COMPACT HERMITIAN SURFACES OF POINTWISE CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

by KOUEI SEKIGAWA and TAKASHI KODA

(Received 7 April, 1994)

1. Introduction. Let $M = (M, J, g)$ be an almost Hermitian manifold and $U(M)$ the unit tangent bundle of M. Then the holomorphic sectional curvature $H = H(x)$ can be regarded as a differentiable function on $U(M)$. If the function H is constant along each fibre, then M is called a space of pointwise constant holomorphic sectional curvature. Especially, if H is constant on the whole $U(M)$, then M is called a space of constant holomorphic sectional curvature. An almost Hermitian manifold with an integrable almost complex structure is called a Hermitian manifold. A real 4-dimensional Hermitian manifold is called a Hermitian surface. Hermitian surfaces of pointwise constant holomorphic sectional curvature have been studied by several authors (cf. [**2**], [**3**], [**5**], [**6**] and so on).

In this paper, we shall prove the following.

THEOREM A. *Let $M = (M, J, g)$ be a compact Hermitian surface of pointwise constant holomorphic sectional curvature. If the scalar curvature of M is nonpositive constant, then M is an Einstein Kähler surface.*

THEOREM B. *Let $M = (M, J, g)$ be a Hermitian surface of pointwise constant holomorphic sectional curvature satisfying the condition*

$$R(X, Y) \cdot R = 0 \textit{ for any differentiable vector fields } X \textit{ and } Y. \tag{1.1}$$

If the curvature operator is non-singular at each point of M, then M is a weakly $*$-Einstein manifold.

Taking account of the solution of Yamabe's problem, the classification problem of compact self-dual (resp. anti-self-dual) Hermitian surfaces can be reduced to the one of compact self-dual (resp. anti-self-dual) Hermitian surfaces with constant scalar curvature. We may easily show that a 4-dimensional almost Hermitian manifold of pointwise constant holomorphic sectional curvature is self-dual (cf. [**2**]). Therefore Theorem A gives a partial solution to the classification problem of compact self-dual Hermitian surfaces and also a partial improvement to the previous result of the present authors ([**3**], Theorem A). In the course of the proof, we have used the following fact ([**3**], Proposition 2.1).

PROPOSITION. [**3**] *Let $M = (M, J, g)$ be a compact Einstein Hermitian surface of pointwise constant holomorphic sectional curvature. Then M is a locally conformal Kähler surface and the tensor field S defined by*

$$S(X, Y) = (\nabla_X \omega)Y - (\nabla_{JX}\omega)JY + \tfrac{1}{2}(\omega(X)\omega(Y) - \omega(JX)\omega(JY))$$

vanishes on M.

However the proof of the proposition is not right (more precisely, the equality (2.19)

1991 *Mathematics Subject Classification.* 53C55, 53C25, 53C15.

Glasgow Math. J. **37** (1995) 343–349.

in [3] is false in sign), which has been pointed out by T. Sato. We give a correct proof of the proposition after proving Theorem A and B.

The authors wish to express their gratitude to Professor T. Sato for his useful advice.

2. Preliminaries. Let $M=(M,J,g)$ be a $2n$-dimensional almost Hermitian manifold with the almost Hermitian structure (J,g), and Ω the Kähler form of M defined by $\Omega(X,Y)=g(X,JY), X, Y\in\mathscr{X}(M)$. We assume that M is oriented by the volume form $dM=\dfrac{(-1)^n}{n!}\Omega^n$. We denote by $\nabla, R, \rho, \tau, \rho^*$ and τ^* the Riemannian connection, the Riemannian curvature tensor, the Ricci tensor, the scalar curvature, the $*$-Ricci tensor and the $*$-scalar curvature of M respectively:

$$R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]},$$
$$\rho(x,y)=\text{trace of }(z\to R(z,x)y),$$
$$\tau=\text{trace of }\rho,$$
$$\rho^*(x,y)=\tfrac{1}{2}\text{trace of }(z\to R(x,Jy)Jz),$$
$$\tau^*=\text{trace of }\rho^*,$$

where $X,Y\in\mathscr{X}(M), x,y,z\in T_p(M), p\in M$.

