var}(\mu_d(C_{t,d}))$$

for all $v \in \mathbb{R}^d$.

Remark 7. By a similar argument, it can be shown that, for every measurable set $S \subset \mathbb{R}^d$ and $v \in \mathbb{R}^d$, $\text{var}(\mu_d(S \cap C_{t,d} \cap (C_{t,d} + v))) \leq 4 \text{var}(\mu_d(C_{t,d}))$, and that, for $k = 1, 2, \dots$ and $v_1, \dots, v_k \in \mathbb{R}^d$,

$$\text{var}(\mu_d(C_{t,d} \cap (C_{t,d} + v_1) \cap \dots \cap (C_{t,d} + v_k))) \leq (k + 1)^2 \text{var}(\mu_d(C_{t,d})).$$

To see why a factor of $(k + 1)^2$ appears in the latter inequality, we extend (7) and (8) to obtain, with $v_0 := \mathbf{0}$, $\mu := \mu_d$, and $C_t := C_{t,d}$,

$$\begin{aligned} \text{var}\left(\mu\left(\bigcap_{i=0}^k (C_t + v_i)\right)\right) &= \int_{(\mathbb{R}^d)^2} \left[\exp\left\{-\mu\left(\left(\bigcup_{i=0}^k B_{x-v_i}\right) \cup \left(\bigcup_{j=0}^k B_{y-v_j}\right)\right)\right\} \right. \\ &\quad \left. - \exp\left\{-\mu\left(\bigcup_{i=0}^k B_{x-v_i}\right) - \mu\left(\bigcup_{j=0}^k B_{y-v_j}\right)\right\} \right] dx dy \end{aligned}$$

and

$$\begin{aligned} &\mu\left(\bigcup_{i=0}^k B_{x-v_i}\right) + \mu\left(\bigcup_{j=0}^k B_{y-v_j}\right) - \mu\left(\left(\bigcup_{i=0}^k B_{x-v_i}\right) \cup \left(\bigcup_{j=0}^k B_{y-v_j}\right)\right) \\ &\leq \sum_{i=0}^k \sum_{j=0}^k \mu(B_{x-v_i} \cap B_{y-v_j}), \end{aligned}$$

which is a sum of $(k + 1)^2$ terms. The desired inequality then follows along the lines of the proof of Theorem 2.

Remark 8. By Theorem 2 we have $\mu_d(C_{t,d} \cap (C_{t,d} + rv_d))$ converges in L^2 to $c(r, \lambda)$ for any sequence of $v_d \in \mathbb{R}^d$ satisfying $\|v_d\|_d = 1$, $d = 1, 2, \dots$, where $c(r, \lambda)$ is given in (2). Furthermore, we can use the argument in Remark 4 to strengthen the L^2 convergence to L^p convergence for all $p > 0$.

4. Volume degeneracy of the typical cell of the k th nearest-point Poisson–Voronoi tessellation

Recall that $C_{t,d}^{(k)}$ as defined in (3) is the typical cell of the k th nearest-point Poisson–Voronoi tessellation. Note that $E[\mu_d(C_{t,d}^{(k)})] = 1/\lambda$. We first prove the following main result of this section.

Theorem 3. For $k = 1, 2, \dots$ and $n = 1, 2, \dots$, we have

$$\lim_{d \rightarrow \infty} E[\mu_d(C_{t,d}^{(k)})]^n = \lambda^{-n}.$$

Consequently, $\mu_d(C_{t,d}^{(k)})$ converges in L^p to λ^{-1} as $d \rightarrow \infty$ for all $p > 0$.

