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ON THE THEOREMS OF BORSUK–ULAM AND LJUSTERNIK–SCHNIRELMANN–BORSUK

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ABSTRACT. Let $p \geq 3$ be a prime number and m a positive integer, and let S be the sphere $S^{(m-1)(p-1)-1}$. Let $f: S \to S$ be a map without fixed points and with $f^p = \mathrm{id}_S$. We show that there exists an $h: S \to \mathbb{R}^m$ with $h(x) \neq h(f(x))$ for all $x \in S$. From this we conclude that there exists a closed cover $U_1, \ldots, U_{4m}$ of S with $U_i \cap f(U_i) = \varnothing$ for $i = 1, \ldots, 4m$. We apply these results to Borsuk–Ulam and Ljusternik–Schnirelmann–Borsuk theorems in the framework of the sectional category and to a problem in asymptotic fixed point theory.

1. Introduction. We consider in this paper a special type of generalizations of the following two classical results (cf. [1, 5]):

BORSUK-ULAM THEOREM. *Let*

$$n \geq m, \tag{1}$$

and let $h: S^n \to \mathbb{R}^m$ be a continuous map. Then there exists an $x \in S^n$ with $h(x) = h(-x)$.

LJUSTERNIK-SCHNIRELMANN-BORSUK THEOREM. *Let $H_1, \ldots, H_k$ be closed subsets of S^n such that $\bigcup_{i=1}^k H_i = S^n$ and $H_i \cap (-H_i) = \varnothing$ for $i = 1, \ldots, k$. Then*

$$n \leq k-2. \tag{2}$$

It is a very natural idea to replace $-\mathrm{id}$ by a map $f: S^n \to S^n$ without fixed points and with $f^p = \mathrm{id}_{S^n}$ for some (prime) number p and to ask for conditions such that

a) for any $h: S^n \to \mathbb{R}^m$, there is an $x \in S^n$ with $h(x) = h(f(x))$ or

b) there exists a covering of S^n by k closed sets $U_1, \ldots, U_k$ with $U_i \cap f(U_i) = \varnothing$ for $i = 1, \ldots, k$.

Both questions have already been extensively studied in literature, not only for spheres, but for more general spaces. For spheres, one has the following results:

GENERALIZED BORSUK-ULAM THEOREM (*cf.* [6, 7]). *Let p be a prime number,*

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$m, n \in \mathbb{N}$ with

(3) $$n \geq (m-1)(p-1)+1,$$

and let $f: S^n \to S^n$ generate a free $\mathbb{Z}_p$-action on S^n (i.e. $f^p = \mathrm{id}_{S^n}$ and f has no fixed points). Then for each continuous $h: S^n \to \mathbb{R}^m$, there exists an $x \in S^n$ with $h(x) = h(f(x))$.

GENERALIZED LJUSTERNIK-SCHNIRELMANN-BORSUK THEOREM (*cf.* [9]). *Let p be a prime number, $n, k \in \mathbb{N}$, and let $f: S^n \to S^n$ generate a free $\mathbb{Z}_p$-action on S^n. Assume that there are closed sets $H_1, \ldots, H_k \subset S^n$ with $\cup_{i=1}^k H_i = S^n$ and $H_i \cap f(H_i) = \emptyset$ for $i = 1, \ldots, k$. Then*

(4) $$n \leq \begin{cases} \frac{p-1}{2}(k-3), & \text{if } p = 3, \\ \frac{p-1}{2}(k-3)+1, & \text{if } p > 3. \end{cases}$$

The purpose of this paper is to consider the question of the optimality of the estimates (3) and (4). It is well known and easy to show that the estimates (1) and (2) are best possible. The optimality of (3) was known only in the case of $p = 3$ [2]. We will construct for all prime numbers p a free $\mathbb{Z}_p$-action φ on $S^{(m-1)(p-1)-1}$ and an $h: S^{(m-1)(p-1)-1} \to \mathbb{R}^m$ with $h(x) \neq h(\varphi(x))$ for all $x \in S^{(m-1)(p-1)-1}$. This yields the optimality of the generalized Borsuk–Ulam theorem, since for $p \geq 3$, there are no free $\mathbb{Z}_p$-actions on the even-dimensional spheres $S^{(m-1)(p-1)}$

Concerning the generalized Ljusternik–Schnirelmann–Borsuk theorem, we will see that the estimate (4) is in fact not best possible, but it is not far from the optimal one. We will see that with $\varphi: S^{(m-1)(p-1)-1} \to S^{(m-1)(p-1)-1}$ as above, there is a cover of $S^{(m-1)(p-1)-1}$ by closed sets $U_1, \ldots, U_{4m}$ with $U_i \cap \varphi(U_i) = \emptyset$ for $i = 1, \ldots, 4m$.