An almost Hermitian manifold $M=(M,J,g)$ is called a weakly $*$-Einstein manifold if it satisfies $\rho^*=\lambda^*g$ for some function λ^* on M.

Now we assume that M is a Hermitian surface. Then we have

$$d\Omega=\omega\wedge\Omega,$$

where $\omega=\delta\Omega\circ J$. The 1-form ω is called the Lee form of M. The Lee form ω satisfies the following (see [7], [8]):

$$\begin{aligned}&J^{ij}\nabla_i\omega_j=0,\\ &2\nabla_iJ_{jk}=\omega_aJ_j{}^ag_{ik}-\omega_aJ_k{}^ag_{ij}+\omega_jJ_{ki}-\omega_kJ_{ji},\\ &\tau-\tau^*=2\delta\omega+\|\omega\|^2.\end{aligned}\tag{2.1}$$

Let M be a Hermitian surface of pointwise constant holomorphic sectional curvature $c=c(p)(p\in M)$. Then we have (see [5])

$$\begin{aligned}R_{ijkl}=&\frac{1}{4}\|\omega\|^2C_{ijkl}+\left(\frac{c}{4}-\frac{1}{16}\|\omega\|^2\right)H_{ijkl}\\ &+\frac{1}{96}\{g_{ik}A_{jl}-g_{il}A_{jk}+g_{jl}A_{ik}-g_{jk}A_{il}\\ &+J_{ik}B_{jl}-J_{il}B_{jk}+J_{jl}B_{ik}-J_{jk}B_{il}\\ &+2J_{ij}B_{kl}+2J_{kl}B_{ij}\},\end{aligned}$$

where

$$\begin{aligned}C_{ijkl}&=g_{il}g_{jk}-g_{ik}g_{jl},\\ H_{ijkl}&=g_{il}g_{jk}-g_{ik}g_{jl}+J_{il}J_{jk}-J_{ik}J_{jl}-2J_{ij}J_{kl},\\ A_{ij}&=21(\nabla_i\omega_j+\nabla_j\omega_i+\omega_i\omega_j)-3J_i{}^aJ_j{}^b(\nabla_a\omega_b+\nabla_b\omega_a+\omega_a\omega_b),\\ B_{ij}&=7(J_j{}^a\nabla_i\omega_a-J_i{}^a\nabla_j\omega_a)-(J_j{}^a\nabla_a\omega_i-J_i{}^a\nabla_a\omega_j)+3(J_j{}^a\omega_i\omega_a-J_i{}^a\omega_j\omega_a).\end{aligned}$$

https://doi.org/10.1017/S0017089500031621 Published online by Cambridge University Press

We put

$$T_{ij} = \nabla_i\omega_j + \nabla_j\omega_i + \omega_i\omega_j - J_i^a J_j^b(\nabla_a\omega_b + \nabla_b\omega_a + \omega_a\omega_b),$$
$$T^*_{ij} = \nabla_i\omega_j - \nabla_j\omega_i - J_i^a J_j^b(\nabla_a\omega_b - \nabla_b\omega_a). \tag{2.2}$$

Then we have

$$\rho = \frac{\tau}{4}g - \frac{1}{4}T,$$
$$\rho^* = \frac{\tau^*}{4}g + \frac{1}{4}T^*. \tag{2.3}$$

We may easily get (see [5])

$$c = \frac{\tau + 3\tau^*}{24}. \tag{2.4}$$

We have the following integral formula (see [5]).

$$\int_M \|T\|^2\,dM = \int_M (4\,\|d\omega\|^2 + 2(\tau - \tau^*)^2 - 4\tau^*\,\|\omega\|^2)\,dM. \tag{2.5}$$

PROPOSITION 2.1. [5] *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature c. Then the Euler class of M is given by*

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{12c^2 - \tfrac{1}{16}(\tau - \tau^*)^2 + \tfrac{1}{2}\tau^*\,\|\omega\|^2\}\,dM. \tag{2.6}$$