Proof. Without loss of generality, assume that $\lambda = 1$. The case $k = 1$ is contained in Theorem 10 of Newman *et al.* [11]. Their proof can be adapted to general k as described below in detail. Letting $D_{k,d} := \bigcup_{i=1}^k C_{t,d}^{(i)}$, since $E[\mu_d(D_{k,d})] = \sum_{i=1}^k E[\mu_d(C_{t,d}^{(i)})] = k$, Lemma 4, below, implies that $\text{var}(\mu_d(D_{k,d})) \rightarrow 0$ as $d \rightarrow \infty$, so that the distribution of $\mu_d(D_{k,d})$ is asymptotically degenerate at k as $d \rightarrow \infty$. By Lemma 5, below, we have, for all k and n ,

$$\sup_d E[\mu_d(D_{k,d})]^n < \infty,$$

which together with the asymptotic degeneracy of $\mu_d(D_{k,d})$ at k implies that

$$\lim_{d \rightarrow \infty} E[\mu_d(D_{k,d})]^n = k^n.$$

Since $\mu_d(C_{t,d}^{(k)}) = \mu_d(D_{k,d}) - \mu_d(D_{k-1,d})$, where $D_{0,d} := \emptyset$, the theorem follows.

Lemma 1. Let $D_{k,d} := \bigcup_{i=1}^k C_{t,d}^{(i)}$. Then, for $x, y \in \mathbb{R}^d$,

$$E[\mathbf{1}_{D_{k,d}}(x)\mathbf{1}_{D_{k,d}}(y)] \geq E[\mathbf{1}_{D_{k,d}}(x)]E[\mathbf{1}_{D_{k,d}}(y)].$$

Proof. Let $N_1 := \#(\Pi \cap (B_x \setminus B_y))$ (the number of Π -points in $B_x \setminus B_y$), $N_2 := \#(\Pi \cap (B_y \setminus B_x))$, and $N_3 := \#(\Pi \cap B_x \cap B_y)$. Then N_1, N_2 , and N_3 are independent Poisson random variables, and $\{x \in D_{k,d}\} = \{N_1 + N_3 \leq k - 1\}$ and $\{y \in D_{k,d}\} = \{N_2 + N_3 \leq k - 1\}$. It can be readily shown that the conditional distribution of N_3 given that $N_1 + N_3 \leq k - 1$ is stochastically smaller than the (unconditional) distribution of N_3 , which implies that the conditional distribution of $N_2 + N_3$ given that $N_1 + N_3 \leq k - 1$ is stochastically smaller than the (unconditional) distribution of $N_2 + N_3$. It follows that

$$P(N_2 + N_3 \leq k - 1 \mid N_1 + N_3 \leq k - 1) \geq P(N_2 + N_3 \leq k - 1),$$

which is equivalent to the desired inequality, completing the proof.

For the lemmas below, assume that $\lambda = 1$. For given $-1 \leq w \leq 1$, $\rho_1 > 0$, and $\rho_2 > 0$, let $x, y \in \mathbb{R}^d$ be any points satisfying

$$\rho_1 = \mu_d(B_x) = v_d \|x\|_d^d, \quad \rho_2 = \mu_d(B_y) = v_d \|y\|_d^d, \quad w = \frac{\langle x, y \rangle_d}{\|x\|_d \|y\|_d}, \tag{9}$$

where $v_d = \pi^{d/2} / \Gamma(d/2 + 1)$ is the d -volume of the unit ball in \mathbb{R}^d . Let

$$P_{k,d}(w, \rho_1, \rho_2) := P\{\#(\Pi \cap B_x) \leq k - 1, \#(\Pi \cap B_y) \leq k - 1\},$$

which depends on x and y only through w, ρ_1 , and ρ_2 . Note that

$$P_{k,d}(w, \rho_1, \rho_2) = E[\mathbf{1}_{D_{k,d}}(x)\mathbf{1}_{D_{k,d}}(y)] \tag{10}$$

for x and y satisfying (9). Let $W_d := \langle X, Y \rangle_d$, where X and Y are independent and uniformly distributed on the unit sphere S^{d-1} in \mathbb{R}^d . It can be shown (cf. [11, p. 742]) that the density function of $W_d, f_d(w)$, is proportional to $(1 - w^2)^{(d-3)/2}$, $-1 \leq w \leq 1$.