Our main motivation for this work was the following conjecture in asymptotic fixed point theory:

ASYMPTOTIC CONJECTURE. *Let E be a normed space, $f: E \to E$ a continuous map such that $\overline{f^{k_0}(E)}$ is compact for some $k_0 \geq 2$. Then there exists an $x \in E$ with $f(x) = x$(?).*

In [8], we described an approach to this problem via estimates of the genus $g(\mathcal{F}[f^p], f)$ (in the sense of A. S. Švarc [10, 11], cf. chapter 3 below) of the fixed point set $\mathcal{F}[f^p]$ of f^p for large prime numbers p. Unfortunately, as a consequence of the above described optimality result for the generalized Ljusternik–Schnirelmann–Borsuk covering property, we will see in chapter 5 that this approach fails. Instead we get another hint that the asymptotic conjecture might be wrong.

2. Optimality results. We start with a result, which shows that estimate (3) in the generalized Borsuk–Ulam theorem cannot be weakened:

THEOREM 1. *Let $m \in \mathbb{N}$, p a prime number,*

$$L := \left\{(x_1, \ldots, x_p) \in (\mathbb{R}^{m-1})^p \;\middle|\; \sum_{j=1}^{p} x_j = 0\right\},$$

$$S := S^{(m-1)p-1} \cap L$$

and

$$\varphi := S \to S, \varphi(x_1, \ldots, x_p) := (x_2, \ldots, x_p, x_1).$$

There exists a continuous map $h : S \to \mathbb{R}^m$ with $h(x) \neq h(\varphi(x))$ for all $x \in S$.

REMARKS. S is an $((m-1)(p-1)-1)$-dimensional sphere and φ generates a free $\mathbb{Z}_p$-action on S. Examples of such maps h on spheres $S^{(m-1)(p-1)-1}$ with a free $\mathbb{Z}_p$-action φ were already known for $p=2$ and any $m \geq 2$ (this is trivial even for S^{m-1} instead of $S^{m-2} = S^{(m-1)(p-1)-1}$) and for $p=3$ and any $m \geq 2$ (cf. [2]). On the other hand, the generalized Borsuk–Ulam theorem of E. Lusk [6] says that on $S^{(m-1)(p-1)+1}$ instead of $S^{(m-1)(p-1)-1}$, one always has a coincidence point x, i.e. $h(x) = h(\varphi(x))$. On $S^{(m-1)(p-1)}$, there are no free $\mathbb{Z}_p$-actions, if $p \geq 3$. We will describe in the next chapter how one can fill this gap.

Proof of Theorem 1. It suffices to consider $m \geq 2$. Let $\tilde{h} : S \to \mathbb{R}^{m-1}$, $\tilde{h}(x_1, \ldots, x_p) := x_1$.
Since $(x_1, \ldots, x_p) \neq 0$ and $\sum_{j=1}^{p} x_j = 0$, we have

$$\begin{aligned} d(x) &:= |(\tilde{h}(\varphi(x)) - \tilde{h}(x), \tilde{h}(\varphi^2(x)) - \tilde{h}(\varphi(x)), \ldots, \tilde{h}(\varphi^p(x)) - \tilde{h}(\varphi^{p-1}(x)))| \\ &= |(x_2 - x_1, x_3 - x_2, \ldots, x_1 - x_p)| \neq 0 \end{aligned}$$

for every $x = (x_1, \ldots, x_p) \in S$. Let $\alpha : \mathbb{R} \to [0, 1]$ be continuous with $\alpha(0) = 1$ and $\alpha(t) = 0$ for $t \geq p^{-1/2}$. Let $g : S \to \mathbb{R}$,

$$g(x) := \sum_{i=0}^{p-2} \prod_{j=0}^{i} \alpha\left(\frac{|\tilde{h}(\varphi^{p-j}(x)) - \tilde{h}(\varphi^{p-j-1}(x))|}{d(x)}\right).$$

Obviously, $d(x) = d(\varphi(x))$ for every $x \in S$. If $x \in S$ with $\tilde{h}(x) = \tilde{h}(\varphi(x))$, we have

$$\alpha\left(\frac{|\tilde{h}(\varphi^{p+1}(x)) - \tilde{h}(\varphi^p(x))|}{d(\varphi(x))}\right) = \alpha\left(\frac{|\tilde{h}(\varphi(x)) - \tilde{h}(x)|}{d(x)}\right) = \alpha(0) = 1.$$

In addition, by the definition of d and α, there must be a $j \in \{0, \ldots, p-1\}$ with

$$\alpha\left(\frac{|\tilde{h}(\varphi^{p-j}(x)) - \tilde{h}(\varphi^{p-j-1}(x))|}{d(x)}\right) = 0.$$

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We have just seen that this cannot be for $j=p-1$, thus

$$\prod_{j=0}^{p-2} \alpha\left(\frac{|\tilde{h}(\varphi^{p-j}(x))-\tilde{h}(\varphi^{p-j-1}(x))|}{d(x)}\right)=0.$$

We obtain in the case of $\tilde{h}(x)=\tilde{h}(\varphi(x))$

$$\begin{aligned} g(\varphi(x)) &= \sum_{i=0}^{p-2}\prod_{j=0}^{i} \alpha\left(\frac{|\tilde{h}(\varphi^{p-j+1}(x))-\tilde{h}(\varphi^{p-j}(x))|}{d(\varphi(x))}\right) \\ &= 1+\sum_{i=1}^{p-2}\prod_{j=1}^{i} \alpha\left(\frac{|\tilde{h}(\varphi^{p-j+1}(x))-\tilde{h}(\varphi^{p-j}(x))|}{d(\varphi(x))}\right) \\ &= 1+\sum_{i=0}^{p-3}\prod_{j=0}^{i} \alpha\left(\frac{|\tilde{h}(\varphi^{p-j}(x))-\tilde{h}(\varphi^{p-j-1}(x))|}{d(x)}\right) \\ &= 1+g(x), \end{aligned}$$

in particular $g(\varphi(x))\neq g(x)$.

