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Parabolic Induction and the Bernstein Decomposition

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Abstract. We use some fundamental work of Bernstein to study parabolic induction in reductive p-adic groups. In particular, we determine when parabolic induction from a component of the Bernstein decomposition of a Levi subgroup to the corresponding component of the full group is an equivalence of categories.

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Let F be a non-Archimedean local field (with finite residue field) and G the group of F-points of a connected reductive algebraic group defined over F. Let Q be a parabolic subgroup of G with Levi factor M. Write i_Q^G for the normalized parabolic induction functor from $\mathfrak{R}(M)$, the category of smooth complex representations of M, to $\mathfrak{R}(G)$, the corresponding category for G.

In this paper, we study the qualitative behaviour of the functors i_Q^G. To state our results, we need to recall the explicit block decomposition of $\mathfrak{R}(G)$ underlying the theory of the Bernstein centre. We refer to this, following [9], as the Bernstein decomposition. (See [2], or the summary account in [9], for more details.)

Let σ be an irreducible supercuspidal representation of a Levi subgroup L of G and write $X(L)$ for the group of unramified characters of L. Define a full subcategory $\mathfrak{R}_{(L,\sigma)}(G)$ of $\mathfrak{R}(G)$ as follows: a smooth representation π of G belongs to $\mathfrak{R}_{(L,\sigma)}(G)$ if and only if each irreducible subquotient of π appears as a subquotient of $i_P^G(\sigma\nu)$ for some $\nu \in X(L)$ and some parabolic subgroup P of G with Levi factor L. The subcategories $\mathfrak{R}_{(L,\sigma)}(G)$, often called components, are indecomposable and split the full smooth category $\mathfrak{R}(G)$:

$$\mathfrak{R}(G) = \prod_{(L,\sigma)} \mathfrak{R}_{(L,\sigma)}(G). \qquad (*)$$

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(Of course, there is considerable redundancy in our indexing of the various subcategories. Indeed, let σ_i be irreducible supercuspidal representations of the Levi subgroups L_i of G, $i = 1, 2$. Then

$$\mathfrak{R}_{(L_1,\sigma_1)}(G) = \mathfrak{R}_{(L_2,\sigma_2)}(G)$$

if and only if there is a $g \in G$ with $gL_1g^{-1} = L_2$, ${}^g\sigma_1 \cong \sigma_2\nu$ for some $\nu \in X(L_2)$, by, e.g., Theorem 2.9 of [4].)

In concrete terms, $(*)$ says that each object V in $\mathfrak{R}(G)$ contains, for each pair (L, σ) as above, a maximal G-subspace $V_{(L,\sigma)}$ in $\mathfrak{R}_{(L,\sigma)}(G)$ and that $V = \sum_{(L,\sigma)} V_{(L,\sigma)}$. The sum is direct since objects in distinct $\mathfrak{R}_{(L,\sigma)}(G)$'s have disjoint sets of irreducible subquotients; for the same reason, G-homomorphisms preserve such sums, and thus one obtains the categorical decomposition $(*)$.

We now fix a Levi subgroup L of G and an irreducible supercuspidal representation σ of L. We assume that the Levi factor M of the parabolic subgroup Q of G contains L. We can thus form the categories $\mathfrak{R}_{(L,\sigma)}(M)$ and $\mathfrak{R}_{(L,\sigma)}(G)$. The restriction of i_Q^G to $\mathfrak{R}_{(L,\sigma)}(M)$ defines a functor from $\mathfrak{R}_{(L,\sigma)}(M)$ to $\mathfrak{R}_{(L,\sigma)}(G)$ which we again denote by i_Q^G.

Our main result, Theorem 3.1, gives a necessary and sufficient condition for

$$i_Q^G\colon \mathfrak{R}_{(L,\sigma)}(M) \to \mathfrak{R}_{(L,\sigma)}(G) \qquad (\star)$$

to be an equivalence of categories. To state the condition, let

$$N(L,\sigma) = \{n \in N_G(L) : {}^n\sigma \cong \sigma\nu, \text{ for some } \nu \in X(L)\}.$$

We also put $W(L,\sigma) = N(L,\sigma)/L$ and write $W^M(L,\sigma)$ for the corresponding object for M. Thus $W^M(L,\sigma) = (N(L,\sigma) \cap M)/L$. We show that $(\star)$ is an equivalence of categories if and only if $W(L,\sigma) = W^M(L,\sigma)$ (equivalently, if and only if $N(L,\sigma)$ is contained in M).

This generalizes Theorem 12.4 of [9] which gives the 'if' direction under the assumption that the components $\mathfrak{R}_{(L,\sigma)}(M)$ and $\mathfrak{R}_{(L,\sigma)}(G)$ admit types satisfying certain auxiliary conditions. In place of types, we use a construction from [3]. Here Bernstein constructs an explicit faithfully projective object $\Pi^G_{(L,\sigma)}$ in each of the categories $\mathfrak{R}_{(L,\sigma)}(G)$. For our purposes, the construction has the key property that $\Pi^G_{(L,\sigma)} \cong i_Q^G(\Pi^M_{(L,\sigma)})$. Theorem 3.1 follows easily using some elementary categorical algebra and the main technical result of [4], Mackey's theorem for the composition of Jacquet restriction and parabolic induction. (In fact, at this stage we only need that the objects $\Pi^G_{(L,\sigma)}$ and $\Pi^M_{(L,\sigma)}$ are faithful.)

Let π be an irreducible object in $\mathfrak{R}_{(L,\sigma)}(M)$. As an immediate consequence of the above, we see that $i_Q^G(\pi)$ is irreducible whenever $W(L,\sigma) = W^M(L,\sigma)$. In particular, if $W(L,s) = \{1\}$, then $i_P^G\sigma$ is irreducible. (We also prove this case more directly in Section 2.) In Section 4, we give an example where $W(L,\sigma) \neq \{1\}$ and $i_P^G(\sigma\nu)$ is irreducible for all $\nu \in X(L)$. Thus parabolic induction can fail to be an equivalence of categories but still always take irreducible objects to irreducible objects.

