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A CHEBYSHEV PSEUDO-SPECTRAL METHOD FOR SOLVING FRACTIONAL-ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

A Chebyshev pseudo-spectral method for solving numerically linear and nonlinear fractional-order integro-differential equations of Volterra type is considered. The fractional derivative is described in the Caputo sense. The suggested method reduces these types of equations to the solution of linear or nonlinear algebraic equations. Special attention is given to study the convergence of the proposed method. Finally, some numerical examples are provided to show that this method is computationally efficient, and a comparison is made with existing results.

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1. Introduction

Many phenomena in engineering, physics, chemistry, and other sciences can be described very successfully by models using fractional calculus, that is, the theory of derivatives and integrals of fractional noninteger order. This allows us to describe real objects more accurately than by the classical "integer" methods; for more details see [7, 15, 19]. Moreover, fractional calculus is applied to model frequency-dependent damping behaviour of many viscoelastic materials [2], economics [3] and dynamics of interfaces between nanoparticles and substrates [5]. Recently, several numerical methods to solve fractional differential equations (FDEs) and fractional integro-differential equations (FIDEs) have been given [2, 4, 6, 8–10, 12, 14–25].

Chebyshev polynomials are a well-known family of orthogonal polynomials on the interval $[-1, 1]$ that have many applications [15]. They are widely used

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because of their properties in the approximation of functions. Recently, Kadem and Baleanu [13] presented a new procedure using Chebyshev polynomials for solving fractional integro-differential equations where a one-dimensional FIDE was converted into a system of FDEs based on the use of the Chebyshev polynomials. Kadem and Baleanu [14] also discussed fractional transport in three dimensions from the perspective of the new method in [13] based on the combined use of the Walsh function and Chebyshev polynomials of the first kind. Yuanlu [25] derived the Chebyshev wavelet operational matrix of the fractional integration and used it to solve a nonlinear fractional differential equation. In this work the FDE is converted into a FIDE via fractional integration; subsequently, the various signals involved in the fractional integral equation are approximated by representing them as linear combinations of the wavelet functions and truncating at optimal levels; finally, the integral equation is converted to an algebraic equation by introducing the wavelet operational matrix of the fractional integration.

In this paper we obtain the solution of FIDEs by using a different expansion. The proposed expansion is more direct or simpler (or both) than the existing methods we have discussed. We are concerned with the numerical study of the following nonlinear fractional integro-differential equation:

$$D^{\alpha} y(x) = F\left(x, y(x), \int_0^x K(t, y(t))\, dt\right), \quad 0 < x < 1, \alpha > 0, \tag{1.1}$$

subject to the following boundary conditions:

$$y(0) = \gamma_0, \quad y''(0) = \gamma_2, \tag{1.2}$$

$$y(1) = \beta_0, \quad y''(1) = \beta_2, \tag{1.3}$$

where $D^{\alpha} y(x)$ indicates the αth Caputo fractional derivative of $y(x)$, F is a nonlinear continuous function, and γ_0, γ_2, β_0 and β_2 are real constants. We point out that, in the case $\alpha = 4$, the fractional equation reduces to the classical fourth-order integro-differential equation. Arikoglu and Ozkol [1] extended the fractional differential transform method (FDTM), which is a semi-analytical numerical technique, to solve the above problem.

This paper is organized as follows. In Section 2 we introduce some necessary definitions and mathematical tools of fractional calculus theory which are required for our subsequent development. In Section 3 we obtain the approximation of the fractional derivative $D^{\alpha} y(x)$. Section 4 summarizes the application of the Chebyshev collocation method to the solution of (1.1). As a result, a set of algebraic equations are formed and the solution of the considered problem is introduced. In Section 5, some numerical results are given to clarify the method.

2. Basic definitions of fractional derivatives

DEFINITION 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_{μ}, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in [0, \infty)$, and it is said to be in the space C_{μ}^{m} if and only if $f^{(m)} \in C_{\mu}$, $m \in \mathbb{N}$.

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DEFINITION 2.2. The fractional-order derivative $D^{\alpha} f(x)$ in the Caputo sense is defined as follows:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt, \qquad x > 0,\, n-1 < \alpha < n, \quad (2.1)$$

where $\alpha > 0$ is the order of the derivative and $n \in \mathbb{N}$ is the smallest integer greater than α, and $f \in C_{-1}^{n}$.