Thus, $h:S\to\mathbb{R}^m$, defined by $h(x):=(\tilde{h}(x),g(x))$, has the desired properties. q.e.d.

Our next result enables us to apply Theorem 1 to the Ljusternik–Schnirelmann–Borsuk covering problem:

THEOREM 2. *Let M be a normal space, p a prime number, and let $f:M\to M$ generate a free $\mathbb{Z}_p$-action on M. Let $m\in\mathbb{N}$ such that there exists a continuous $h:M\to\mathbb{R}^m$ with $h(x)\neq h(f(x))$ for all $x\in M$. Then there exist closed sets $U_1,\ldots,U_{4m}\subset M$ with $\bigcup_{n=1}^{4m} U_n=M$ and $U_n\cap f(U_n)=\varnothing$ for $n=1,\ldots,4m$.*

Proof. Let $g:M\to S^{m-1}$,

$$g(x):=\frac{h(f(x))-h(x)}{|h(f(x))-h(x)|}.$$

For $l\in\{1,\ldots,m\}$, we define

$$\begin{aligned} R_l &:=\{x\in M \mid |(g(x))_l|\geq m^{-1/2}\}, \\ W_l^+ &:=\{x\in M \mid (g(x))_l\geq 0\}, \\ W_l^- &:=\{x\in M \mid (g(x))_l\leq 0\} \end{aligned}$$

and

$$R_l^\pm:=R_l\cap W_l^\pm.$$

We fix $l\in\{1,\ldots,m\}$ and consider first only R_l^+:

Let $a_+:M\to[0,1]$ be a continuous map with $a_+|_{R_l^+}=1$ and $a_+|_{W_l^-}=0$. We define $b_+:M\to\mathbb{R}$,

$$b_+(x):=\sum_{k=0}^{p-2}\prod_{j=0}^{k} a_+(f^{p-j}(x)).$$

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We first prove that for all $x \in M$ with $f(x) \in R_l^+$, we have $b_+(f(x)) - b_+(x) = 1$: Since $a_+(f(x)) = 1$ and since for some $j \in \{0, \ldots, p-1\}$, we must have

$$(h(f^{p-j+1}(x)) - h(f^{p-j}(x)))_l \leq 0,$$

i.e. $f^{p-j}(x) \in W_l^-$, it follows that

$$\prod_{j=0}^{p-2} a_+(f^{p-j}(x)) = \prod_{j=0}^{p-1} a_+(f^{p-j}(x)) = 0.$$

Thus,

$$\begin{aligned} b_+(f(x)) - b_+(x) &= \sum_{k=0}^{p-2} \prod_{j=0}^{k} a_+(f^{p-j+1}(x)) - \sum_{k=0}^{p-2} \prod_{j=0}^{k} a_+(f^{p-j}(x)) \\ &= 1 + \sum_{k=1}^{p-2} \prod_{j=1}^{k} a_+(f^{p-j+1}(x)) - \sum_{k=0}^{p-3} \prod_{j=0}^{k} a_+(f^{p-j}(x)) \\ &= 1 + \sum_{k=0}^{p-3} \prod_{j=0}^{k} a_+(f^{p-j}(x)) - \sum_{k=0}^{p-3} \prod_{j=0}^{k} a_+(f^{p-j}(x)) \\ &= 1. \end{aligned}$$

We define $c_+ : M \to S^1 \subset \mathbb{C}$, $c_+(x) := e^{b_+(x)\pi i}$,

$$D_d := \left\{ e^{2\pi i t} \,\middle|\, \frac{d-1}{3} \leq t \leq \frac{d}{3} \right\}$$

and

$$G_d^+ := R_l^+ \cap c_+^{-1}(D_d)$$

for $d = 1, 2, 3$.

Doing the same with a function $a_- : M \to [0, 1]$ with $a_-|_{R_l^-} = 1$ and $a_-|_{W_l^+} = 0$, we obtain in an analogous way a function $b_- : M \to \mathbb{R}$, such that for $x \in M$ with $f(x) \in R_l^-$, we have

$$b_-(f(x)) - b_-(x) = 1.$$

With $c_- : M \to S^1$, $c_-(x) := e^{b_-(x)\pi i}$, we then define for $d = 1, 2, 3$

$$G_d^- := R_l^- \cap c_-^{-1}(D_d).$$

Let $x \in M$ with $f(x) \in R_l^+$. Then

$$b_+(f(x)) = b_+(x) + 1$$

and hence $c_+(f(x)) = -c_+(x)$, which implies that there is no $d \in \{1, 2, 3\}$ with $c_+(x), c_+(f(x)) \in D_d$ simultaneously. It follows that for $d = 1, 2, 3$