The example arose from our work on SL_N [13, 14]. There we were led to consider whether the action (induced by conjugation) of $W(L, \sigma)$ on the set $\mathrm{Irr}(L, \sigma)$ of equivalence classes of irreducible objects in $\mathfrak{R}_{(L,\sigma)}(L)$ always admits a fixed point. In other words, given an irreducible supercuspidal representation σ of a Levi subgroup L of $\mathrm{SL}_N(F)$, can one always find an unramified twist of σ that is fixed (up to equivalence) by $W(L, \sigma)$? P. Kutzko quickly pointed out an example where the answer is negative. For this example, the argument establishing that the fixed-point set $\mathrm{Irr}(L, \sigma)^{W(L,\sigma)}$ is empty also easily implies that $i_P^G(\sigma\nu)$ is irreducible for all $\nu \in X(L)$.

We also note that if L is a maximal Levi subgroup of a general group G, then the feature we have emphasized in the example, that $\mathrm{Irr}(L, \sigma)^{W(L,\sigma)}$ is empty, in fact characterizes all such examples. That is, we show that if L is maximal in G, then i_P^G: $\mathfrak{R}_{(L,\sigma)}(L) \to \mathfrak{R}_{(L,\sigma)}(G)$ takes irreducible objects to irreducible objects but is not an equivalence of categories if and only if $\mathrm{Irr}(L, \sigma)^{W(L,\sigma)}$ is empty.

In the final part of the paper, we look at parabolic induction in the context of the equivalences of categories, due to Bernstein, recalled in Section 1. More precisely, by a general result in categorical algebra, Theorem 1.3 of [1] or Theorem 1.1 below, the faithfully projective object $\Pi^G_{(L,\sigma)}$ induces an equivalence of categories

$$\mathfrak{R}_{(L,\sigma)}(G) \cong \mathfrak{mod} - \mathrm{End}_G\, \Pi^G_{(L,\sigma)},$$

the category of right modules over the endomorphism ring of $\Pi^G_{(L,\sigma)}$. Hence i_Q^G: $\mathfrak{R}_{(L,\sigma)}(M) \to \mathfrak{R}_{(L,\sigma)}(G)$ corresponds to a certain functor t: $\mathfrak{mod} - \mathcal{B} \to \mathfrak{mod} - \mathcal{A}$ where $\mathcal{A} = \mathrm{End}_G\, \Pi^G_{(L,\sigma)}$ and $\mathcal{B} = \mathrm{End}_M\, \Pi^M_{(L,\sigma)}$. Now i_Q^G gives rise to a homomorphism of rings $t_Q: \mathcal{B} \to \mathcal{A}$ (in fact, an embedding of rings, as i_Q^G is faithful). We show that t is equivalent to the functor

$$M \mapsto M \otimes_{\mathcal{B}} \mathcal{A}: \mathfrak{mod} - \mathcal{B} \to \mathfrak{mod} - \mathcal{A}, \qquad (\dagger)$$

where $\mathcal{A}$ is viewed as a left $\mathcal{B}$-module via t_Q. This description is of course reminiscent of, and indeed was motivated by, the treatment of parabolic induction in [9]. In an earlier version of the paper, we used it, along with the general observation that a functor of the form (†) is an equivalence of categories if and only if t_Q is an isomorphism, to give a more roundabout proof of the main result. Here we simply remark that it yields a transparent proof that parabolic induction takes finitely generated objects to finitely generated objects, which is proved by other means in [2, 3]. (Remark 1.3 below gives yet another proof.)

1. Some Results of Bernstein

As above, let σ be an irreducible supercuspidal representation of a Levi subgroup L of G. In this section, we recall Bernstein's construction of an explicit faithfully projective object in the category $\mathfrak{R}_{(L,\sigma)}(G)$. (We review the relevant elementary categorical algebra in subsection 1.1 below.)

The construction has two stages. The first, when $L = G$, is straightforward. Let Σ denote the resulting faithfully projective object in $\mathfrak{R}_{(L,\sigma)}(L)$. The more difficult stage, when $L \neq G$, relies crucially on the observation that parabolic induction takes projective objects to projective objects. This property of induction is a formal consequence of Bernstein's second adjoint theorem (see Subsection 1.3 below).

It follows easily that $\bigoplus_{P \in \mathcal{P}(L)} i_P^G \Sigma$ is then a faithfully projective object in $\mathfrak{R}_{(L,\sigma)}(G)$, where $\mathcal{P}(L)$ denotes the set of parabolic subgroups of G with Levi component L. In fact, Bernstein proves that the isomorphism class of $i_P^G \Sigma$ is independent of the choice of $P \in \mathcal{P}(L)$. Hence, for any $P \in \mathcal{P}(L)$, $i_P^G \Sigma$ is itself faithfully projective in $\mathfrak{R}_{(L,\sigma)}(G)$. This more precise result is central to our applications.

A proof of the second adjoint theorem is outlined in Rumelhart's notes [3]. Bushnell's paper [5] contains a different proof. In view of this reference, we simply state the result here and provide full proofs for the remainder of Bernstein's construction. We emphasize, therefore, that the proofs in this section are due to Bernstein and are taken, with a few (minor) changes in detail and exposition, from the account in [3]. They are included for completeness and for the reader's convenience. In addition, we use a part of one proof later in the paper (in the proof of Proposition 4.3).

1.1. Let $\underline{A}$ be an Abelian category with direct sums and let $\underline{Ab}$ denote the category of Abelian groups. We recall some elementary categorical algebra.

An object P in $\underline{A}$ is projective if the functor $\mathrm{Hom}(P, -): \underline{A} \to \underline{Ab}$ is exact. An object F in $\underline{A}$ is faithful, or a generator, if $\mathrm{Hom}(F, -): \underline{A} \to \underline{Ab}$ is a faithful functor, i.e., is injective on morphisms. It is easy to see that an exact functor between Abelian categories is faithful if and only if it takes nonzero objects to nonzero objects. Thus, if P is projective in $\underline{A}$, then P is faithful if and only if $\mathrm{Hom}(P, X) \neq 0$ for each nonzero object X in $\underline{A}$. An object S in $\underline{A}$ is small if the functor $\mathrm{Hom}(S, -): \underline{A} \to \underline{Ab}$ preserves direct sums, i.e., for any family of objects $\{A_i\}_{i \in I}$ in $\underline{A}$, the natural morphism

$$\bigoplus_{i \in I} \mathrm{Hom}(S, A_i) \longrightarrow \mathrm{Hom}\left(S, \bigoplus_{i \in I} A_i\right)$$

is an isomorphism.