Lemma 2. Assume that $\lambda = 1$. We have

$$E[\mu_d(D_{k,d})]^2 = \int_0^\infty \int_0^\infty \int_{-1}^1 P_{k,d}(w, \rho_1, \rho_2) f_d(w) dw d\rho_1 d\rho_2.$$

Proof. For notational simplicity, write $\mu = \mu_d$, $D = D_{k,d}$, and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_d$. We have

$$\begin{aligned} E[\mu(D)]^2 &= \int_{(\mathbb{R}^d)^2} E[\mathbf{1}_D(x)\mathbf{1}_D(y)] dx dy \\ &= \int_0^\infty \int_0^\infty M(r_1, r_2) \sigma_d r_1^{d-1} dr_1 \sigma_d r_2^{d-1} dr_2, \end{aligned} \tag{11}$$

where $\sigma_d = 2\pi^{d/2} / \Gamma(d/2)$ is the $(d - 1)$ -volume (surface area) of the unit sphere S^{d-1} , and $M(r_1, r_2)$ is the average value of $E[\mathbf{1}_D(x)\mathbf{1}_D(y)]$ over $x \in r_1 S^{d-1}$ and $y \in r_2 S^{d-1}$ (spheres of radii r_1 and r_2). To find $M(r_1, r_2)$, note that $E[\mathbf{1}_D(x)\mathbf{1}_D(y)]$ depends on $x \in r_1 S^{d-1}$ and $y \in r_2 S^{d-1}$ only through r_1, r_2 , and $w = \langle x, y \rangle / r_1 r_2$. For given w, r_1 , and r_2 , let

$$L(w, r_1, r_2) := E[\mathbf{1}_D(x)\mathbf{1}_D(y)] \tag{12}$$

for any $x \in r_1 S^{d-1}$ and $y \in r_2 S^{d-1}$ satisfying $w = \langle x, y \rangle / r_1 r_2$. Then the average value $M(r_1, r_2)$ of $E[\mathbf{1}_D(x)\mathbf{1}_D(y)]$ over $x \in r_1 S^{d-1}$ and $y \in r_2 S^{d-1}$ equals

$$\int_{-1}^1 L(w, r_1, r_2) f_d(w) dw,$$

since $f_d(w)$ is the density of $w = \langle x, y \rangle / r_1 r_2$ with x/r_1 and y/r_2 interpreted as independent random variables uniformly distributed on S^{d-1} . Thus, by (11),

$$E[\mu(D)]^2 = \int_0^\infty \int_0^\infty \int_{-1}^1 L(w, r_1, r_2) f_d(w) dw \sigma_d r_1^{d-1} dr_1 \sigma_d r_2^{d-1} dr_2. \tag{13}$$

Letting $\rho_i := \sigma_d r_i^d / d (= v_d r_i^d)$, $i = 1, 2$, note that x and y satisfy $x \in r_1 S^{d-1}$, $y \in r_2 S^{d-1}$, and $w = \langle x, y \rangle / r_1 r_2$ if and only if x and y satisfy (9). So, by (10) and (12), $L(w, r_1, r_2) = P_{k,d}(w, \rho_1, \rho_2)$, which together with (13) completes the proof.

Lemma 3. For $-1 \leq w \leq 1$, $\rho_1 > 0$, and $\rho_2 > 0$, let x and y be any points satisfying (9). Then

$$\mu_d(B_x \cap B_y) \leq \left(\frac{1+w}{2}\right)^{d/2} \max\{\rho_1, \rho_2\}.$$

Proof. Let $\rho := \max\{\rho_1, \rho_2\}$, $x' := (\rho/\rho_1)^{1/d} x$, $y' := (\rho/\rho_2)^{1/d} y$, so that $\mu_d(B_{x'}) = \mu_d(B_{y'}) = \rho$ and $\langle x', y' \rangle_d / \|x'\|_d \|y'\|_d = w$, and $B_x \cap B_y \subset B_{x'} \cap B_{y'}$. Since the angle θ between the line segments $\overline{x'\mathbf{0}}$ and $\overline{y'\mathbf{0}}$ satisfies $\cos \theta = w$ ($0 \leq \theta \leq \pi$), the intersection of $B_{x'} \cap B_{y'}$ with the $(d - 1)$ -dimensional hyperplane passing through $(x' + y')/2$ and perpendicular to $\overline{x'y'}$, is a $(d - 1)$ -dimensional ball centered at $(x' + y')/2$ with radius $\|(x' + y')/2\|_d = \|x'\|_d \cos(\theta/2) = (\rho/v_d)^{1/d} \cos(\theta/2)$. It is readily seen that $B_{x'} \cap B_{y'}$ is contained in the (d) -dimensional ball centered at $(x' + y')/2$ with radius $(\rho/v_d)^{1/d} \cos(\theta/2)$. It follows that

