Can. J. Math., Vol. XXXVI, No. 6, 1984, pp. 1021-1030

MODULARITY IN THE LATTICE OF PROJECTIONS OF A VON NEUMANN ALGEBRA

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Introduction. We say that two elements e and f of a lattice are *moderately separated* provided $e \wedge f = 0$ and both (e', f') and (f', e') are modular pairs for all $e' \leqq e$ and $f' \leqq f$. Here (e', f') a *modular pair* means that, for all $g \geqq e'$,

$$g \wedge (e' \vee f') = e' \vee (g \wedge f').$$

In the lattice of projections of a factor we show that e and f, with $e \wedge f = 0$, are modularly separated if and only if $\| (e - k)f \| < 1$ for some finite projection $k \leqq e$. From there we can show that a kind of "independence property" holds for modular separation in this case: if e and f are modularly separated and if $e \vee f$ and g are modularly separated, then e and $f \wedge g$ are modularly separated. In the lattice of projections of a von Neumann algebra which is not a factor, the necessary and sufficient condition for modular separation becomes

$$\| (e - k)fc_n \| < 1$$

for some finite projection $k \leqq e$ and some sequence (c_n) of central projections with $\sum c_n = 1$.

The condition "e and f are modularly separated" arises naturally in the study of co-ordinatization of lattices which are not modular. It may be possible to prove a generalized von Neumann co-ordinatization theorem [**9**] for lattices in which modular separation has reasonable properties (strong enough to imply, for example, the independence property of Section 3). In this general geometric context, the lattice of projections of a von Neumann algebra can be regarded as a testing ground.

On the other hand, if we are interested in the von Neumann algebras themselves, the properties of modular separation proved in this paper may be regarded as the first step in determining when an isomorphism of projection lattices "extends" to an isomorphism (not necessarily a *-isomorphism) of algebras. In fact, H. A. Dye, [**3**], and J. Feldman, [**4**] and [**5**], utilized certain features of the von Neumann co-ordinatization theorem in their proof that (in the absence of a type I_2 part) an

Received July 8, 1982 and in revised form November 2, 1983.

orthomorphism of projection lattices extends to a *-isomorphism of algebras. In this connection it is important that the lattice theoretic property modular separation is equivalent to an algebraic property. For example, if ϕ is an automorphism of the lattice of projections of a type III factor, we see immediately that $\|ef\| < 1$ implies

$$\| (\phi(e))(\phi(f)) \| < 1.$$

We begin, in Section 1, with a proof that the condition

$$\| (e - k)f\| < 1 \quad \text{for some finite } k \leqq e$$

is sufficient for e and f to be modularly separated. The proof proceeds easily by standard techniques (see especially [**6**]). In Section 2 we prove the necessity of the condition in the factor case. The heart of the proof is a delicate adaptation of the classical construction in a Hilbert space of two closed subspaces whose linear sum fails to be closed. In Section 3 we show that modular separation has an independence property in the case of projection lattices of factors. In Section 4 we discuss the non-factor case.

Notations. Our definition of modular pair is the dual of Birkhoff's [**1**] but coincides with Mackey's [7]. We denote the Hilbert space on which the von Neumann algebra $\mathscr{A}$ acts by $\mathscr{H}$. We write $\mathscr{L}(H)$ for the algebra of all bounded linear operators on H, and $\mathscr{A}'$ for the commutant of $\mathscr{A}$ in $\mathscr{L}(H)$. $\mathscr{A}^p$ will denote the lattice of projections of $\mathscr{A}$. For e in $\mathscr{A}^p$ we write $[e]$ to mean eH, the range of e. $\sim$ denotes the Murray-von Neumann equivalence relation on $\mathscr{A}^p$ [**8**]: $e \sim f$ means that there exists u in $\mathscr{A}$ with $e = u^*u$ and $f = uu^*$. e in $\mathscr{A}^p$ is called *finite* provided $e \sim f \leqq e$ implies $e = f$. If $\mathscr{A}$ is a factor dim denotes a dimension function on $\mathscr{A}^p$; we assume, in the type I case, that $\dim(e) = 1$ if e is an atom. In general we follow the standard notation of [**2**].