Finally, an object P in $\underline{A}$ is, by definition, faithfully projective (or a progenerator) if P is projective, faithful *and* small. The following is a special case of a theorem of Gabriel. (Versions are also due independently to Freyd and Mitchell.)

THEOREM. *Let $\underline{A}$ be an Abelian category with direct sums and a faithfully projective object P. Then the functor*

$$A \mapsto \mathrm{Hom}(P, A): \underline{A} \to \mathfrak{mod} - \mathrm{End}\, P$$

is an equivalence of categories.

Here, of course, $\mathrm{End}\, P = \mathrm{Hom}(P, P)$ and the right $\mathrm{End}\, P$-module structure on $\mathrm{Hom}(P, A)$ is given by composition:

$$f \cdot g = f \circ g, \quad f \in \mathrm{Hom}(P, A),\ g \in \mathrm{End}\, P.$$

1.2. We first recall the construction of a faithfully projective object in $\mathfrak{R}_{(L,\sigma)}(L)$.

Let A_L denote the maximal split torus in the centre of L. Then $A_L L^0$ has finite index in L where $L^0 = \bigcap_{\nu\in X(L)} \operatorname{Ker}\nu$. Hence

$$\sigma|L^0 = \sigma_1 \oplus \cdots \oplus \sigma_r$$

with each σ_i irreducible (and supercuspidal). Let Σ denote the compactly induced representation $\operatorname{ind}_{L^0}^{L} \sigma_1$. (As the representations σ_i are L-conjugate (up to isomorphism), the isomorphism class of Σ is independent of the choice of irreducible component σ_1 of $\sigma|L^0$.)

We verify that Σ is faithfully projective in $\mathfrak{R}_{(L,\sigma)}(L)$. Let Π be an object in $\mathfrak{R}_{(L,\sigma)}(L)$. Then $\Pi|L^0$ is a direct sum of copies of the various σ_i and

$$\operatorname{Hom}_L(\Sigma, \Pi) = \operatorname{Hom}_L(\operatorname{ind}_{L^0}^{L} \sigma_1, \Pi) \cong \operatorname{Hom}_{L^0}(\sigma_1, \Pi|L^0).$$

It follows that $\operatorname{Hom}_L(\Sigma, -)$ is exact, i.e., Σ is projective.

If $\Pi \neq 0$, then Π has an irreducible subquotient isomorphic to $\sigma\nu$ for some $\nu \in X(L)$. Thus there is an L-subspace Π_1 of Π and a surjective L-homomorphism from $\Pi_1 \to \sigma\nu$. Since

$$\operatorname{Hom}_L(\Sigma, \sigma\nu) \cong \operatorname{Hom}_{L^0}(\sigma_1, \sigma|L^0) \neq 0,$$

Σ also surjects onto $\sigma\nu$. By the projectivity of Σ, $\operatorname{Hom}_L(\Sigma, \Pi_1) \neq 0$ and so, *a fortiori*, $\operatorname{Hom}_L(\Sigma, \Pi) \neq 0$.

Finally, since σ_1 is irreducible and therefore finitely generated and since compact induction takes finitely generated objects to finitely generated objects, we see that Σ is finitely generated. It follows immediately that Σ is small in $\mathfrak{R}(L)$ or, equivalently, in $\mathfrak{R}_{(L,\sigma)}(L)$.

Theorem 1.1 now yields the following:

PROPOSITION. *With notation as above, the functor*

$$\Pi \mapsto \operatorname{Hom}_L(\Sigma, \Pi)\colon\ \mathfrak{R}_{(L,\sigma)}(L) \to \mathfrak{mod} - \operatorname{End}_L\Sigma$$

is an equivalence of categories.

1.3. Next let L be a proper Levi subgroup of G. Let $P \in \mathcal{P}(L)$ where, as above, $\mathcal{P}(L)$ denotes the set of parabolic subgroups of G with Levi component L. Write U for the unipotent radical of P and $\bar{U}$ for the unipotent radical of the L-opposite $\bar{P}$ of P.

General existence theorems show easily that the parabolic induction functor $i_P^G\colon \mathfrak{R}(L) \to \mathfrak{R}(G)$ admits a right adjoint. However, identifying this adjoint in explicit representation-theoretic terms is decidedly nontrivial. This is the content of the following theorem, often called Bernstein's second adjoint theorem or, simply, the second adjoint theorem.

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THEOREM. *The normalized Jacquet functor* $r_{\bar{U}}$: $\mathfrak{R}(G) \to \mathfrak{R}(L)$ *is right adjoint to* i_P^G: $\mathfrak{R}(L) \to \mathfrak{R}(G)$, *i.e., for objects* V *in* $\mathfrak{R}(G)$ *and* W *in* $\mathfrak{R}(L)$, *there exist natural isomorphisms, in* V *and* W,

$$\mathrm{Hom}_G(i_P^G W, V) \cong \mathrm{Hom}_L(W, r_{\bar{U}} V).$$

COROLLARY. *The functor* i_P^G: $\mathfrak{R}(L) \to \mathfrak{R}(G)$ *takes projective objects to projective objects.*

Proof. Let W be an object in $\mathfrak{R}(L)$. By the second adjoint theorem,

$$\mathrm{Hom}_G(i_P^G W, -) \cong \mathrm{Hom}_L(W, -) \circ r_{\bar{U}}.$$

Elementary arguments show that the functor $r_{\bar{U}}$ is exact. It follows that $\mathrm{Hom}_G(i_P^G W, -)$ is exact whenever W is projective in $\mathfrak{R}(L)$. □