$$\mu_d(B_x \cap B_y) \leq \mu_d(B_{x'} \cap B_{y'}) \leq \rho \cos^d\left(\frac{\theta}{2}\right) = \rho \left(\frac{1+w}{2}\right)^{d/2},$$

completing the proof.

Lemma 4. Assume that $\lambda = 1$. As $d \rightarrow \infty$, $E[\mu_d(D_{k,d})]^2 \rightarrow k^2$.

Proof. For $\rho_1 > 0$ and $\rho_2 > 0$, let

$$G_d(\rho_1, \rho_2) := \int_{-1}^1 P_{k,d}(w, \rho_1, \rho_2) f_d(w) dw, \tag{14}$$

so that, by Lemma 2,

$$E[\mu_d(D_{k,d})]^2 = \int_0^\infty \int_0^\infty G_d(\rho_1, \rho_2) d\rho_1 d\rho_2. \tag{15}$$

By (10), for x and y satisfying (9),

$$P_{k,d}(w, \rho_1, \rho_2) = E[\mathbf{1}_{D_{k,d}}(x)\mathbf{1}_{D_{k,d}}(y)] \leq E[\mathbf{1}_{D_{k,d}}(z)] = P\{\#\{\Pi \cap B_z\} \leq k - 1\},$$

where $z := x\mathbf{1}_{\rho_1 \geq \rho_2} + y\mathbf{1}_{\rho_1 < \rho_2}$. Since $\mu_d(B_z) = \max\{\rho_1, \rho_2\}$,

$$P_{k,d}(w, \rho_1, \rho_2) \leq \sum_{i=0}^{k-1} \frac{(\max\{\rho_1, \rho_2\})^i}{i!} e^{-\max\{\rho_1, \rho_2\}} \leq c_k e^{-\max\{\rho_1, \rho_2\}/2}, \tag{16}$$

where

$$c_k := \sup_{u>0} \sum_{i=0}^{k-1} \frac{u^i}{i!} e^{-u/2} < \infty. \tag{17}$$

It follows by (14) that $G_d(\rho_1, \rho_2) \leq c_k e^{-\max\{\rho_1, \rho_2\}/2}$. If we can show that

$$\lim_{d \rightarrow \infty} G_d(\rho_1, \rho_2) = \left(\sum_{i=0}^{k-1} \frac{\rho_1^i}{i!} e^{-\rho_1}\right) \left(\sum_{i=0}^{k-1} \frac{\rho_2^i}{i!} e^{-\rho_2}\right) \text{ for } \rho_1 > 0 \text{ and } \rho_2 > 0, \tag{18}$$

then it follows from the dominated convergence theorem and (15) that

$$\lim_{d \rightarrow \infty} E[\mu_d(D_{k,d})]^2 = \int_0^\infty \int_0^\infty \left(\sum_{i=0}^{k-1} \frac{\rho_1^i}{i!} e^{-\rho_1}\right) \left(\sum_{i=0}^{k-1} \frac{\rho_2^i}{i!} e^{-\rho_2}\right) d\rho_1 d\rho_2 = k^2.$$

It remains to establish (18). For fixed ρ_1 and ρ_2 , we have, by (16),

$$\int_{1/2}^1 P_{k,d}(w, \rho_1, \rho_2) f_d(w) dw \leq c_k e^{-\max\{\rho_1, \rho_2\}/2} \int_{1/2}^1 f_d(w) dw \rightarrow 0 \text{ as } d \rightarrow \infty. \tag{19}$$