1. Sufficiency.

THEOREM 1. *Suppose that $\mathscr{A}$ is a von Neumann algebra and that e and f are projections in $\mathscr{A}$ with*

$$e \wedge f = 0 \quad \textit{and} \quad \| (e - e_1)f\| < 1$$

for some finite e_1 in $\mathscr{A}^P$ with $e_1 \leqq e$. Then e and f are modularly separated in $\mathscr{A}^P$.

Proof. Let us assume that we have proved Lemmas 1 to 5 which follow. Because the condition is symmetric on e and f (Lemma 2) and evidently holds for $e' \leqq e$ and $f' \leqq f$, it is sufficient to show that (e, f) is a modular pair. Suppose that $g \geqq e$. We note that $(e_1,(e - e_1) \vee f)$ and $(e - e_1, f)$ are modular pairs by Lemmas 5 and 1. Therefore:

$$
\begin{aligned}
g \wedge (e \vee f) &= g \wedge (e_1 \vee (e - e_1) \vee f) \\
&= e_1 \vee (g \wedge ((e - e_1) \vee f)) \\
&= e_1 \vee (e - e_1) \vee (g \wedge f) = e \vee (g \wedge f).
\end{aligned}
$$

Then the proof is completed by the following lemmas, in which we are working within the lattice $\mathscr{A}^P$ and e, f and g_i denote elements of $\mathscr{A}^P$.

LEMMA 1. *$\|ef\| < 1$ if and only if $[e] + [f]$ is closed. If $\|ef\| < 1$ then (e, f) is a modular pair and $[e \vee f] = [e] + [f]$.*

Proof. This is well known. See [7] for the lattice $(\mathscr{L}(\mathscr{H}))^P$, and notice that $\mathscr{A}^P$ is a sub-lattice of $(\mathscr{L}(\mathscr{H}))^P$.

LEMMA 2. *If f_1 is finite with*

$$f_1 \leqq f \quad \text{and} \quad \|e(f - f_1)\| < 1,$$

then there exists a finite e_1 with

$$e_1 \leqq e \quad \text{and} \quad \|(e - e_1)f\| < 1.$$

Proof.

$$
\begin{aligned}
(e \wedge (1 - f_1))f &= (e \wedge (1 - f_1))(e(1 - f_1)f) \\
&= (e \wedge (1 - f_1))(e(f - f_1)).
\end{aligned}
$$

Therefore $\|e(f - f_1)\| < 1$ implies

$$\|(e \wedge (1 - f_1))f\| < 1.$$

Take $e_1 = e - e \wedge (1 - f_1)$. Then $\|(e - e_1)f\| < 1$ and

$$e_1 = e - e \wedge (1 - f_1) \sim e \vee (1 - f_1) - (1 - f_1) \leqq f_1$$

so e_1 is finite if f_1 is finite.

LEMMA 3. *If f is finite and $g_1 \leqq g_2$ with each g_i a complement for f in some $g \geqq f$, then $g_1 = g_2$.*

Proof. Since each g_i is a complement for f in g, we have

$$f \sim g - g_1 \text{ and } f \sim g - g_2.$$

Therefore $g - g_1$ is finite and

$$g - g_1 \sim g - g_2 \leqq g - g_1,$$

which shows that $g_1 = g_2$.

LEMMA 4. *If f is finite then (e, f) is a modular pair for all e.*

Proof. Suppose $g \geqq e$. Let

$$h = g \wedge (e \vee f) \text{ and } k = e \vee (g \wedge f).$$

Evidently

$$h \geqq k, h \wedge f = g \wedge f \leqq k \wedge f, k \vee f = e \vee f \geqq h \vee f:$$

thus we conclude that $k - h \wedge f$ and $h - h \wedge f$ are complements for f in $e \vee f$.