PROPOSITION 2.2. [5] *Let M be a compact Hermitian surface of pointwise constant holomorphic sectional curvature. Then the square of the first Chern class of M is given by*

$$c_1(M)^2 = \frac{1}{32\pi^2}\int_M \{(\tau^*)^2 + \tau^*\,\|\omega\|^2 + \|d\omega\|^2\}\,dM. \tag{2.7}$$

THEOREM 2.3. [4] *Let $M = (M, J)$ be a compact connected complex surface. Then we have*

$$c_1(M)^2 \leqq \max\{2c_2(M), 3c_2(M)\}. \tag{2.8}$$

3. Proof of Theorem A. In this section, we shall prove Theorem A. Before proceeding to the proof, we recall the following fact.

THEOREM 3.1. [5] *Let $M = (M, J, g)$ be a compact Hermitian surface of constant nonpositive holomorphic sectional curvature. Then M is a Kähler surface.*

We assume that $M = (M, J, g)$ is a compact Hermitian surface of pointwise constant holomorphic sectional curvature $c = c(p)$, $p \in M$.

First we assume that $c_2(M)(=\chi(M))<0$. Then Miyaoka's inequality (2.8) implies $c_1(M)^2 \leqq 2c_2(M)$. Then by (2.4), (2.6) and (2.7), we have

$$\begin{aligned} 0 &\leqq \int_M \left\{\frac{1}{24}(\tau+3\tau^*)^2 - \frac{1}{8}(\tau-\tau^*)^2 + \tau^* \|\omega\|^2 - (\tau^*)^2 - \tau^* \|\omega\|^2 - \|d\omega\|^2\right\} dM \\ &= \int_M \left\{\frac{1}{24}(-2\tau^2 + 12\tau\tau^* - 18(\tau^*)^2) - \|d\omega\|^2\right\} dM \\ &= \int_M \left\{-\frac{1}{12}(\tau-3\tau^*)^2 - \|d\omega\|^2\right\} dM \leqq 0. \end{aligned}$$

Thus we have

$$\tau = 3\tau^* \quad \text{and} \quad d\omega = 0. \tag{3.1}$$

In this case, by the assumption that M has nonpositive constant scalar curvature τ, c is nonpositive constant on M. By Theorem 3.1, M is a Kähler surface. And then we have $\tau = \tau^* = c = 0$. This contradicts $\chi(M) < 0$.

Hence it follows that $c_2(M)(=\chi(M)) \geqq 0$. Then Miyaoka's inequality implies

$$c_1(M)^2 \leqq 3c_2(M).$$

Then from (2.4), (2.6) and (2.7), we have

$$\begin{aligned} 0 \leqq \int_M \Big\{&\frac{1}{16}(\tau+3\tau^*)^2 - \frac{3}{16}(\tau-\tau^*)^2 + \frac{3}{2}\tau^* \|\omega\|^2 - (\tau^*)^2 - \tau^* \|\omega\|^2 \\ &- \frac{1}{4}\|T\|^2 + \frac{1}{2}(\tau-\tau^*)^2 - \tau^* \|\omega\|^2\Big\} dM. \end{aligned} \tag{3.2}$$

From (3.2) and (2.5) we have

$$\begin{aligned} &\int_M \left\{\frac{1}{16}(\tau+3\tau^*)^2 - (\tau^*)^2 + \frac{1}{16}(\tau-\tau^*)^2\right\} dM \\ &\qquad \geqq \int_M \left\{\frac{1}{2}\tau^* \|\omega\|^2 - \frac{1}{4}(\tau-\tau^*)^2 + \frac{1}{8}\|T\|^2\right\} dM + \frac{1}{8}\int_M \|T\|^2 \, dM \\ &\qquad = \frac{1}{2}\int_M \|d\omega\|^2 \, dM + \frac{1}{8}\int_M \|T\|^2 \, dM \geqq 0. \end{aligned} \tag{3.3}$$