For $-1 \leq w \leq \frac{1}{2}$, we have, by (10) with x and y satisfying (9),

$$\begin{aligned} &P_{k,d}(w, \rho_1, \rho_2) \\ &= P\{\#\{\Pi \cap B_x\} \leq k - 1, \#\{\Pi \cap B_y\} \leq k - 1\} \\ &\leq P\{\#\{\Pi \cap B_x \cap B_y\} > 0\} \\ &\quad + P\{\#\{\Pi \cap (B_x \setminus B_y)\} \leq k - 1, \#\{\Pi \cap (B_y \setminus B_x)\} \leq k - 1\} \\ &= 1 - e^{-u(w)} + \left(\sum_{i=0}^{k-1} \frac{(\rho_1 - u(w))^i}{i!} e^{-\rho_1 + u(w)}\right) \left(\sum_{i=0}^{k-1} \frac{(\rho_2 - u(w))^i}{i!} e^{-\rho_2 + u(w)}\right) \\ &\rightarrow \left(\sum_{i=0}^{k-1} \frac{\rho_1^i}{i!} e^{-\rho_1}\right) \left(\sum_{i=0}^{k-1} \frac{\rho_2^i}{i!} e^{-\rho_2}\right) \text{ as } d \rightarrow \infty, \end{aligned} \tag{20}$$

uniformly in $-1 \leq w \leq \frac{1}{2}$, where $u(w) := \mu_d(B_x \cap B_y) \leq [(1 + w)/2]^{d/2} \max\{\rho_1, \rho_2\}$ by Lemma 3. On the other hand, by Lemma 1 we have, for $-1 \leq w \leq 1$ and x and y satisfying (9),

$$\begin{aligned} P_{k,d}(w, \rho_1, \rho_2) &= E[\mathbf{1}_{D_{k,d}}(x)\mathbf{1}_{D_{k,d}}(y)] \\ &\geq E[\mathbf{1}_{D_{k,d}}(x)]E[\mathbf{1}_{D_{k,d}}(y)] \\ &= \left(\sum_{i=0}^{k-1} \frac{\rho_1^i}{i!} e^{-\rho_1}\right) \left(\sum_{i=0}^{k-1} \frac{\rho_2^i}{i!} e^{-\rho_2}\right), \end{aligned}$$

which together with (19) and (20) implies (18). The proof is complete.

Lemma 5. Assume that $\lambda = 1$. We have

$$E[\mu_d(D_{k,d})]^n \leq c_k 2^n n!, \quad n = 1, 2, \dots,$$

where c_k is as defined in (17).

Proof. Write $\mu = \mu_d, \|\cdot\| = \|\cdot\|_d$, and $D = D_{k,d}$ for notational simplicity. By Fubini’s theorem,

$$\begin{aligned} E[\mu(D)]^n &= \int_{(\mathbb{R}^d)^n} E[\mathbf{1}_D(x_1) \cdots \mathbf{1}_D(x_n)] dx_1 \cdots dx_n \\ &\leq \int_{(\mathbb{R}^d)^n} E[\mathbf{1}_D(x_{I(n)})] dx_1 \cdots dx_n, \end{aligned} \tag{21}$$

where $I(n) = I(n, x_1, \dots, x_n) := \min\{1 \leq i \leq n : \|x_i\| \geq \|x_j\| \text{ for all } 1 \leq j \leq n\}$. We have

$$E[\mathbf{1}_D(x_{I(n)})] = P\{\#\{\Pi \cap B_{x_{I(n)}}\} \leq k - 1\} = \sum_{i=0}^{k-1} \frac{u^i}{i!} e^{-u} \leq c_k e^{-u/2}, \tag{22}$$

where $u := \mu(B_{x_{I(n)}}) = \max\{\mu(B_{x_i}) : i = 1, \dots, n\}$. It follows from (21) and (22) that

$$E[\mu(D)]^n \leq c_k \int_{(\mathbb{R}^d)^n} \exp\left(-\frac{1}{2} \max_{1 \leq i \leq n} \mu(B_{x_i})\right) dx_1 \cdots dx_n = c_k n! \left(\frac{1}{2}\right)^{-n},$$

where the equality is derived in the proof of Lemma 3.1 in [1, pp. 922, 923]. The proof is complete.