By Lemma 3 then $k - h \wedge f = h - h \wedge f$ so $k = h$.

LEMMA 5. *If f is finite then (f, e) is a modular pair for all e.*

Proof. We can assume $e \vee f = 1$ and then $1 - e$ is finite. We want to show that, for all $g \geqq f$:

$$g \wedge (f \vee e) = f \vee (g \wedge e).$$

Using orthocomplementation, this is equivalent to;

$$(1 - g) \vee ((1 - f) \wedge (1 - e)) = (1 - f) \wedge ((1 - g) \vee (1 - e))$$

which is true because $(1 - g, 1 - e)$ is a modular pair by Lemma 4.

COROLLARY TO THEOREM 1. *Suppose that $\mathscr{A}$ is a von Neumann algebra and that e and f are in $\mathscr{A}^P$ with $e \wedge f = 0$. Suppose there exists a finite e_1 in $\mathscr{A}^P$ with $e_1 \leqq e$, and a sequence $(c_n)_{n \in \mathbf{N}}$ of central projections of $\mathscr{A}$ with $\sum c_n = 1$, such that for all $n \in \mathbf{N}$,*

$$\|(e - e_1)fc_n\| < 1.$$

Then e and f are modularly separated.

2. Necessity of the condition in the factor case.

THEOREM 2. *Suppose that $\mathscr{A}$ is a von Neumann algebra and that e and f are projections of $\mathscr{A}$ with $e \wedge f = 0$. Suppose further that there exist sequences $e_1, e_2, \ldots$ and $f_1, f_2, \ldots$ of non-zero projections of $\mathscr{A}$ with the following properties*:

(2.1) *For all n, $e_n \leqq e, f_n \leqq f, e_n \sim f_n \sim f_1$.*

(2.2) *$e_n \vee f_n \perp e_m \vee f_m$ for all $n \neq m$.*

(2.3) *For all n,*

$$\|e_nf_n\| < 1 \text{ and } f_ne_nf_n \geqq (1 - \delta_n^2)f_n$$

where $\delta_n \rightarrow 0$.

Then there exist projections e', f' and g in $\mathscr{A}$ with $e' \leqq e, f' \leqq f, e' < g \leqq e' \vee f'$ and $g \wedge f' = 0$. In particular e and f fail to be modularly separated in $\mathscr{A}^P$.

Proof. We can assume $\delta_n < n^{-1}$. We set $e' = \sum e_n$ and $f' = \sum f_n$. Then

https://doi.org/10.4153/CJM-1984-058-6 Published online by Cambridge University Press

$$e' \wedge f' = 0 \quad \text{and} \quad e' \vee f' = (\vee e_n) \vee (\vee f_n) = \vee (e_n \vee f_n)$$

so that by (2.2):

$$[e' \vee f'] = \bigoplus_{n=1}^{\infty} [e_n \vee f_n].$$

By (2.1) and $e \wedge f = 0$, there exist partial isometries

$$u_n:[e_1 \vee f_1] \rightarrow [e_n \vee f_n]$$

with $u_n[e_1] = [e_n]$; i.e.,

$$u_n u_n^* = e_n \vee f_n,\ u_n^* u_n = e_1 \vee f_1 \text{ and } u_n e_1 u_n^* = e_n.$$

Fix $\phi_1 \in [e_1 \vee f_1]$ with $\|\phi_1\| = 1$ and $e_1\phi_1 = 0$. Let $\phi_n = u_n\phi_1$. Then

$$\phi_n \in [e_n \vee f_n] \text{ with } \|\phi_n\| = 1 \text{ and } e_n\phi_n = 0.$$

Let h denote the orthogonal projection onto the subspace generated by $\mathscr{A}' \sum n^{-1}\phi_n$. Evidently $h \in \mathscr{A}$, $0 < h \leqq e' \vee f'$ and $he' = 0$.