$$G_d^+ \cap f(G_d^+) = \varnothing$$

and (with the same proof)

$$G_d^- \cap f(G_d^-) = \varnothing.$$

In addition, we get for $d=1$ and $d=3$

$$G_d^+\cap f(G_d^-)=\emptyset \text{ and } G_d^-\cap f(G_d^+)=\emptyset:$$

Let $x\in G_d^-\subset R_l^-\subset W_l^-$ such that $f(x)\in R_l^+$. Then $a_+(x)=0$ and therefore

$$b_+(f(x))=\sum_{k=0}^{p-2}\prod_{j=0}^{k} a_+(f^{p-j+1}(x))=a_+(f(x))=1.$$

It follows $c_+(f(x))=e^{\pi i}\in D_2\setminus(D_1\cup D_3)$, which implies

$$f(x)\in G_2^+\setminus(G_1^+\cup G_3^+).$$

Thus, $G_d^+\cap f(G_d^-)=\emptyset$. The proof of $G_d^-\cap f(G_d^+)=\emptyset$ is identical.

We define the closed sets

$$H_1:=G_1^+\cup G_1^-, H_2:=G_2^+, H_3:=G_2^-, H_4:=G_3^+\cup G_3^-.$$

By definition, it is obvious that

$$R_l=\bigcup_{i=1}^{4} H_i,$$

and we have just proved that

$$H_i\cap f(H_i)=\emptyset \text{ for } i=1,\dots,4.$$

This finishes the proof, since we know that

$$\bigcup_{l=1}^{m} R_l=M. \quad \text{q.e.d.}$$

Combining Theorem 1 and 2, we obtain

THEOREM 3. *Let m, p, S and $\varphi: S\to S$ be as in Theorem 1. Then there exist closed sets $U_1,\dots,U_{4m}\subset S$ with $\cup_{n=1}^{4m} U_n=S$ and $U_n\cap\varphi(U_n)=\emptyset$ for $n=1,\dots,4m$.*

3. Sectional category and Borsuk–Ulam and Ljusternik–Schnirelmann–Borsuk theorems. Let us first recall the notion of sectional category (this term is due to I. M. James [3]) or genus in the sense of A. S. Švarc [10, 11]:

DEFINITION 1. Let $\mathcal{H}_p$ be the class of pairs (M, f), where M is a Hausdorff space and $f: M\to M$ generates a free $\mathbb{Z}_p$-action on M. Let $\mathcal{N}_p:=\{(M,f)\in\mathcal{H}_p \mid M \text{ normal}\}$.

DEFINITION 2. Let M be a Hausdorff space. We call a set $\mathfrak{D}\subset 2^M$ an admissible covering of M, if

a) $\mathfrak{D}$ is an open covering of M,

b) there exists a family $(t_D)_{D\in\mathfrak{D}}$ of continuous maps $t_D: M\to[0,1]$ such that

i) $t_D|_{M\setminus D}=0$,

ii) for every $x\in M$ there is a $D\in\mathfrak{D}$ with $t_D(x)=1$.

Let $\mathcal{K}$ be the class of cardinal numbers and ∞ an object not in $\mathcal{K}$. Let $\mathcal{K}' := \mathcal{K} \cup \{\infty\}$ with the ordering induced by the well-ordering of $\mathcal{K}$ together with $\alpha < \infty$ for all $\alpha \in \mathcal{K}$.

Definition 3. Let p be a prime number and $(M, f) \in \mathcal{H}_p$. Then the sectional category or genus $\tilde{g}(M, f)$ is defined by

$$\tilde{g}(M,f) := \min\Big\{\operatorname{card} \mathcal{U} \;\Big|\; \mathcal{U} \subset 2^M \text{ is an admissible covering, for every } U \in \mathcal{U} \text{ there exist disjoint open sets } U_0, \ldots, U_{p-1} \text{ with } \bigcup_{j=0}^{p-1} U_j = U \text{ and } U_j = f^j(U_0) \text{ for } j = 1, \ldots, p-1\Big\},$$

if such a $\mathcal{U}$ exists, and $\tilde{g}(M, f) := \infty$ otherwise.

In [8, 9], we used another genus $g(M, f)$, for which one knows that $g(M, f) \le \tilde{g}(M, f)$ and that it coincides with $\tilde{g}(M, f)$, if M is normal. We prefer below $\tilde{g}(M, f)$ instead of $g(M, f)$, because it is more suitable for Borsuk–Ulam theorems. That does not matter for Ljusternik–Schnirelmann–Borsuk theorems, since we can consider these only in normal spaces. We shall need several elementary properties of the genus:

Lemma 1 (*cf.* [11, 8]). *Let p be a prime number and (M_1, f_1), $(M_2, f_2) \in \mathcal{H}_p$. Let $P : (M_1, f_1) \to (M_2, f_2)$ be an equivariant map (i.e. $P : M_1 \to M_2$ with $P \circ f_1 = f_2 \circ P$). Then we have $\tilde{g}(M_1, f_1) \le \tilde{g}(M_2, f_2)$.*