We now show that $\bigoplus_{P \in \mathcal{P}(L)} i_P^G \Sigma$ is faithfully projective in $\mathfrak{R}_{(L,\sigma)}(G)$. It is immediate, from the above corollary, that this object is projective. To see that it is faithful, let Π be an object in $\mathfrak{R}_{(L,\sigma)}(G)$ with irreducible subquotient π. Thus Π contains a G-subspace Π_1 such that there is a G-surjection $\Pi_1 \to \pi$. There is a parabolic sub-group P_1 of G with Levi factor L_1 and an irreducible supercuspidal representation σ_1 of L_1 such that π is a quotient of $i_{P_1}^G \sigma_1$. Since π is in $\mathfrak{R}_{(L,\sigma)}(G)$, it is also a subquotient of $i_{P'}^G(\sigma\nu)$, for some $P' \in \mathcal{P}(L)$ and some $\nu \in X(L)$. Theorem 2.9 of [4] then implies that there is a $g \in G$ with ${}^g L_1 = L, {}^g\sigma_1 \cong \sigma\nu$ and so π is a quotient of $i_P^G(\sigma\nu)$ for some $P \in \mathcal{P}(L)$. The second adjoint theorem then implies that $\mathrm{Hom}_L(\sigma\nu, r_{\bar{U}}\pi) \neq 0$. Hence, since Σ surjects onto $\sigma\nu$, $\mathrm{Hom}_L(\Sigma, r_{\bar{U}}\pi) \neq 0$. Applying the second adjoint theorem once more, we deduce that there is a nonzero, and therefore surjective, G-homomorphism from $i_P^G \Sigma$ to π. Since $i_P^G \Sigma$ is projective, $\mathrm{Hom}_G(i_P^G \Sigma, \Pi_1) \neq 0$ and so, *a fortiori*, $\mathrm{Hom}_G(\bigoplus_{P \in \mathcal{P}(L)} i_P^G \Sigma, \Pi) \neq 0$.

To see that $\bigoplus_{P \in \mathcal{P}(L)} i_P^G \Sigma$ is small in $\mathfrak{R}_{(L,\sigma)}(G)$ (or, equivalently, in $\mathfrak{R}(G)$), it clearly suffices to show that $i_P^G \Sigma$ is small for each $P \in \mathcal{P}(L)$. It is immediate that the functor $r_{\bar{U}}$: $\mathfrak{R}(G) \to \mathfrak{R}(L)$ preserves direct sums. (This is also a formal consequence of the (elementary) fact that $r_{\bar{U}}$ admits a right adjoint.) Hence, for any family of objects $\{V_i\}_{i \in I}$ in $\mathfrak{R}_{(L,\sigma)}(G)$, there is a natural isomorphism

$$\mathrm{Hom}_G\left(i_P^G \Sigma, \bigoplus_{i \in I} V_i\right) \cong \mathrm{Hom}_L\left(\Sigma, \bigoplus_{i \in I} r_{\bar{U}} V_i\right).$$

We therefore obtain the following diagram

$$\begin{array}{ccc} \bigoplus_{i \in I} \mathrm{Hom}_G(i_P^G \Sigma, V_i) & \longrightarrow & \mathrm{Hom}_G(i_P^G \Sigma, \bigoplus_{i \in I} V_i) \\ \simeq\downarrow & & \downarrow\simeq \\ \bigoplus_{i \in I} \mathrm{Hom}_L(\Sigma, r_{\bar{U}} V_i) & \xrightarrow{\simeq} & \mathrm{Hom}_L(\Sigma, \bigoplus_{i \in I} r_{\bar{U}} V_i). \end{array}$$

where the bottom horizontal arrow is an isomorphism since Σ is small in $\mathfrak{R}(L)$. A routine computation, using naturality of the isomorphism in the second adjoint

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theorem, shows that the diagram commutes. It follows that the top horizontal arrow must be an isomorphism and, hence, that $i_P^G\,\Sigma$ is small.

Remark. Alternatively, one could observe, as in [2] or [3], that parabolic induction takes finitely generated objects to finitely generated objects, whence $i_P^G\,\Sigma$ is finitely generated and therefore small.

On the other hand, this property of parabolic induction is itself a consequence of the above argument showing that $i_P^G\,\Sigma$ is small. Indeed, the argument clearly shows that $i_P^G\,W$ is small in $\mathfrak{R}(G)$ whenever W is small in $\mathfrak{R}(L)$. It is not hard to verify that an object V in $\mathfrak{R}(G)$ is small if and only if the union of a countable chain of proper G-subspaces of V is also proper. (For module categories, this is exercise (b) on page 54 of [1].) In particular, a countably generated small object in $\mathfrak{R}(G)$ is in fact finitely generated. Now let W be a finitely generated object in $\mathfrak{R}(L)$. Then $i_P^G\,W$ is clearly countably generated. Since W is small, $i_P^G\,W$ is also small and so must be finitely generated.

1.4. We have noted that $\oplus_{P\in\mathcal{P}(L)} i_P^G\,\Sigma$ is a faithfully projective object in $\mathfrak{R}_{(L,\sigma)}(G)$. For our applications we will need Bernstein's significantly sharper result:

THEOREM. *The isomorphism class of $i_P^G\,\Sigma$ is independent of the parabolic subgroup $P\in\mathcal{P}(L)$.*

Proof. Let P, $P'\in\mathcal{P}(L)$. Then there is a sequence $P_1=P,\dots,P_n=P'$ in $\mathcal{P}(L)$ such that the following holds: for each $i=1,\dots,n-1$, P_i and P_{i+1} are contained in a parabolic subgroup $Q=Q_i$ of G such that Q has a Levi factor M containing L as a maximal (proper) Levi subgroup. Note that $M\cap P_i$ and $M\cap P_{i+1}$ are both parabolic subgroups of M having Levi component L. By transitivity of parabolic induction,

$$i_{P_i}^G\,\Sigma\cong i_Q^G(i_{M\cap P_i}^M\,\Sigma),\qquad i_{P_{i+1}}^G\,\Sigma\cong i_Q^G(i_{M\cap P_{i+1}}^M\,\Sigma).$$

Thus, to prove the theorem, we may assume that L is a maximal Levi subgroup of G. In this case, $\mathcal{P}(L)=\{P,\bar{P}\}$ for any fixed element P of $\mathcal{P}(L)$.

Let $N(L,\sigma)$ be the group of elements n in $N_G(L)$ such that ${}^n\sigma\cong\sigma\nu$ for some $\nu\in X(L)$ and set $W(L,\sigma)=N(L,\sigma)/L$. Since L is maximal, $|W(L,\sigma)|\leqslant 2$. We assume first that $W(L,\sigma)\neq\{1\}$. Then ${}^wP=\bar{P}$ and ${}^w\Sigma\cong\Sigma$ where w is the unique nontrivial element of $W(L,\sigma)$. Hence

$$i_P^G\,\Sigma\cong{}^w i_P^G\,\Sigma\cong i_{\bar{P}}^G\,{}^w\Sigma\cong i_{\bar{P}}^G\,\Sigma.$$

1.5. Suppose now that $W(L,\sigma)=\{1\}$ (and L is maximal). This case requires a more elaborate argument. The key step is contained in the following proposition:

PROPOSITION. *If L is a maximal Levi subgroup of G and $W(L,\sigma)=\{1\}$, then $i_P^G\sigma$ is irreducible.*

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Proof. Suppose that π is a nonzero subquotient of $i_P^G\sigma$. Then, by Corollary 7.2.2 of [11], $r_U\pi \neq 0$. Hence, the length of $i_P^G\sigma$ is less than or equal to the length of $r_U i_P^G\sigma$. Since the composition factors of $r_U i_P^G\sigma$ are the various ${}^w\sigma$ for $w \in W(L) = N_G(L)/L$ (e.g., by 2.12 of [4]), we see that the length of $i_P^G\sigma$ is at most $|W(L)|$.