By assumption (3.3) and Lemma 1,

$$[e_n \vee f_n] = [e_n] + [f_n].$$

Let p_n denote the skew projection of $[e_n \vee f_n]$ onto $[f_n]$ along $[e_n]$. Then:

(2.4) $\quad \|p_n \psi_n\| \geqq n\|\psi_n\|$ for all $\psi_n \in [e_n \vee f_n]$ with $e_n\psi_n = 0$.

For we can write

$$\psi_n = (1 - p_n)\psi_n + p_n\psi_n$$

and applying $1 - e_n$ we obtain:

$$\psi_n = (1 - e_n)p_n\psi_n$$

$$\|\psi_n\|^2 + \|e_n p_n\psi_n\|^2 = \|p_n\psi_n\|^2$$
$$\|\psi_n\|^2 = ((f_n - f_n e_n f_n)p_n\psi_n | p_n\psi_n)$$
$$\leqq \delta_n^2 \|p_n\psi_n\|^2 \leqq n^{-2}\|p_n\psi_n\|^2.$$

For ξ in $[e' \vee f'] = \oplus [e_n \vee f_n]$ write

$$\xi = \sum \xi_n \quad \text{with } \xi_n \in [e_n \vee f_n].$$

We will show the following:

(2.5) $\quad$ For all $\xi \in [h]$, $n\|\xi_n\| = \|\xi_1\|$ and $e_n\xi_n = 0$.

(2.6) $\quad$ For all $\xi \in [h] \cap ([e'] + [f'])$, $\sum n^2 \|\xi_n\|^2 < \infty$.

Evidently (2.5) and (2.6) together demonstrate that

$$[h] \cap ([e'] + [f']) = 0;$$

hence that the subspaces $[h]$, $[e']$ and $[f']$ are linearly independent. Since $[h] \perp [e']$,

$$[h] + [e'] = [h \vee e']$$

and we conclude

$$(h \vee e') \wedge f' = 0.$$

Taking $g = h \vee e'$ we have:

$$e' < g \leqq e' \vee f' \text{ and } g \wedge f' = 0.$$

To demonstrate (2.5) it evidently suffices to show that it holds for ξ of the form

$$\xi = T' \sum n^{-1} \phi_n \text{ with } T' \in \mathscr{A}'.$$

For such an ξ,

$$\xi_n = n^{-1} T' \phi_n,$$

so $e_n \xi_n = 0$ and:

$$\|\xi_n\| = n^{-1}\|T'u_n \phi_1\| = n^{-1}\|T' \phi_1\| = n^{-1}\|\xi_1\|.$$

That proves (2.5).

To demonstrate (2.6), suppose that $\xi = \xi' + \xi''$ is in $[h]$ with $\xi' \in [e']$ and $\xi'' = [f']$. Then $\xi' = \sum \xi_n'$ with $\xi_n' \in [e_n]$ and $\xi'' = \sum \xi_n''$ with $\xi_n'' \in [f_n]$: therefore

$$\xi_n = \xi_n' + \xi_n'' \text{ and } \xi_n'' = p_n\xi_n$$

in the notation of (2.4). Now by (2.5) we have $e_n\xi_n = 0$: therefore (2.4) shows

$$\|\xi_n''\| = \|p_n\xi_n\| \geqq n\|\xi_n\|.$$

From there

$$\sum n^2 \|\xi_n\|^2 \leqq \sum \|\xi_n''\|^2 = \|\xi''\|^2 < \infty$$

and we have obtained (2.6).

THEOREM 3. *Suppose that $\mathscr{A}$ is a semi-finite factor and that e and f are projections of $\mathscr{A}$ with*

$$e \wedge f = 0 \text{ and } \|(e - k)f\| = 1$$

for all finite projections k in $\mathscr{A}$ with $k \leqq e$. Then e and f fail to be modularly separated.