Definition 4. Let $n \in \{0, 1, 2, \ldots\}$ and p a prime number. Let

$$l_2 := \left\{ z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}} \;\middle|\; \sum_{j=1}^{\infty} |z_j|^2 < \infty \right\}$$

with the usual Hilbert-space norm. We define

$$F_{n,p} := \begin{cases} \varnothing, \text{ if } n = 0, \\ \{z = (z_1, \ldots, z_{n/2}, 0, \ldots) \in l_2 \mid \|z\| = 1\}, \text{ if } n \in \{2, 4, 6, \ldots\}, \\ \{z = (z_1, \ldots, z_{(n+1)/2}, 0, \ldots) \in l_2 \mid \|z\| = 1,\ z_{(n+1)/2} = |z_{(n+1)/2}|\, e^{(k/p)2\pi i} \text{ for} \\ \qquad \text{some } k \in \{0, \ldots, p-1\}\}, \text{ if } n \in \{1, 3, 5, \ldots\} \end{cases}$$

and $\varphi_p : F_{n,p} \to F_{n,p}$.

$$\varphi_p(z) := (e^{2\pi i/p} z_1, e^{2\pi i/p} z_2, \ldots).$$

If n is even, $F_{n,p}$ is a sphere S^{n-1}. In addition, $(F_{n,p}, \varphi_p) \in \mathcal{N}_p$.

Lemma 2 (*cf.* [10,11,8,7]). *Let p be a prime number and $(M, f) \in \mathcal{H}_p$. If $n := \tilde{g}(M, f)$ is finite, then n is the minimal number such that there exists an equivariant map $P : (M, f) \to (F_{n,p}, \varphi_p)$.*

Due to Krasnosel'skiĭ [4] is the following result:

LEMMA 3. *For any prime number p and any map* $f: S^n \to S^n$ *with* $(S^n, f) \in \mathcal{N}_p$, *one has* $\tilde{g}(S^n, f) = n+1$. *Furthermore,* $\tilde{g}(F_{n,p}, \varphi_p) = n$.

LEMMA 4 (*cf.* [11,8]). *Let* $n \in \{0, 1, 2, \ldots\}$ *and p a prime number. Let* $(M, f) \in \mathcal{H}_p$ *with M nonempty and n-connected. Then there exists an equivariant map* $P: (F_{n+2,p}, \varphi_p) \to (M, f)$, *in particular* $\tilde{g}(M, f) \geq n+2$.

We obtain as a simple corollary:

LEMMA 5. *Let* $m \geq 2$ *be an integer and let p be a prime number. Let* (S, φ) *be as in Theorem* 1 *and* $(M, f) \in \mathcal{H}_p$. *Then* $\tilde{g}(M, f) \leq (m-1)(p-1)$ *if and only if there exists an equivariant map* $P: (M, f) \to (S, \varphi)$.

Proof. Let $n := \tilde{g}(M, f) \leq (m-1)(p-1)$. By Lemma 2, there exists an equivariant map $P_1: (M, f) \to (F_{n,p}, \varphi_p)$. Let $j: F_{n,p} \to F_{(m-1)(p-1),p}$ be the inclusion map. S is an $((m-1)(p-1)-1)$-dimensional sphere, hence $((m-1)\times(p-1)-2)$-connected. It follows by Lemma 4 that there exists an equivariant map $P_2: (F_{(m-1)(p-1),p}, \varphi_p) \to (S, \varphi)$. Then $P := P_2 \circ j \circ P_1: (M, f) \to (S, \varphi)$ is equivariant.

The other direction of the proof is obvious by Lemma 2 and 3. q.e.d.

For $m \in \mathbb{N}$ and p a prime number, we define:

$$q_1(m, p) := \max\{n \in \{0, 1, 2, \ldots\} \mid \text{For every } (M, f) \in \mathcal{H}_p \text{ with } \tilde{g}(M, f) \leq n, \text{ there exists an } h: M \to \mathbb{R}^m \text{ with } h(x) \neq h(f(x)) \text{ for all } x \in M\},$$

$$q_2(m, p) := \max\{n \in \{0, 1, 2, \ldots\} \mid \text{There exists an } (M, f) \in \mathcal{H}_p \text{ with } \tilde{g}(M, f) = n, \text{ such that there exists an } h: M \to \mathbb{R}^m \text{ with } h(x) \neq h(f(x)) \text{ for all } x \in M\},$$

$$r_1(m, p) := \max\{n \in \{0, 1, 2, \ldots\} \mid \text{For every } (M, f) \in \mathcal{N}_p \text{ with } \tilde{g}(M, f) \leq n, \text{ there exist closed sets } U_1, \ldots, U_m \text{ with } \bigcup_{i=1}^m U_i = M \text{ and } U_i \cap f(U_i) = \emptyset \text{ for } i = 1, \ldots, m\},$$

$$r_2(m, p) := \max\{n \in \{0, 1, 2, \ldots\} \mid \text{There exists an } (M, f) \in \mathcal{N}_p \text{ with } \tilde{g}(M, f) = n, \text{ such that there exist closed sets } U_1, \ldots, U_m \text{ with } \bigcup_{i=1}^m U_i = M \text{ and } U_i \cap f(U_i) = \emptyset \text{ for } i = 1, \ldots, m\}.$$