For the Caputo derivative we have [19]

$$D^{\alpha} C = 0, \qquad C \text{ is a constant}, \quad (2.2)$$

$$D^{\alpha} x^{\beta} = \begin{cases} 0 & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \alpha \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha} & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq \lceil \alpha \rceil, \end{cases} \quad (2.3)$$

where we use the ceiling function $\lceil \alpha \rceil$ to denote the smallest integer greater than or equal to α, and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order. Like integer-order differentiation, Caputo's fractional differentiation is a linear operation:

$$D^{\alpha}(\lambda f(x) + \mu g(x)) = \lambda D^{\alpha} f(x) + \mu D^{\alpha} g(x),$$

where λ and μ are constants.

3. An approximate Caputo derivative using Chebyshev series expansion

The well-known Chebyshev polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula [11, 15]:

$$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x), \qquad n = 1, 2, \ldots, \quad (3.1)$$

where $T_0(x) = 1$ and $T_1(x) = x$. The analytical form of the Chebyshev polynomial of degree n is given by [15]

$$T_n(x) = \frac{n}{2} \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \frac{(n-r-1)!}{r!(n-2r)!} (2x)^{n-2r}. \quad (3.2)$$

In order to apply the Chebyshev polynomials in the interval $[0, 1]$, we define the shifted Chebyshev polynomials $T_n^*(x)$. These are defined in terms of the Chebyshev polynomials $T_n(x)$ by the following relation [15]:

$$T_n^*(x) = T_n(2x-1), \quad (3.3)$$

and by the following recurrence formula:

$$T_{n+1}^*(x) = 2(2x-1) T_n^*(x) - T_{n-1}^*(x), \qquad n = 1, 2, \ldots, \quad (3.4)$$

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where $T_0^*(x) = 1$ and $T_1^*(x) = 2x - 1$. The orthogonality condition is [14]

$$\int_0^1 \frac{T_n^*(x) T_m^*(x)}{\sqrt{x - x^2}}\, dx = \begin{cases} 0 & m \neq n, \\ \dfrac{\pi}{2} & m = n \neq 0, \\ \pi & m = n = 0. \end{cases}$$

Now, we can use the well-known relation,

$$T_n^*(x) = T_{2n}(\sqrt{x}),$$

and (3.2) to obtain the analytical form of the shifted Chebyshev polynomials of order n as follows:

$$T_n^*(x) = \sum_{r=0}^{n} (-1)^r 2^{2n-2r} \frac{n(2n - r - 1)!}{r!(2n - 2r)!} x^{n-r}. \tag{3.5}$$

A function $y(x) \in L_2[0, 1]$ may be expressed in terms of the shifted Chebyshev polynomials as follows:

$$y(x) = \sum_{n=0}^{\infty} c_n T_n^*(x), \tag{3.6}$$

where the coefficients c_n, $n = 1, 2, \ldots$ are given by

$$c_0 = \frac{1}{\pi} \int_0^1 \frac{f(x) T_0^*(x)}{\sqrt{x - x^2}}\, dx \quad \text{and} \quad c_n = \frac{2}{\pi} \int_0^1 \frac{f(x) T_n^*(x)}{\sqrt{x - x^2}}\, dx. \tag{3.7}$$

In practice, only the first $(m + 1)$-terms are considered. That is, for some m, $y(x)$ is approximated by

$$y_m(x) = \sum_{n=0}^{m} c_n T_n^*(x). \tag{3.8}$$

THEOREM 3.1 (Chebyshev truncation theorem). *The error in approximating $y(x)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. That is, if*

$$y_m(x) = \sum_{k=0}^{m} c_k T_k(x), \tag{3.9}$$

then, for all $y(x)$, all m, and all $x \in [-1, 1]$, we have

$$E_T(m) \equiv |y(x) - y_m(x)| \leq \sum_{k=m+1}^{\infty} |c_k|. \tag{3.10}$$

PROOF. The Chebyshev polynomials are bounded by 1, that is, $|T_k(x)| \leq 1$ for all $x \in [-1, 1]$ and for all k. This implies that the kth term is bounded by $|c_k|$. Subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds gives the theorem. □

The main approximate formula for the fractional derivative of $y(x)$ is given in the following theorem.