We may assume therefore that $W(L) \neq \{1\}$. Hence, as L is maximal, $W(L) = \{1, w\}$ with $w \neq 1$. Suppose that $i_P^G\sigma$ is reducible. Then we have a short exact sequence

$$0 \to \pi_1 \to i_P^G\sigma \to \pi_2 \to 0 \tag{1.5.1}$$

with π_i irreducible for $i = 1, 2$. Applying the exact functor r_U, we obtain another short exact sequence

$$0 \to r_U\pi_1 \to r_U i_P^G\sigma \to r_U\pi_2 \to 0.$$

Since $r_U\pi_i \neq 0$ for $i = 1, 2$ and $r_U i_P^G\sigma$ has composition factors σ and ${}^w\sigma$, we must have

$$r_U\pi_1 \cong {}^w\sigma \quad \text{and} \quad r_U\pi_2 \cong \sigma, \tag{1.5.2}$$

or

$$r_U\pi_1 \cong \sigma \quad \text{and} \quad r_U\pi_2 \cong {}^w\sigma.$$

Note that ${}^w\sigma$ is not isomorphic to σ as, by assumption, $w \notin W(L, \sigma)$. It follows that the sequence (1.5.1) cannot split. Indeed, if it did split, we would have nonzero G-homomorphisms $\pi_i \to i_P^G\sigma$ $(i = 1, 2)$ and thus also nonzero L-homomorphisms $r_U\pi_i \to \sigma$ $(i = 1, 2)$. Then (1.5.2) would imply ${}^w\sigma \cong \sigma$. We conclude that (1.5.1) does not split and so, in particular, σ cannot be unitary.

Since σ is supercuspidal, this simply means that A_L, the maximal split torus in the centre of L, acts by a nonunitary character $\chi_\sigma\colon A_L \to \mathbb{C}^\times$. Let A_G denote the maximal split torus in the centre of G. Then the restriction homomorphism

$$\nu \mapsto \nu|A_G\colon X(G) \to X(A_G) \tag{1.5.3}$$

is surjective. (Indeed, it is easy to verify that the $A_G^0 = A_G \cap G^0$, whence A_G/A_G^0 embeds into G/G^0. Applying the exact functor $\mathrm{Hom}_Z(-, \mathbb{C}^\times)$, we obtain surjectivity of (1.5.3).) In particular, there exists a $\nu \in X(G)$ such that $\nu = |\chi_\sigma|^{-1}$ on A_G. Therefore, by tensoring the sequence (1.5.1) with ν and adjusting σ, we may assume that $|\chi_\sigma| = 1$ on A_G.

As L is maximal, the split torus A_L/A_G is one-dimensional. Since the action of w on this torus (induced by conjugation) is nontrivial of order two, we have

$$waw^{-1} \equiv a^{-1} \pmod{A_G}, \quad a \in A_L.$$

Hence $|{}^w\chi_\sigma| = |\chi_\sigma|^{-1}$. Let A_L^+ denote the semigroup of strictly positive (with respect to P) elements in A_L. (Thus $a \in A_L^+$ if and only if $a^nKa^{-n} \searrow \{1\}$ as $n \to \infty$ for some, or equivalently for all, compact open subgroups K of U.) Since χ_σ is nonunitary and A_L/A_G is one-dimensional, we see that $|\chi_\sigma| > 1$ or $|\chi_\sigma| < 1$ on A_L^+. Applying Casselman's square-integrability criterion [11] and (1.5.2), it follows that one of the representations π_i $(i = 1, 2)$, and only one, is square-integrable-mod-centre. Let π denote this representation. Of course, π is, in particular, unitary.

Let $X_r(*) = \mathrm{Hom}(*, \mathbb{R}_{>0})$ where $\mathbb{R}_{>0}$ denotes the (divisible) multiplicative group of nonzero positive real numbers. Then, as in (1.5.3), the restriction homomorphism

$$\eta \mapsto \eta | A_L : X_r(L) \to X_r(A_L)$$

is surjective. In fact, since $X_r(A_L)$ is torsion-free and $A_L L^0$ has finite index in L, this map is an isomorphism.

We may therefore write $\sigma = \sigma_0 \eta$ where σ_0 is unitary and η is the unique element of $X_r(L)$ whose restriction to A_L equals $|\chi_\sigma|$. Since π is a subquotient of $i_P^G(\sigma_0\eta)$, the contragredient $\pi^\vee$ of π is a subquotient of $i_P^G(\sigma_0^\vee \eta^{-1})$. Similarly, $\bar{\pi}$, the complex conjugate of π, is a subquotient of $i_P^G(\bar{\sigma}_0\eta)$. As π and σ_0 are unitary, $\pi^\vee \cong \bar{\pi}$, $\sigma_0^\vee \cong \bar{\sigma}_0$. Therefore, $\pi^\vee$ is a subquotient both of $i_P^G(\sigma_0^\vee\eta^{-1})$ and $i_P^G(\sigma_0^\vee\eta)$, whence

$$\sigma_0^\vee \eta^{-1} \cong \sigma_0^\vee \eta \quad \text{or} \quad {}^w(\sigma_0^\vee \eta^{-1}) \cong \sigma_0^\vee \eta.$$

Taking contragredients and rearranging, $\sigma_0 \cong \sigma_0 \eta^2$ or ${}^w\sigma_0 \cong \sigma_0 \zeta$ where $\zeta = \eta^{-1}({}^w\eta)^{-1} \in X_r(L)$. On examining central characters, the first possibility implies that $\eta = 1$ (since $X_r(A_L) \cong X_r(L)$ is torsion-free). This in turn implies that σ is unitary. However, we have already noted that σ must be nonunitary. Therefore this possibility cannot occur. The second possibility also cannot occur since, by assumption, $w \notin W(L, \sigma)$.