For $s \in \mathbb{R}$, let

$$[s] := \max\{n \in \mathbb{Z} \mid n \leq s\}.$$

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Now we can formulate the main result of this chapter:

THEOREM 4. *Let* $m \in \mathbb{N}$ *and* $p \geq 3$ *a prime number. Then*

$$(m-1)(p-1) \leq q_1(m, p) \leq q_2(m, p) = (m-1)(p-1)+1$$

and, if $m \geq 3$,

$$\left(\left[\frac{m}{4}\right]-1\right)(p-1) \leq r_1(m, p) \leq r_2(m, p) \leq (m-3)\frac{p-1}{2}+\begin{cases}1, & \text{if } p=3\\ 2, & \text{if } p>3.\end{cases}$$

REMARK. In the case of $m=2$, one knows that (cf. [7], 7.13)

$$q_1(2, p) = p-1 < p = q_2(2, p).$$

It is not known whether q_1 and q_2 differ for all $p \geq 3$ and $m \geq 3$.

Proof of Theorem 4. $(m-1)(p-1) \leq q_1(m, p)$: Let (S, φ) be as in Theorem 1 and let $(M, f) \in \mathcal{H}_p$, such that $\tilde{g}(M, f) \leq (m-1)(p-1)$. By Lemma 5, there exists an equivariant map $P:(M, f) \to (S, \varphi)$. From Theorem 1 we know that there exists a continuous map $h: S \to \mathbb{R}^m$ with $h(\varphi(y)) \neq h(y)$ for all $y \in S$. It follows for every $x \in M$

$$(h \circ P)(f(x)) = h((P \circ f)(x)) = h(\varphi(P(x))) \neq h(P(x)) = (h \circ P)(x).$$

$q_1(m, p) \leq q_2(m, p)$ is obvious.

$q_2(m, p) = (m-1)(p-1)+1$: See Schupp [7], 5.2 and 6.1.

$([m/4]-1)(p-1) \leq r_1(m, p)$: It suffices to consider $m \geq 4$. Let $(M, f) \in \mathcal{N}_p$ with $\tilde{g}(M, f) \leq ([m/4]-1)(p-1)$, and let (S, φ) be as in Theorem 1 with $[m/4]$ instead of m. By Lemma 5, there exists an equivariant map $P:(M, f) \to (S, \varphi)$. From Theorem 3, we know that S can be covered by $4[m/4] \leq m$ closed sets $V_1, \ldots, V_{4[m/4]}$ with $V_i \cap \varphi(V_i) = \emptyset$ for $i=1, \ldots, 4[m/4]$. Let $U_i := P^{-1}(V_i)$. Then we have $U_i \cap f(U_i) = P^{-1}(V_i) \cap f(P^{-1}(V_i)) = P^{-1}(V_i) \cap P^{-1}(\varphi(V_i)) = \emptyset$ for $i=1, \ldots, 4[m/4]$ and

$$M = \bigcup_{i=1}^{4[m/4]} U_i.$$

$r_1(m, p) \leq r_2(m, p)$ is obvious.

$$r_2(m, p) \leq (m-3)\frac{p-1}{2}+\left\{\begin{matrix}1, \text{ if } p=3\\ 2, \text{ if } p>3\end{matrix}\right\}:$$

This has been shown in [9], Theorem 5. q.e.d.

4. The case $p=7$. In the last chapter, we obtained a lower estimate for the Ljusternik–Schnirelmann–Borsuk covering property, which is approximately one half of the upper estimate proved in [9]. In this chapter, we want to show that this lower estimate actually is not too bad. We improve the upper estimate considerably in the special case of $p=7$.

We start with some wellknown elementary facts and with some notations:

LEMMA 6 (*cf.* [7]). *Let p be a prime number and* $(M,f)\in\mathcal{N}_p$. *Let* $A_0,\ldots,A_n\subset M$ *be closed sets with* $\emptyset=A_0\subset A_1\subset\cdots\subset A_n=M$, $f(A_i)=A_i$ *and* $\tilde{g}(A_i\setminus A_{i-1},f)$ *finite for* $i=1,\ldots,n$. *Then* $\tilde{g}(M,f)\le\sum_{i=1}^n\tilde{g}(A_i\setminus A_{i-1},f)$.

The proof of Lemma 6 is based on the fact that for $k=1,\ldots,n-1$, there exists an open neighborhood B_k of A_k with $f(B_k)=B_k$ and $\tilde{g}(B_k,f)=\tilde{g}(A_k,f)$.

In addition to this property, we will use below only the trivial fact that for every free $\mathbb{Z}_p$-action $f:S^1\to S^1$, we have $\tilde{g}(S^1,f)\le 2$.