Remark 9. Similarly to Theorem 1, it is easily shown from Lemma 1 that the covariance of $\mu_d(D_{k,d} \cap S_1)$ and $\mu_d(D_{k,d} \cap S_2)$ is nonnegative for any measurable subsets S_1 and S_2 of \mathbb{R}^d . (Note that the nonnegative covariance property does not hold for $C_{t,d}^{(k)}$ with $k \geq 2$.) It then follows that

$$\text{var}(\mu_d(D_{k,d} \cap S_1)) \leq \text{var}(\mu_d(D_{k,d} \cap S_2)) \leq \text{var}(\mu_d(D_{k,d})),$$

if $S_1 \subset S_2 \subset \mathbb{R}^d$. So, by Theorem 3 for any measurable sets $A_d \subset \mathbb{R}^d, \mu_d(D_{k,d} \cap A_d) - E[\mu_d(D_{k,d} \cap A_d)]$ converges in L^2 to 0 as $d \rightarrow \infty$. Since $\mu_d(C_{t,d}^{(k)} \cap A_d) = \mu_d(D_{k,d} \cap A_d) - \mu_d(D_{k-1,d} \cap A_d)$, we have $\mu_d(C_{t,d}^{(k)} \cap A_d) - E[\mu_d(C_{t,d}^{(k)} \cap A_d)]$ converges in L^2 to 0 as $d \rightarrow \infty$. By the argument in Remark 4, the L^2 convergence can be strengthened to L^p convergence for all $p > 0$.

Remark 10. Along the lines of the proof of Lemma 4 with more refined calculations, it can be shown that $\text{var}(\mu_d(D_{k,d}))$ decays to 0 at a geometric rate as $d \rightarrow \infty$, which in turn implies that $\text{var}(\mu_d(C_{t,d}^{(k)}))$ decays to 0 at a geometric rate as $d \rightarrow \infty$. Define, for $k = 1, 2, \dots$,

$$\rho_k := \inf \left\{ \rho > 0 : \sup_d \frac{\text{var}(\mu_d(C_{t,d}^{(k)}))}{\rho^d} < \infty \right\}.$$

While Alishahi and Sharifitabar [1] obtained $\rho_1 = 4/3\sqrt{3}$, it is of theoretical interest to find the exact value of ρ_k for $k \geq 2$ as well as to study the qualitative behavior of the ρ_k . In particular, it seems of interest to know whether the ρ_k are monotone in k .

Remark 11. Newman *et al.* [11, Theorem 10] considered a two-ensemble d -dimensional Poisson point process with Π_1 -points of intensity λ_1 and Π_2 -points of intensity λ_2 . Let Q_{Π_1} denote a typical Π_1 -point, and let $N_{k,d}$, $k = 1, 2, \dots$, be the number of Π_2 -points that have Q_{Π_1} as their k th nearest Π_1 -neighbor. Newman *et al.* proved that, as $d \rightarrow \infty$, $N_{1,d}$ is asymptotically Poisson with mean λ_2/λ_1 . Denote by $C_{t,d,\Pi_1}^{(k)}$ the (typical) k th nearest-point Voronoi cell centered at Q_{Π_1} generated by the Π_1 -points. Then $N_{k,d}$ is the number of Π_2 -points in $C_{t,d,\Pi_1}^{(k)}$. Since, as $d \rightarrow \infty$, $\mu_d(C_{t,d,\Pi_1}^{(k)})$ is asymptotically degenerate at $1/\lambda_1$, $k = 1, 2, \dots$, and since, conditional on $\mu_d(C_{t,d,\Pi_1}^{(k)})$, $k = 1, 2, \dots$, the $N_{k,d}$ are independent Poisson random variables with means $\lambda_2 \mu_d(C_{t,d,\Pi_1}^{(k)})$, it follows that, as $d \rightarrow \infty$, the $N_{k,d}$, $k = 1, 2, \dots$, are asymptotically independent and identically distributed Poisson with mean λ_2/λ_1 .

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