This contradiction shows that our original assumption that $i_P^G\sigma$ is reducible is false and thus completes the proof of the proposition. □

Write $p_{(L,\sigma)}: \mathfrak{R}(L) \to \mathfrak{R}_{(L,\sigma)}(L)$ for the projection functor (implied by the Bernstein decomposition of L) and set $r_U^{(L,\sigma)} = p_{(L,\sigma)} \circ r_U$.

COROLLARY. *With the same hypotheses as above, the functor*

$$r_U^{(L,\sigma)}: \ \mathfrak{R}_{(L,\sigma)}(G) \to \mathfrak{R}_{(L,\sigma)}(L)$$

takes nonzero objects to nonzero objects.

Proof. Let π be an irreducible object in $\mathfrak{R}_{(L,\sigma)}(G)$. It suffices to show that $r_U^{(L,\sigma)}\pi \neq 0$. However, from the proposition, $\pi \cong i_P^G(\sigma\nu)$ for some $\nu \in X(L)$. Therefore, $r_U^{(L,\sigma)}\pi \cong \sigma\nu \neq 0$. □

1.6. We show now that the functor $r_U^{(L,\sigma)}$: $\mathfrak{R}_{(L,\sigma)}(G) \to \mathfrak{R}_{(L,\sigma)}(L)$ is an equivalence of categories (keeping the hypotheses of Proposition 1.5). This will suffice to complete the proof of the theorem. Indeed, it is clear that the representation Σ is quasi-cuspidal. Further the hypothesis $W(L, \sigma) = \{1\}$ implies that ${}^w\Sigma \notin \mathfrak{R}_{(L,\sigma)}(L)$. Combining these two observations with 5.2 of [4], we see that

$$r_U^{(L,\sigma)}\, i_P^G\, \Sigma \cong \Sigma, \qquad r_U^{(L,\sigma)}\, i_{\bar{P}}^G\, \Sigma \cong \Sigma.$$

Therefore, if $r_U^{(L,\sigma)}$ is an equivalence of categories, then $i_P^G\,\Sigma \cong i_{\bar{P}}^G\,\Sigma$, as required.

Let V be an object in $\mathfrak{R}_{(L,\sigma)}(G)$ and W an object in $\mathfrak{R}_{(L,\sigma)}(L)$. The adjoint relation

$$\mathrm{Hom}_G(V, i_P^G W) \xrightarrow{\simeq} \mathrm{Hom}_L(r_U^{(L,\sigma)} V, W) \tag{1.6.1}$$

maps $\alpha \in \mathrm{Hom}_G(V, i_P^G W)$ to $e \circ r_U^{(L,\sigma)}(\alpha)$ where $e: r_U^{(L,\sigma)} i_P^G W \to W$ is induced by the natural L-homomorphism

$$f \mapsto f(1): i_P^G W \to W \otimes \delta_P^{1/2}.$$

In particular, the identity map of $i_P^G W$ corresponds to the natural L-homomorphism $e: r_U^{(L,\sigma)} i_P^G W \to W$. The hypothesis $W(L, \sigma) = \{1\}$ and 5.2 of [4] imply that this map is an isomorphism. Hence $r_U^{(L,\sigma)} \circ i_P^G \cong \mathrm{id}$, the identity functor of $\mathfrak{R}_{(L,\sigma)}(L)$.

Let $\alpha_V: V \to i_P^G r_U^{(L,\sigma)} V$ be the natural G-homomorphism corresponding, under (1.6.1), to $\mathrm{id}: r_U^{(L,\sigma)} V \to r_U^{(L,\sigma)} V$, the identity map of $r_U^{(L,\sigma)} V$. Thus $e \circ r_U^{(L,\sigma)}(\alpha_V) = \mathrm{id}$ where

$$e:\ r_U^{(L,\sigma)} i_P^G (r_U^{(L,\sigma)} V) \to r_U^{(L,\sigma)} V.$$

By the previous paragraph, e is an isomorphism, whence $r_U^{(L,\sigma)}(\alpha_V)$ is also an isomorphism. We now apply the exact functor $r_U^{(L,\sigma)}$ to the exact sequence

$$0 \longrightarrow \mathrm{Ker}\, \alpha_V \longrightarrow V \xrightarrow{\alpha_V} i_P^G r_U^{(L,\sigma)} V \longrightarrow \mathrm{Coker}\, \alpha_V \longrightarrow 0.$$

Since $r_U^{(L,\sigma)}(\alpha_V)$ is an isomorphism, the extreme terms

$$r_U^{(L,\sigma)}(\mathrm{Ker}\, \alpha_V) \quad \text{and} \quad r_U^{(L,\sigma)}(\mathrm{Coker}\, \alpha_V)$$

must both be zero. Using Corollary 1.5, we deduce that

$$\mathrm{Ker}\, \alpha_V = 0 \quad \text{and} \quad \mathrm{Coker}\, \alpha_V = 0.$$

Thus $\alpha_V: V \to i_P^G r_U^{(L,\sigma)} V$ is an isomorphism and $i_P^G \circ r_U^{(L,\sigma)} \cong \mathrm{id}$, the identity functor of $\mathfrak{R}_{(L,\sigma)}(G)$. Therefore $r_U^{(L,\sigma)}$ is indeed an equivalence of categories and we have completed the proof of the theorem. □

COROLLARY. *For any $P \in \mathcal{P}(L)$, the representation $i_P^G \Sigma$ is faithfully projective in $\mathfrak{R}_{(L,\sigma)}(G)$. In particular, the functor*

$$V \mapsto \mathrm{Hom}_G(i_P^G \Sigma, V):\ \mathfrak{R}_{(L,\sigma)}(G) \to \mathfrak{mod} - \mathrm{End}_G(i_P^G \Sigma)$$

is an equivalence of categories.

2. An Estimate on Length

Again let σ be an irreducible supercuspidal representation of a Levi subgroup L of G. Let P be a parabolic subgroup of G with Levi component L and write U for the unipotent radical of P. Let $W(L) = N(L)/L$ and set

$$\begin{aligned} N(L, \sigma) &= \{n \in N_G(L):\ {}^n\sigma \cong \sigma\nu,\ \text{for some } \nu \in X(L)\}, \\ W(L, \sigma) &= N(L, \sigma)/L. \end{aligned} \tag{2.0.2}$$

We now use Corollary 1.6 to show that the length of the induced representation $i_P^G\sigma$ is at most $|W(L,\sigma)|$. This slightly sharpens the bound $|W(L)|$ of 7.2.3 of [11].