Let $(M,f)\in\mathcal{N}_p$, and let $M_1,\ldots,M_m\subset M$ be closed sets with $\cup_{i=1}^m M_i=M$ and $M_i\cap f(M_i)=\emptyset$ for $i=1,\ldots,m$. We define for $j=0,\ldots,m$

$$A_j:=\left\{x\in M\mid\{x,f(x),\ldots,f^{p-1}(x)\}\subset\bigcup_{l=1}^{j}M_l\right\}.$$

Obviously we have (if $p\ge 3$, which will be assumed in the sequel)

$$\emptyset=A_0=A_1=A_2\subset A_3\subset\cdots\subset A_m=M,$$

and all A_j are closed and $f(A_j)=A_j$ for $j=0,\ldots,m$. By Lemma 6, we have

$$\tilde{g}(M,f)\le\sum_{j=1}^{m}\tilde{g}(A_j\setminus A_{j-1},f)=\sum_{j=3}^{m}\tilde{g}(A_j\setminus A_{j-1},f).$$

Our aim is to estimate $\tilde{g}(A_j\setminus A_{j-1},f)$ by some $s(p)$. Before, we need some further notation.

We fix j and define a special subdivision of $A_j\setminus A_{j-1}$. It is obvious from the definition that for any $x\in A_j\setminus A_{j-1}$, $\{x,f(x),\ldots,f^{p-1}(x)\}\cap M_j\ne\emptyset$. Let $\rho:\mathbb{Z}\to\mathbb{Z}_p$ be the canonical homomorphism. We shall use the notation $\bar{a}:=\rho(a)$ for $a\in\mathbb{Z}$. For any $N\subset\mathbb{Z}_p$, we define

$$\langle N\rangle:=\{x\in A_j\setminus A_{j-1}\mid f^a(x)\in M_j\Leftrightarrow\bar{a}\in N\}.$$

Some elementary remarks:

a) $M_j\cap f(M_j)=\emptyset$ implies that

$$\langle N\rangle\ne\emptyset\Rightarrow a\pm\bar{1}\notin N\text{ for all }a\in N.$$

In particular $\langle N\rangle=\emptyset$, if card $N\ge(p+1)/2$.

b) From the continuity of f and the closedness of M_j, we obtain for all $N\subset\mathbb{Z}_p$ that

$$\bigcup_{N\subset V\subset\mathbb{Z}_p}\langle V\rangle$$

is closed.

For $a_1,\ldots,a_l\in\mathbb{Z}_p$, we write $\langle a_1,\ldots,a_l\rangle$ instead of $\langle\{a_1,\ldots,a_l\}\rangle$.

We consider now the case of $p=7$:

THEOREM 5. *If* $p=7$, *then* $g(A_j\setminus A_{j-1},f)\le 2$ *for* $j=3,\ldots,m$.

Proof. We define an equivariant map $P:(A_j\setminus A_{j-1},f)\to(S^1,\varphi)$, where $\varphi:S^1\to S^1$, $\varphi(z):=ze^{(4/7)2\pi i}$.

It is easy to see that

$$\begin{aligned}A_j\setminus A_{j-1}&=\bigcup_{l\in\mathbb{Z}_7}(\langle l,l+\bar{2},l+\bar{4}\rangle\cup\langle l,l+\bar{2}\rangle\cup\langle l,l+\bar{4}\rangle\cup\langle l\rangle)\\&=\langle\bar{1},\bar{3},\bar{5}\rangle\cup\langle\bar{2},\bar{4},\bar{6}\rangle\cup\cdots\cup\langle\bar{7},\bar{2},\bar{4}\rangle\cup\langle\bar{1},\bar{3}\rangle\cup\cdots\cup\langle\bar{7},\bar{2}\rangle\\&\quad\cup\langle\bar{1},\bar{5}\rangle\cup\cdots\cup\langle\bar{7},\bar{4}\rangle\cup\langle\bar{1}\rangle\cup\cdots\cup\langle\bar{7}\rangle.\end{aligned}$$

We define for $l=0,\ldots,6$

$$P(f^l(\langle\bar{1},\bar{3},\bar{5}\rangle\cup\langle\bar{1},\bar{5}\rangle)):=\{e^{l(4/7)2\pi i}\}.$$

In particular, we have

$$P(\langle\bar{1},\bar{3},\bar{5}\rangle\cup\langle\bar{1},\bar{5}\rangle)=\{e^0\}$$

and

$$P(\langle\bar{6},\bar{1},\bar{3}\rangle\cup\langle\bar{6},\bar{3}\rangle)=\{e^{(8/7)2\pi i}\}=\{e^{(1/7)2\pi i}\}.$$

By the Tietze–Urysohn theorem, we can extend P continuously to

$$\langle\bar{1},\bar{3}\rangle\cup\bigcup_{l=0}^{6}f^l(\langle\bar{1},\bar{3},\bar{5}\rangle\cup\langle\bar{1},\bar{5}\rangle),$$

such that

$$P(\langle\bar{1},\bar{3}\rangle)\subset\left\{e^{d2\pi i}\mid 0\le d\le\frac{1}{7}\right\}$$

(cf. Remark b). We then extend P to an equivariant map on

$$R:=\bigcup_{l=0}^{6}f^l(\langle\bar{1},\bar{3},\bar{5}\rangle\cup\langle\bar{1},\bar{5}\rangle\cup\langle\bar{1},\bar{3}\rangle).$$

A similar argument allows us to extend P to $\langle\bar{1}\rangle\cup R$ such that

$$P(\langle\bar{1}\rangle)\subset\left\{e^{d2\pi i}\mid 0\le d\le\frac{2}{7}\right\}.$$

Again, P can be extended to an equivariant map on $A_j\setminus A_{j-1}$. q.e.d.