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THEOREM 3.2. *Let $y(x)$ be approximated by the shifted Chebyshev polynomials as in (3.8) and also suppose $\alpha > 0$. Then*

$$D^{\alpha}(y_m(x)) = \sum_{n=\lceil\alpha\rceil}^{m} \sum_{r=0}^{n-\lceil\alpha\rceil} c_n b_{n,r}^{(\alpha)} x^{n-r-\alpha}, \tag{3.11}$$

where $b_{n,r}^{(\alpha)}$ is given by

$$b_{n,r}^{(\alpha)} = (-1)^r 2^{2n-2r} \frac{n(2n-r-1)!(n-r)!}{r!(2n-2r)!\Gamma(n-r+1-\alpha)}. \tag{3.12}$$

PROOF. Since Caputo fractional differentiation is a linear operation we have

$$D^{\alpha}(y_m(x)) = \sum_{n=0}^{m} c_n D^{\alpha}(T_n^*(x)). \tag{3.13}$$

Now, to evaluate $D^{\alpha}(T_n^*(x))$, applying Equations (2.2), (2.3) to (3.5):

$$D^{\alpha}(T_n^*(x)) = \sum_{r=0}^{n} (-1)^r 2^{2n-2r} \frac{n(2n-r-1)!}{r!(2n-2r)!} D^{\alpha}(x^{n-r}),$$
$$n = \lceil\alpha\rceil, \lceil\alpha\rceil+1, \dots, m. \tag{3.14}$$

Since $T_n^*(x)$ is a polynomial of degree n, we have:

$$D^{\alpha}(T_n^*(x)) = 0 \quad \text{for all } n = 0, 1, 2, \dots, \lceil\alpha\rceil - 1, \alpha > 0. \tag{3.15}$$

A combination of (3.13)–(3.15) leads to the following form:

$$\begin{aligned} D^{\alpha}(y_m(x)) &= \sum_{n=\lceil\alpha\rceil}^{m} \sum_{r=0}^{n-\lceil\alpha\rceil} \frac{c_n(-1)^r 2^{2n-2r} n(2n-r-1)!(n-r)!}{r!(2n-2r)!\Gamma(n-r+1-\alpha)} x^{n-r-\alpha} \\ &= \sum_{n=\lceil\alpha\rceil}^{m} \sum_{r=0}^{n-\lceil\alpha\rceil} c_n b_{n,r}^{(\alpha)} x^{n-2r-\alpha} \end{aligned} \tag{3.16}$$

which is the desired result. □

Test example. Consider the formula (3.11) with $y(x) = x^2$, $m = 2$. The shifted series of x^2 is

$$x^2 = c_0 T_0^*(x) + c_1 T_1^*(x) + c_2 T_2^*(x) = \tfrac{3}{8} T_0^*(x) + \tfrac{1}{2} T_1^*(x) + \tfrac{1}{8} T_2^*(x),$$

and

$$\begin{aligned} D^{1/2}(x^2) &= \sum_{n=1}^{2} \sum_{r=0}^{n-1} c_n b_{n,r}^{(1/2)} x^{n-r-1/2} = c_1 b_{1,0}^{(1/2)} x^{1/2} + c_2 b_{2,0}^{(1/2)} x^{3/2} + c_2 b_{2,1}^{(1/2)}\, x^{1/2} \\ &= \frac{2}{\sqrt{\pi}} x^{1/2} + \frac{8}{3\sqrt{\pi}} x^{3/2} - \frac{2}{\sqrt{\pi}} x^{1/2} = \frac{8}{3\sqrt{\pi}} x^{3/2}, \end{aligned}$$

which is the same result as if we evaluate $D^{1/2}(x^2)$ by relation (2.3).

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4. Chebyshev collocation method

In this section we introduce a discretization formula of (1.1) using the Chebyshev collocation method. To achieve this aim, we approximate $y(x)$ as

$$y_m(x) = \sum_{n=0}^{m} c_n T_n^*(x). \tag{4.1}$$

From Equations (1.1), (4.1) and Theorem 3.2 we have

$$\sum_{n=\lceil\alpha\rceil}^{m} \sum_{r=0}^{n-\lceil\alpha\rceil} c_n b_{n,r}^{(\alpha)} x^{n-r-\alpha} = F\left(x, \sum_{n=0}^{m} c_n T_n^*(x), \int_0^x K\left(t, \sum_{n=0}^{m} c_n T_n^*(t)\right) dt\right). \tag{4.2}$$