Proof. We can, by replacing e by $e - e \wedge (1 - f)$ and f by $f - f \wedge (1 - e)$, assume that

$$e \wedge (1 - f) = f \wedge (1 - e) = 0.$$

Then, letting ef have polar decomposition $ef = up$, we obtain a partial isometry $u:f \to e$, i.e., with $uu^* = e$ and $u^*u = f$. Let $p^2 = fef$ have spectral resolution $E(\lambda)$. Suppose $0 < \lambda < 1$ and $k = 1 - E(\lambda)$: then evidently

$$\|e(f - k)\| < 1$$

so by assumption k must be infinite. Thus given any sequence $\delta_n > 0$, $\delta_n \rightarrow 0$ we can choose a sequence f'_n of mutually orthogonal projections of $\mathscr{A}$ such that:

(2.8) $\quad f'_n \leqq f$ and $\dim(f'_n) \geqq 1$

(2.9) $\quad pf'_n = f'_n p$

(2.10) $\quad f'_n \leqq E(1 - \epsilon_n) - E(1 - \delta_n^2)$ where $\epsilon_n > 0$.

In fact f'_n can be taken to be $E(\mu_n) - E(\lambda_n)$ where $0 \leqq \lambda_n < \mu_n < 1$ and λ_n and μ_n are chosen inductively:

$$\lambda_n = \max (1 - \delta_n^2, \mu_{n-1})$$

and then μ_n large enough to satisfy (2.8). This uses the fact that $1 - E(\lambda_n)$ is infinite and that

$$E(\mu) - E(\lambda_n) \uparrow 1 - E(\lambda_n) \quad \text{as } \mu \uparrow 1,$$

which holds because 1 is not an eigenvalue of fef.

Now, since $\mathscr{A}$ is a semi-finite factor, we can choose f_n in $\mathscr{A}$ so that $f_n \leqq f'_n$ and $\dim f_n = 1$. Let $e_n = u f_n u^*$. We proceed to verify that e_n and f_n satisfy the hypothesis of Theorem 2. Evidently (2.1) holds. Since

$$e_n \vee f_n \leqq uf'_n u^* \vee f'_n$$

and since the f'_n's are mutually orthogonal, to verify (2.2) it suffices to show that, for $n \neq m$,

$$f'_n(u f'_m u^*) = 0.$$

Using (2.9), we see

$$f'_n(uf'_m u^*) = f'_n(fe)(uf'_m u^*) = f'_n(pu^*)(uf'_m u^*)$$
$$= f'_n p f'_m u^* = 0.$$

To verify (2.3) we begin with (2.11). First

$$f'_n p^2 = f'_n e f'_n \leqq 1 - \epsilon_n$$

so

$$\|f'_n e\| < 1 \text{ and } \|f_n e_n\| < 1.$$

Secondly:

$$f'_n p^2 f'_n \geqq (1 - \delta_n^2) f'_n$$

$$f'_n p f'_n \geqq (1 - \delta_n^2)^{1/2} f'_n$$

$$f_n p f_n \geqq (1 - \delta_n^2)^{1/2} f_n$$

$$f_n p f_n p f_n \geqq (1 - \delta_n^2) f_n$$

$$\begin{aligned} f_n e_n f_n &= f_n (fe)(u f_n u^*)(ef) f_n \\ &= f_n p u^* u f_n u^* u p f_n = f_n p f_n p f_n. \end{aligned}$$

We have now constructed e_n and f_n satisfying all the conditions of Theorem 2. Therefore e and f fail to be modularly separated.

THEOREM 4. *Suppose that $\mathscr{A}$ is a type* III *factor and that e and f are projections of $\mathscr{A}$ with $e \wedge f = 0$ and $\|ef\| = 1$. Then e and f fail to be modularly separated.*

Proof. We proceed as in the preceding proof, except that (2.8) is replaced by:

(2.11) $f'_n \leqq f$ and $f'_n \neq 0$.

In a type III factor every projection majorizes a separable non-zero projection (take a cyclic projection) and all separable non-zero projections are equivalent [**2**]. Thus we choose $f_n \leqq f'_n$ such that $f_n \in \mathscr{A}$ is separable and non-zero, and the proof proceeds as before.