The left hand side of the above inequality reduces to

$$\begin{aligned} &\int_M \left(\left(\frac{1}{4}(\tau+3\tau^*) - \tau^*\right)\left(\frac{1}{4}(\tau+3\tau^*) + \tau^*\right) + \frac{1}{16}(\tau-\tau^*)^2\right) dM \\ &= \frac{1}{16}\int_M ((\tau-\tau^*)(\tau+7\tau^*) + (\tau-\tau^*)^2) \, dM \\ &= -\frac{3}{8}\int_M (\tau-\tau^*)^2 \, dM + \frac{\tau}{2}\int_M (\tau-\tau^*) \, dM \\ &= -\frac{3}{8}\int_M (\tau-\tau^*)^2 \, dM + \frac{\tau}{2}\int_M \|\omega\|^2 \, dM \leqq 0. \end{aligned} \tag{3.4}$$

https://doi.org/10.1017/S0017089500031621 Published online by Cambridge University Press

Thus by (3.3) and (3.4), we have finally $d\omega = 0$, $T = 0$ and hence $S = 0$, where S is the tensor field defined by

$$S(X, Y) = (\nabla_X \omega)Y - (\nabla_{JX}\omega)JY + \tfrac{1}{2}(\omega(X)\omega(Y) - \omega(JX)\omega(JY)). \tag{3.5}$$

Thus, from (2.3), we see that M is an Einstein locally conformal Kähler surface and the tensor field S vanishes on M. In particular, Proposition 1.2 of [**3**] is valid in the case where the Einstein constant is nonpositive. Thus by the argument after Proposition 2.1 of [**3**], we may conclude that M is Kähler surface.

This completes the proof of Theorem A.

4. Proof of Theorem B. In this section, we shall prove Theorem B. The condition (1.1) implies

$$R_{ija}{}^{t}R_{tbcd} + R_{ijb}{}^{t}R_{atcd} + R_{ijc}{}^{t}R_{abtd} + R_{ijd}{}^{t}R_{abct} = 0. \tag{4.1}$$

Now by (2.3) we have

$$\begin{aligned}
J^{ia}J^{jc}R_{ija}{}^{t}R_{tbcd} &= \frac{1}{2}J^{ia}J^{jc}(R_{ija}{}^{t} - R_{aji}{}^{t})R_{tbcd} \\
&= -\frac{1}{2}J^{ia}J^{jc}R_{aij}{}^{t}R_{tbcd} \\
&= -\rho^{*tc}R_{tbcd} \\
&= \frac{\tau^*}{4}\rho_{bd} - \frac{1}{4}T^{*tc}R_{tbcd} \\
&= \frac{\tau^*}{4}\rho_{bd} - \frac{1}{8}T^{*tc}(R_{tbcd} - R_{cbtd}) \\
&= \frac{\tau^*}{4}\rho_{bd} - \frac{1}{8}T^{*tc}R_{tcbd},
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
J^{ia}J^{jc}R_{ijc}{}^{t}R_{abtd} &= \frac{1}{2}J^{ia}J^{jc}(R_{ijc}{}^{t} - R_{icj}{}^{t})R_{abtd} \\
&= -\frac{1}{2}J^{ia}J^{jc}R_{jci}{}^{t}R_{abtd} \\
&= \rho^{*ta}R_{abtd} \\
&= -\frac{\tau^*}{4}\rho_{bd} - \frac{1}{8}T^{*ta}R_{tabd},
\end{aligned} \tag{4.3}$$

https://doi.org/10.1017/S0017089500031621 Published online by Cambridge University Press

$$
\begin{aligned}
J^{ia}J^{jc}R_{ijb}{}^{t}R_{atcd} &= \tfrac{1}{2}J^{ia}J^{jc}R_{ijb}{}^{t}(R_{atcd}-R_{ctad})\\
&= -\tfrac{1}{2}J^{ia}J^{jc}R_{ijb}{}^{t}R_{catd}\\
&= -\tfrac{1}{2}J^{ia}J^{jc}R_{ijb}{}^{t}R_{acdt},
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
J^{ia}J^{jc}R_{ijd}{}^{t}R_{abct} &= \tfrac{1}{2}J^{ia}J^{jc}R_{ijd}{}^{t}(R_{abct}-R_{cbat})\\
&= -\tfrac{1}{2}J^{ia}J^{jc}R_{ijd}{}^{t}R_{cabt}\\
&= \tfrac{1}{2}J^{ai}J^{cj}R_{ijd}{}^{t}R_{acbt}\\
&= \tfrac{1}{2}J^{ia}J^{jc}R_{acd}{}^{t}R_{ijbt}.
\end{aligned}
\tag{4.5}
$$