We now collocate (4.2) at points x_p, $p = 0, 1, \ldots, m - \lceil\alpha\rceil$:

$$\sum_{n=\lceil\alpha\rceil}^{m} \sum_{r=0}^{n-\lceil\alpha\rceil} c_n b_{n,r}^{(\alpha)} x_p^{n-r-\alpha} = F\left(x_p, \sum_{n=0}^{m} c_n T_n^*(x_p), \int_0^{x_p} K\left(t, \sum_{n=0}^{m} c_n T_n^*(t)\right) dt\right). \tag{4.3}$$

For suitable collocation points we use the roots of the shifted Chebyshev polynomial $T^*_{m+1-\lceil\alpha\rceil}(x)$. In order to use the Gaussian integration formula for (4.3), we transform the t-interval $[0, x_p]$ into the τ-interval $[-1, 1]$ by means of the transformation

$$\tau = \frac{x_p}{2} t - 1.$$

Equation (4.3), for $p = 0, 1, \ldots, m - \lceil\alpha\rceil$, may be restated as

$$\begin{aligned}\sum_{n=\lceil\alpha\rceil}^{m} \sum_{r=0}^{n-\lceil\alpha\rceil} & c_n b_{n,r}^{(\alpha)}\, x_p^{n-r-\alpha} \\ &= F\left(x_p, \sum_{n=0}^{m} c_n T_n^*(x_p), \frac{x_p}{2} \int_{-1}^{1} K\left(\frac{x_p}{2}(\tau+1), \sum_{n=0}^{m} c_n T_n^*\left(\frac{x_p}{2}(\tau+1)\right)\right) d\tau\right).\end{aligned} \tag{4.4}$$

By using the Gaussian integration formula, for $p = 0, 1, \ldots, m - \lceil\alpha\rceil$, we get

$$\begin{aligned}\sum_{n=\lceil\alpha\rceil}^{m} \sum_{r=0}^{n-\lceil\alpha\rceil} & c_n b_{n,r}^{(\alpha)}\, x_p^{n-r-\alpha} \\ &= F\left(x_p, \sum_{n=0}^{m} c_n T_n^*(x_p), \frac{x_p}{2} \sum_{q=0}^{m} \omega_q K\left(\frac{x_p}{2}(\tau_q+1), \sum_{n=0}^{m} c_n T_n^*\left(\frac{x_p}{2}(\tau_q+1)\right)\right)\right),\end{aligned} \tag{4.5}$$

where the τ_q are the $m + 1$ zeros of the Chebyshev polynomial $T_{m+1}(t)$ and w_q are the corresponding weights given in [6]. The idea behind the above approximation is

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the exactness of the Gaussian integration formula for polynomials of degree not exceeding $2m+1$.

By substituting (3.16) into initial conditions or boundary conditions, we can find a further $\lceil\alpha\rceil$ equations. For example, by substituting (3.16) into the boundary conditions (1.2) and (1.3) we obtain

$$\sum_{r=0}^{m}(-1)^r c_r = \gamma_0, \quad \sum_{n=2}^{m} c_n b_{n,n}^{(2)} = \gamma_2, \tag{4.6}$$

$$\sum_{r=0}^{m} c_r = \beta_0, \quad \sum_{n=2}^{m}\sum_{k=2}^{n} c_n b_{n,k}^{(2)} = \beta_2. \tag{4.7}$$

Equation (4.5), together with $\lceil\alpha\rceil$ equations of initial conditions or boundary conditions, give $(m+1)$ nonlinear algebraic equations which can be solved for the unknowns c_n, $n = 0, 1, 2, \ldots, m$, using Newton's iterative method. Consequently, the function $y(x)$ given in (1.1) can be calculated.

5. Comparison with numerical results

In this section, we implement our proposed method to solve three examples. These examples can be used as a basis for comparison with other methods such as the Adomian decomposition method (ADM) and fractional differential transform method (FDTM).