We obtain

THEOREM 6. *$r_2(m,7)\le 2(m-2)$ for all $m\ge 2$.*

For the next prime number $p=11$, it is not difficult to prove $\tilde{g}(A_3\setminus A_2,f)\le 2$ (cf. [9]) and $\tilde{g}(A_j\setminus A_{j-1},f)\le 4$ for $j=4,\ldots,m$, and hence $r_2(m,11)\le 4(m-3)+2$ for $m\ge 3$, whereas it seems to be impossible to show that $\tilde{g}(A_j\setminus A_{j-1},f)\le 3$.

5. An application to asymptotic fixed point theory. In [8], we developed the following strategy to prove the "asymptotic conjecture" (cf. the introduction):

One assumes that $f(x) \neq x$ for all $x \in E$. Then, for all prime numbers p, $(\mathscr{F}[f^p], f) \in \mathcal{N}_p$. In [8], Satz 14, we proved that

$$\tilde{g}(\mathscr{F}[f^p], f) > \frac{2}{k_0} p - 2.$$

We would get a contradiction, if we could prove that

$$\tilde{g}(\mathscr{F}[f^p], f) = o(p).$$

We tried to derive this only from the fact that

$$\bigcup_{p \text{ prime}} \mathscr{F}[f^p] \subset \overline{f^{k_0}(E)},$$

where $\overline{f^{k_0}(E)}$ is a compact set. But this is not sufficient. We will give an example of a compact subset K of a normed space and a continuous map $\varphi : K \to K$ with $\mathscr{F}[\varphi] = \varnothing$ and $\tilde{g}(\mathscr{F}[\varphi^p], \varphi)$ increases linearly with p (p prime):

Let $\Delta_{m-1} := \text{co}\{E_1, \ldots, E_m\}$ be the standard $(m-1)$-simplex spanned by the points $E_i \in \mathbb{R}^m$ with the components $(E_i)_k := \delta_{ik}$. Let

$$\Delta_{m-1;i} := \text{co}(\{E_1, \ldots, E_m\} \backslash \{E_i\})$$

be the $(m-2)$-dimensional face of Δ_{m-1}, which is opposite to E_i, and let $\partial\Delta_{m-1}$ be the boundary of Δ_{m-1}, i.e.

$$\partial\Delta_{m-1} := \bigcup_{i=1}^{m} \Delta_{m-1;i}.$$

In addition, we need $\Sigma_{m-1;i}$, which is the union of all simplices σ of the first barycentric subdivision of $\partial\Delta_{m-1}$ with $\bar{\sigma} \cap \Delta_{m-1;i} = \varnothing$. Let

$$l_2(\mathbb{R}^m) := \left\{ (x_n) \in (\mathbb{R}^m)^{\mathbb{N}} \;\middle|\; \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$$

with the usual l_2-norm and let

$$L := \left\{ \left(x_1, \frac{1}{2} x_2, \frac{1}{3} x_3, \ldots \right) \in l_2(\mathbb{R}^m) \;\middle|\; x_n \in \partial\Delta_{m-1} \right.$$
$$\left. \text{and} \quad (x_n \in \Delta_{m-1;i} \Rightarrow x_{n+1} \in \Sigma_{m-1;i}) \quad \text{for all } n \in \mathbb{N} \right\},$$

$K := \bar{L}$ and $\varphi : K \to K$,

$$\varphi\left(x_1, \frac{1}{2} x_2, \frac{1}{3} x_3, \ldots \right) := \left(x_2, \frac{1}{2} x_3, \frac{1}{3} x_4, \ldots \right).$$

Obviously, $\varphi(x) \neq x$ for all $x \in K$, and in addition, there is a canonical equivariant map $h:(\tilde{L}_{m,p}, \varphi_{m,p}) \to (\mathcal{F}[\varphi^p], \varphi)$, where (cf. [9])

$$\tilde{L}_{m,p} := \{(x_1, \ldots, x_p) \in (\partial\Delta_{m-1})^p \mid \text{If } j, k \in \{1, \ldots, p\} \text{ with } k \equiv j+1 \pmod p \text{ and } x_j \in \Delta_{m-1;i}, \text{ then } x_k \in \Sigma_{m-1;i}\}$$

and $\varphi_{m,p}: \tilde{L}_{m,p} \to \tilde{L}_{m,p}$,

$$\varphi_{m,p}(x_1, \ldots, x_p) := (x_2, \ldots, x_p, x_1).$$

By Theorem 2 in [9] and by Theorem 4, we know that

$$\tilde{g}(\mathcal{F}[\varphi^p], \varphi) \geq \tilde{g}(\tilde{L}_{m,p}, \varphi_{m,p}) = r_2(m, p) \geq \left(\left[\frac{m}{4}\right] - 1\right)(p-1).$$

This example shows that one would have to use additional properties of f to prove the asymptotic conjecture. But moreover, it is a strong hint that the conjecture is wrong.

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