As in Section 1, we write $p_{(L,\sigma)}\colon \mathfrak{R}(L) \to \mathfrak{R}_{(L,\sigma)}(L)$ for the projection functor (implied by the Bernstein decomposition of L) and set $r_U^{(L,\sigma)} = p_{(L,\sigma)} \circ r_U$.

PROPOSITION. *The functor* $r_U^{(L,\sigma)}\colon \mathfrak{R}_{(L,\sigma)}(G) \to \mathfrak{R}_{(L,\sigma)}(L)$ *takes nonzero objects to nonzero objects.*

Proof. Let Π be a nonzero object in $\mathfrak{R}_{(L,\sigma)}(G)$. By Corollary 1.6, $\mathrm{Hom}_G(i_{\bar{P}}^G\Sigma, \Pi) \neq 0$. Hence, by the second adjoint theorem, $\mathrm{Hom}_L(\Sigma, r_U^{(L,\sigma)}\Pi) \neq 0$. In particular, $r_U^{(L,\sigma)}\Pi \neq 0$. □

COROLLARY. *The length of the representation* $i_P^G\sigma$ *is at most* $|W(L,\sigma)|$. *In particular, if* $W(L,\sigma) = \{1\}$, *then* $i_P^G\sigma$ *is irreducible.*

Proof. By the proposition, $\mathrm{lt}\, i_P^G\sigma \leqslant \mathrm{lt}\, r_U^{(L,\sigma)} i_P^G\sigma$, where lt denotes length. By 2.12 of [4], $r_U^{(L,\sigma)} i_P^G\sigma$ has a filtration by L-subspaces whose associated graded module is $\bigoplus_{w\in W(L,\sigma)} {}^w\sigma$ and, hence, $\mathrm{lt}\, r_U^{(L,\sigma)} i_P^G\sigma = |W(L,\sigma)|$. □

3. Equivalences of Categories

Let L and M be Levi subgroups of G with L contained in M. We fix a parabolic subgroup P of G with Levi component L and write Q for the unique parabolic subgroup of G containing P and having Levi component M. Note that $M \cap P$ is then a parabolic subgroup of M with Levi component L. We again let σ be an irreducible supercuspidal representation of L. The parabolic induction functor $i_Q^G\colon \mathfrak{R}(M) \to \mathfrak{R}(G)$ then yields a functor

$$i_Q^G\colon\ \mathfrak{R}_{(L,\sigma)}(M) \to \mathfrak{R}_{(L,\sigma)}(G).$$

3.1. We now determine when this is an equivalence of categories. Our criterion is stated in terms of the group $W(L,\sigma)$ of (2.0.2) and the corresponding object for M which we denote by $W^M(L,\sigma)$. Thus $W^M(L,\sigma) = N^M(L,\sigma)/L$ where $N^M(L,\sigma) = M \cap N(L,\sigma)$.

THEOREM. *The functor* $i_Q^G\colon \mathfrak{R}_{(L,\sigma)}(M) \to \mathfrak{R}_{(L,\sigma)}(G)$ *is an equivalence of categories if and only if* $W(L,\sigma) = W^M(L,\sigma)$.

Proof. We first prove necessity. Thus suppose

$$i_Q^G\colon\ \mathfrak{R}_{(L,\sigma)}(M) \to \mathfrak{R}_{(L,\sigma)}(G)$$

is an equivalence of categories. Then there exists a functor

$$F\colon\ \mathfrak{R}_{(L,\sigma)}(G) \to \mathfrak{R}_{(L,\sigma)}(M)$$

such that

$$F \circ i_Q^G \cong \mathrm{id}, \qquad i_Q^G \circ F \cong \mathrm{id},$$

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where id denotes the identity functors, respectively, of $\mathfrak{R}_{(L,\sigma)}(M)$ and $\mathfrak{R}_{(L,\sigma)}(G)$. It follows that F furnishes a left (and a right) adjoint to i_Q^G.

Write N for the unipotent radical of Q and

$$r_N^{(L,\sigma)}\colon\ \mathfrak{R}_{(L,\sigma)}(G) \to \mathfrak{R}_{(L,\sigma)}(M)$$

for the composition of the projection functor

$$p_{(L,\sigma)}\colon\ \mathfrak{R}(M) \to \mathfrak{R}_{(L,\sigma)}(M)$$

and the restriction of the Jacquet functor $r_N\colon\ \mathfrak{R}(G) \to \mathfrak{R}(M)$ to $\mathfrak{R}_{(L,\sigma)}(G)$. Then $r_N^{(L,\sigma)}$ is also a left adjoint to i_Q^G, whence, by uniqueness of adjoints, $F \cong r_N^{(L,\sigma)}$. In particular, $r_N^{(L,\sigma)} \circ i_Q^G \cong \mathrm{id}$, the identity functor of $\mathfrak{R}_{(L,\sigma)}(M)$.

Evaluating at $i_{M\cap P}^M\sigma$, and using transitivity of parabolic induction, we obtain $r_N^{(L,\sigma)} i_P^G\sigma \cong i_{M\cap P}^M\sigma$. We next apply the functor

$$r_{U\cap M}^{(L,\sigma)}\colon\ \mathfrak{R}_{(L,\sigma)}(M) \to \mathfrak{R}_{(L,\sigma)}(L)$$

(which is left adjoint to $i_{M\cap P}^M\colon\ \mathfrak{R}_{(L,\sigma)}(L) \to \mathfrak{R}_{(L,\sigma)}(M)$). Using transitivity of Jacquet modules, this yields

$$r_U^{(L,\sigma)} i_P^G\sigma \cong r_{U\cap M}^{(L,\sigma)}\, i_{M\cap P}^M\sigma.$$

By [4] 2.12, the left side has length $|W(L,\sigma)|$ and the right side length $|W^M(L,\sigma)|$. We conclude that $W(L,\sigma) = W^M(L,\sigma)$.