EXAMPLE 1. Consider the following fractional integro-differential equation [1, 18]:

$$D^{0.75}y(x) = -\frac{e^x x^2}{5}y(x) + \frac{6x^{2.25}}{3.25} + \int_0^x e^t t y(t)\,dt, \tag{5.1}$$

with the initial condition,

$$y(0) = 0. \tag{5.2}$$

We implement the suggested method with $m = 6$, and we approximate the solution as

$$y_6(x) = \sum_{n=0}^{6} c_n T_n^*(x). \tag{5.3}$$

Using (4.5) we get

$$\sum_{n=1}^{6}\sum_{r=0}^{n-1} c_n b_{n,r}^{(0.75)} x_p^{n-r-0.75} \simeq -\frac{e^{x_p}x_p^2}{5}\sum_{n=0}^{6} c_n T_n^*(x_p) + \frac{6x_p^{2.25}}{3.25} + \frac{x_p}{2}e^{x_p}\sum_{q=0}^{6} w_q \frac{x_p}{2}(\tau_q+1)\sum_{j=0}^{6} c_n T_n^*\left(\frac{x_p}{2}(\tau_q+1)\right), \tag{5.4}$$

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with $p = 0, 1, \ldots, 5$, where x_p are roots of the shifted Chebyshev polynomial $T_6^*(x)$ and their values are

$$x_0 = \tfrac{1}{2} + \tfrac{1}{4}\sqrt{2}, \quad x_1 = \tfrac{1}{2} - \tfrac{1}{4}\sqrt{2}, \quad x_2 = \tfrac{1}{2} - \tfrac{1}{8}\sqrt{6} - \tfrac{1}{8}\sqrt{2},$$
$$x_3 = \tfrac{1}{2} + \tfrac{1}{8}\sqrt{6} + \tfrac{1}{8}\sqrt{2}, \quad x_4 = \tfrac{1}{2} - \tfrac{1}{8}\sqrt{6} + \tfrac{1}{8}\sqrt{2}, \quad x_5 = \tfrac{1}{2} + \tfrac{1}{8}\sqrt{6} - \tfrac{1}{8}\sqrt{2}.$$

Also τ_q are the roots of Chebyshev polynomial $T_7(x)$ and their values are

$$\tau_0 = -0.974\,927\,91, \quad \tau_1 = -0.781\,831\,48, \quad \tau_2 = -0.433\,883\,74,$$
$$\tau_3 = 0.433\,883\,739, \quad \tau_4 = 0.781\,831\,482, \quad \tau_5 = 0.974\,927\,912.$$

The w_q are the corresponding weights and their values are

$$w_0 = 0.086\,716\,18, \quad w_1 = 0.287\,831\,39,$$
$$w_2 = 0.398\,241\,54, \quad w_3 = 0.454\,421\,77,$$
$$w_4 = 0.398\,241\,54, \quad w_5 = 0.287\,831\,39, \quad w_6 = 0.086\,716\,18.$$

By using (3.16) and (5.2) we get

$$c_0 - c_1 + c_2 - c_3 + c_4 - c_5 + c_6 = 0. \tag{5.5}$$

Now solving Equations (5.4) and (5.5) we find

$$c_0 = 0.3125, \quad c_1 = 0.4688, \quad c_2 = 0.1875, \quad c_3 = 0.0313, \quad c_4 = c_5 = c_6 = 0.$$

Thus, using (5.3), we get

$$y(x) = 7 \times 10^{-21} - 2 \times 10^{-19}x + 1.1 \times 10^{-18}x^2 + x^3 + 8.96 \times 10^{-19}x^4 \simeq x^3,$$

which is the exact solution of (5.1).

EXAMPLE 2. Consider the linear fourth-order fractional integro-differential equation [1, 17],

$$D^\alpha y(x) = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(t)\,dt, \quad 0 < x < 1,\ 3 < \alpha \le 4, \tag{5.6}$$

with the following boundary conditions:

$$y(0) = 1, \quad y''(0) = 2, \quad y(1) = 1 + e, \quad y''(1) = 3e. \tag{5.7}$$

In the case $\alpha = 4$, the exact solution is known and it is given by $y(x) = 1 + xe^x$. To solve this example, we implement the method suggested in Section 4 for $\alpha = 3.25$ with $m = 20$ and for $\alpha = 3.75$ with $m = 10$. The numerical results of our method and those obtained from ADM used in [17] and FDTM considered in [1] are presented in Table 1. The behaviour of the exact and approximate solutions (our method) of this example when $\alpha = 4$ is presented in Figure 1.