THEOREM 5. *Suppose that $\mathscr{A}$ is a factor and that e and f are projections of $\mathscr{A}$ with $e \wedge f = 0$. Then e and f are modularly separated in $\mathscr{A}^P$ if and only if there exists a finite projection k in $\mathscr{A}$, $k \leqq e$ and $\|(e - k)f\| < 1$.*

COROLLARY. *Suppose that $\mathscr{A}$ is a factor of type* I *or* III, *and that ϕ is a lattice automorphism of $\mathscr{A}^P$. Then $\|ef\| = 1$ if and only if*

$$\|\phi(e)\phi(f)\| = 1.$$

Remark. The corollary fails for $\mathscr{A}$ of type II ([**4**, p. 62]).

3. The independence property for modular separation.

THEOREM 6. *Suppose that $\mathscr{A}$ is a factor and that e, f and g are elements of $\mathscr{A}^P$. If e and f are modularly separated and $e \vee f$ and g are modularly separated, then e and $f \vee g$ are modularly separated.*

Proof. Assume that e and f are modularly separated and that $e \vee f$ and g are modularly separated. Then

$$e \wedge f = 0 \text{ and } (e \vee f) \wedge g = 0.$$

It follows that:

$$e \wedge (f \vee g) = e \wedge (f \vee ((e \vee f) \wedge g)) = e \wedge f = 0$$

where we have used that (g, f) is a modular pair, true because $e \vee f$ and g are modularly separated.

Now we will confirm the condition of Theorem 5 for $e, f \vee g$.

Suppose first that $\|ef\| < 1$ and $\|(e \vee f)g\| < 1$. By Lemma 1,

$$[e \vee f] = [e] + [f], [(e \vee f) \vee g)] = [e \vee f] + [g],$$

and, because $\|fg\| < 1$,

$$[f \vee g] = [f] + [g].$$

Therefore

$$\begin{aligned}[e] + [f \vee g] &= [e] + ([f] + [g]) \\ &= [e \vee f] + [g] = [(e \vee f) \vee g]\end{aligned}$$

is closed. Hence $\|e(f \vee g)\| < 1$ by Lemma 1 again.

In the general case we have e_1 finite with $e_1 \leqq e$, and g_1 finite with $g_1 \leqq g$ such that

$$\|(e - e_1)f\| < 1 \text{ and } \|(g - g_1)((e - e_1) \vee f)\| < 1.$$

According to the preceding paragraph, then,

$$\|(e - e_1)(f \vee (g - g_1))\| < 1.$$

Now $(f \vee (g - g_1)) \vee g_1 = f \vee g$ implies that

$$f \vee (g - g_1) = f \vee g - k$$

with k finite. By Theorem 5 we conclude that e is modularly separated from $f \vee g$.

4. The non-factor case.

THEOREM 7. *Suppose $\mathscr{A}$ is a von Neumann algebra, and that e and f are in $\mathscr{A}^P$ with $e \wedge f = 0$. Then e and f are modularly separated if and only if there exits a finite e_1 in $\mathscr{A}^P$ with $e_1 \leqq e$ and a sequence $(c_n)_{n \in \mathbf{N}}$ of central projections of $\mathscr{A}$ such that $\sum c_n = 1$ and:*

$$\|(e - e_1)f\, c_n\| < 1 \text{ for all } n \in \mathbf{N}.$$

COROLLARY. *If e is modularly separated from f, and $e \vee f$ is modularly separated from g, then e is modularly separated from $f \vee g$.*

Notes on the proof. The corollary can be proved from the theorem as in the proof of Theorem 6.

The sufficiency of the condition for modular separation in Theorem 7 is the corollary to Theorem 1. If the condition fails, then it is easy to see that there exists a central projection $c \neq 0$ such that

$$\|(e - e_1)f\, c_1\| = 1$$

for all central projections $c_1 \leqq c$ and all finite projections $e_1 \leqq e$. We can then construct sequences of projections satisfying the conditions of Theorem 2 as we did in the proofs of Theorems 3 and 4. This seems to require the use of the center-valued trace and a measure on the center.

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