Thus, transvecting (4.1) with $J^{ia}J^{jc}$ and taking account of (4.2)–(4.5), we have

$$R_{abcd}T^{*ab}=0. \tag{4.6}$$

Since the curvature operator is non-singular at each point of M, (4.6) implies $T^*=0$ on M. Hence by (2.3) we see that M is a weakly $*$-Einstein manifold.

This completes the proof of Theorem B.

Finally we shall prove Proposition 2.1 of [**3**]. We assume that M is a compact Einstein Hermitian surface of pointwise constant holomorphic sectional curvature $c=c(p)(p\in M)$. Taking account of the proof of Theorem A in Section 3, it suffices to consider the case where $\tau>0$. N. Hitchin proved the following.

THEOREM 4.1. [**1**] *Let $M=(M,g)$ be a 4-dimensional half-conformally flat Einstein manifold of positive scalar curvature. Then M is isometric to a 4-dimensional sphere or a complex projective space with the respective standard metric.*

Since a 4-dimensional almost Hermitian manifold of pointwise constant holomorphic sectional curvature is self-dual, then by Theorem 4.1, the manifold $M=(M,J,g)$ under consideration satisfies the conditions of Theorem B. Then from Theorem B we get $T^*=0$. On the other hand, we have (see (3.13) of [**5**])

$$\int_M J^{ia}J^{jb}\nabla_a\omega_b\nabla_i\omega_j\,dM=\int_M J^{ia}J^{jb}\nabla_a\omega_b\nabla_j\omega_i\,dM. \tag{4.7}$$

By (2.2) and (4.7) we obtain

$$\int_M \|T^*\|^2\,dM=4\int_M \|d\omega\|^2\,dM. \tag{4.8}$$

Hence we have $d\omega=0$, that is M is a locally conformal Kähler surface. Furthermore by (2.2) and (2.3) we have $S=0$, since M is assumed to be Einstein.

This completes the proof of Proposition 2.1 of [**3**].

https://doi.org/10.1017/S0017089500031621 Published online by Cambridge University Press

REFERENCES

1. N. Hitchin, Kählerian twistor spaces, *Proc. London Math. Soc.* **43** (1981), 133–150.
2. T. Koda, Self-dual and anti-self-dual Hermitian surfaces, *Kodai Math. J.* **10** (1987), 335–342.
3. T. Koda and K. Sekigawa, Self-dual Einstein Hermitian surfaces, in *Progress in Differential Geometry*, Advanced Studies in Pure Mathematics **22** (1993), 123–131.
4. Y. Miyaoka, On the Chern numbers of surfaces of general type, *Invent. Math.* **42** (1977), 225–237.
5. T. Sato and K. Sekigawa, Hermitian surfaces of constant holomorphic sectional curvature, *Math. Z.* **205** (1990), 659–668.
6. T. Sato and K. Sekigawa, Hermitian surfaces of constant holomorphic sectional curvature II, *Tamkang J. Math.* **23** (1992), 137–143.
7. K. Sekigawa, On some 4-dimensional compact almost Hermitian manifolds, *J. Ramanujan Math. Soc.* **2** (1987), 101–116.
8. F. Tricceri and I. Vaisman, On some 2-dimensional Hermitian manifolds, *Math. Z.* **192** (1986), 205–216.

Kouei Sekigawa
Department of Mathematics
Faculty of Science
Niigata University
Niigata 950-21
Japan

Takashi Koda
Department of Mathematics
Faculty of Science
Toyama University
Gofuku
Toyama 930
Japan
E-mail address: koda@sci.toyama-u.ac.jp (Takashi Koda)