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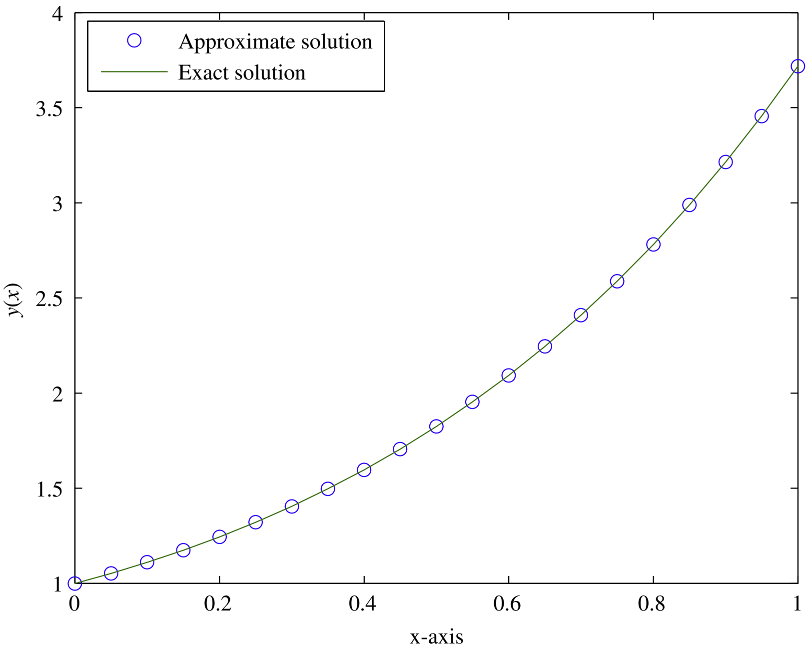

FIGURE 1. The behaviour of the exact and approximate solutions of Example 2 when $\alpha = 4$.

TABLE 1. Comparison of $y(x)$ for $\alpha = 3.25$ (left) and $\alpha = 3.75$ (right) for Example 2.

x	ADM	FDTM	Our method	x	ADM	FDTM	Our method
0.0	1.000 0000	1.000 0000	1.000 0000	0.0	1.000 0000	1.000 0000	1.000 0000
0.1	1.120 2485	1.120 8247	1.127 1325	0.1	1.115 5766	1.115 7640	1.115 7416
0.2	1.262 4009	1.263 5120	1.274 1287	0.2	1.253 7482	1.254 1102	1.254 0705
0.3	1.429 3560	1.430 9200	1.444 1923	0.3	1.417 7579	1.418 2688	1.418 2241
0.4	1.624 8579	1.626 7534	1.642 1688	0.4	1.611 5327	1.612 1543	1.612 1058
0.5	1.853 4780	1.855 5465	1.869 8097	0.5	1.839 7180	1.840 3995	1.840 3452
0.6	2.120 6551	2.122 7071	2.136 7908	0.6	2.107 7352	2.108 4152	2.108 3613
0.7	2.432 7621	2.434 5869	2.445 3183	0.7	2.421 8567	2.422 4656	2.422 4222
0.8	2.797 1905	2.798 5742	2.806 7522	0.8	2.789 2943	2.789 7592	2.789 7307
0.9	3.222 4499	3.223 2029	3.227 4548	0.9	3.218 3001	3.218 5547	3.218 5401
1.0	3.718 2818	3.718 2818	3.718 2818	1.0	3.718 2818	3.718 2818	3.718 2822

EXAMPLE 3. Consider the fourth-order, nonlinear fractional integro-differential equation [9, 17],

$$D^{\alpha} y(x) = 1 + \int_0^x e^{-t} y^2(t)\, dt, \quad 0 < x < 1,\ 3 < \alpha \le 4, \tag{5.8}$$

subject to the following boundary conditions:

$$y(0) = 1, \quad y(1) = e, \quad y''(0) = 1, \quad y''(1) = e. \tag{5.9}$$

When $\alpha = 4$ the exact solution is known and given by $y(x) = e^x$.

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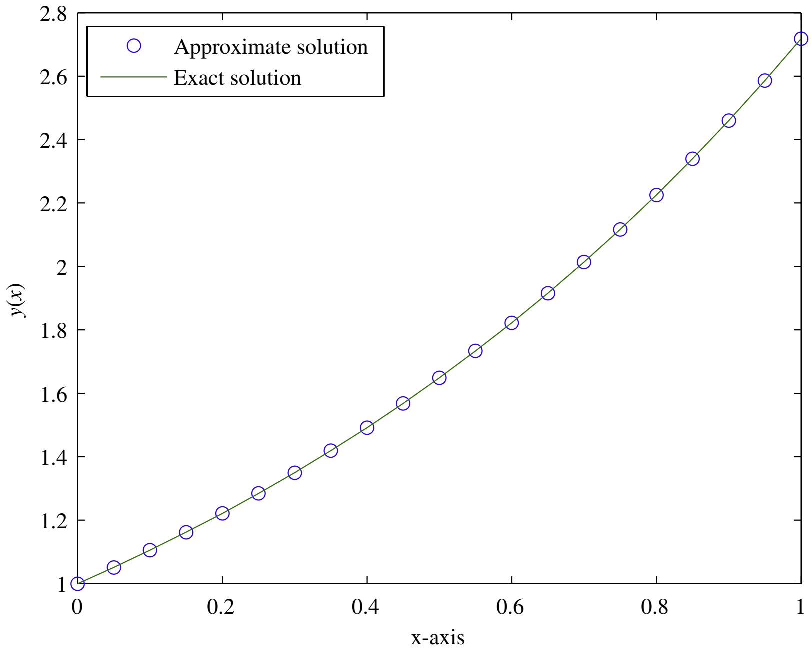

FIGURE 2. The behaviour of the exact and approximate solutions of Example 3 when $\alpha = 4$.

To solve this example, we implement the method suggested in Section 4 for $\alpha = 3.25$ with $m = 18$ and for $\alpha = 3.75$ with $m = 22$. The numerical results of our method and those obtained from ADM used in [17] and FDTM considered in [1] are given in Table 2. The behaviour of the exact solution and approximate solution (our method) of this example when $\alpha = 4$ is presented in Figure 2.

From the obtained results we can draw the following conclusions.

(1) The approximate solutions given by the proposed method are in high agreement with the exact solution for $\alpha = 4$.
(2) The results obtained by the proposed method are in excellent agreement with the results of both ADM and FDTM.
(3) The accuracy of the proposed method can be improved by using more terms of shifted Chebyshev polynomials, that is, by increasing m.

6. Summary and conclusions

In this paper, we presented a numerical method for solving the linear and nonlinear fractional integro-differential equations of Volterra type. An approximate formula for the Caputo derivative using Chebyshev series expansion was derived. The properties of Chebyshev polynomials together with the Gaussian integration method were utilized to reduce the fractional integro-differential equations to the solution of algebraic

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TABLE 2. Comparison of $y(x)$ for $\alpha = 3.25$ (left) and $\alpha = 3.75$ (right) for Example 3.

x	ADM	FDTM	Our method	x	ADM	FDTM	Our method
0.0	1.000 0000	1.000 0000	1.000 0000	0.0	1.000 0000	1.000 0000	1.000 0000
0.1	1.106 5516	1.106 6188	1.100 9350	0.1	1.106 1515	1.106 1771	1.106 5979
0.2	1.223 9319	1.224 0617	1.214 6541	0.2	1.223 2273	1.223 2768	1.224 0247
0.3	1.353 2000	1.353 3836	1.340 2542	0.3	1.352 3083	1.352 3784	1.353 3381
0.4	1.495 6009	1.495 8249	1.482 7189	0.4	1.494 6360	1.494 7215	1.495 7752
0.5	1.652 5535	1.652 8003	1.638 3820	0.5	1.651 6152	1.651 7095	1.652 7487
0.6	1.825 6547	1.825 9025	1.813 8275	0.6	1.824 8237	1.824 9185	1.825 8580
0.7	2.016 6876	2.016 9112	2.006 1210	0.7	2.016 0237	2.016 1093	2.016 8727
0.8	2.227 6345	2.227 8067	2.220 6096	0.8	2.227 1764	2.227 2425	2.227 7794
0.9	2.460 6910	2.460 7861	2.456 7504	0.9	2.460 4588	2.460 4953	2.460 7721
1.0	2.718 2818	2.718 2818	2.718 2818	1.0	2.718 2818	2.718 2818	2.718 2818

equations. Illustrative examples with comparisons between ADM, FDTM, the exact solution and the proposed method were presented.

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