3.2. We now begin the proof of sufficiency. Let V be an object in $\mathfrak{R}_{(L,\sigma)}(G)$ and W an object in $\mathfrak{R}_{(L,\sigma)}(M)$. The adjoint relation

$$\mathrm{Hom}_G(V, i_Q^G\, W) \cong \mathrm{Hom}_M(r_N^{(L,\sigma)} V, W), \tag{3.2.1}$$

gives rise to a natural transformation

$$e\colon\ r_N^{(L,\sigma)} \circ i_Q^G \to \mathrm{id}, \tag{3.2.2}$$

the identity functor of $\mathfrak{R}_{(L,\sigma)}(M)$. For each object W in $\mathfrak{R}_{(L,\sigma)}(M)$,

$$e = e_W\colon r_N^{(L,\sigma)} i_Q^G\, W \to W$$

corresponds, under (3.2.1), to the identity map of $i_Q^G\, W$: it is induced by the map

$$f \mapsto f(1)\colon\ i_Q^G\, W \to W \otimes \delta_Q^{1/2}.$$

In these terms, (3.2.1) is described by

$$s \mapsto e \circ r_N^{(L,\sigma)}(s), \qquad s \in \mathrm{Hom}_G(V, i_Q^G\, W).$$

Similarly, (3.2.1) gives rise to a natural transformation

$$\alpha\colon\ \mathrm{id} \to i_Q^G \circ r_N^{(L,\sigma)}, \tag{3.2.3}$$

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where id now denotes the identity functor of $\mathfrak{R}_{(L,\sigma)}(G)$. Again, for each object V in $\mathfrak{R}_{(L,\sigma)}(G)$,

$$\alpha = \alpha_V\colon V \to i_Q^G\, r_N^{(L,\sigma)} V$$

corresponds, under (3.2.1), to the identity map of $r_N^{(L,\sigma)} V$ and (3.2.1) is described by

$$t \longmapsto i_Q^G(t) \circ \alpha, \quad t \in \mathrm{Hom}_M(r_N^{(L,\sigma)} V, W). \tag{3.2.4}$$

3.3. We will show that (3.2.2) and (3.2.3) are equivalences under the hypothesis $W(L, \sigma) = W^M(L, \sigma)$. Our proof relies on the following general result:

PROPOSITION. *Let $\underline{A}$ and $\underline{B}$ be abelian categories with direct sums. Let f, g be right exact functors from $\underline{A}$ to $\underline{B}$ that preserve direct sums. Let $t\colon f \to g$ be a natural transformation such that, for some faithful object F in $\underline{A}$, $t_F\colon fF \to gF$ is an isomorphism. Then t is a natural equivalence.*

This follows from the five-lemma; see, for example, [12] p. 24.

The functors $r_N^{(L,\sigma)}$ and i_Q^G are exact. They also preserve direct sums, since, for example, they admit right adjoints. Therefore the functors $r_N^{(L,\sigma)} \circ i_Q^G$ and $i_Q^G \circ r_N^{(L,\sigma)}$ are each (right) exact and preserve direct sums.

3.4. We now prove that

$$e\colon\ r_N^{(L,\sigma)} i_Q^G(i_{M\cap P}^M \Sigma) \to i_{M\cap P}^M \Sigma$$

is an isomorphism. Since $i_{M\cap P}^M \Sigma$ is faithful in $\mathfrak{R}_{(L,\sigma)}(M)$, Proposition 3.3 will then imply that (3.2.2) is an equivalence.

First, we need some notation. Let A_0 denote a maximal split torus in L. Then A_0 is also a maximal split torus in M and in G. Let

$$W^G = N_G(A_0)/C_G(A_0), \qquad W^M = N_M(A_0)/C_M(A_0),$$
$$W^L = N_L(A_0)/C_L(A_0).$$

Note that, since $C_G(A_0)$ is contained in L,

$$C_G(A_0) = C_M(A_0) = C_L(A_0),$$

and, hence, that $W^G \supset W^M \supset W^L$. Let

$$N_0(L, \sigma) = \{n \in N_G(A_0) \cap N_G(L)\colon {}^n\sigma \cong \sigma\nu, \quad \text{for some } \nu \in X(L)\},$$
$$W_0(L, \sigma) = N_0(L, \sigma)/C_L(A_0).$$

The hypothesis $W(L, \sigma) = W^M(L, \sigma)$ implies that

$$W_0(L, \sigma) \subset W^M. \tag{3.4.1}$$

Consider the representation

$$r_N\, i_Q^G\, i_{M\cap P}^M \Sigma \cong r_N\, i_P^G \Sigma.$$

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By Theorem 5.2 of [4], this has a filtration by M-subspaces such that the associated graded object is

$$\bigoplus_{W^M \backslash \{w \in W^G : {}^wL \subset M\}/W^L} i^M_{M\cap {}^wP}({}^w\Sigma).$$

For any w in the index set, $i^M_{M\cap {}^wP}({}^w\Sigma) \in \mathfrak{R}_{({}^wL, {}^w\sigma)}(M)$. Thus, if $i^M_{M\cap {}^wP}({}^w\Sigma) \in \mathfrak{R}_{(L,\sigma)}(M)$, then there is a $w_1 \in W^M$ such that ${}^{w_1w}L = L$, ${}^{w_1w}\sigma \cong \sigma\nu$ for some $\nu \in X(L)$, i.e., $w_1 w \in W_0(L,\sigma)$. By (3.4.1), this forces $w \in W^M$. We conclude that

$$r_N^{(L,\sigma)} i_Q^G (i^M_{M\cap P}\Sigma) \cong i^M_{M\cap P}\Sigma.$$

By inspection of [4] 5.5, this isomorphism is e (up to a nonzero scalar).

3.5. It remains to show that (3.2.3) is an equivalence. We could deduce this from Proposition 2 by means of an argument from Section 1.6. Instead, we show that $\alpha\colon i_P^G\Sigma \to i_Q^G r_N^{(L,\sigma)} i_P^G\Sigma$ is an isomorphism. Using Proposition 3.3 once more, this will imply that (3.2.3) is an equivalence and so will complete the proof.

By (3.2.4), the adjoint relation (3.2.1) maps $e\colon r_N^{(L,\sigma)} i_P^G \Sigma \to i^M_{M\cap P}\Sigma$ to $i_Q^G(e)\circ\alpha$. Therefore $i_Q^G(e)\circ\alpha = \mathrm{id}$, the identity map of $i_P^G\Sigma$. Since e is an isomorphism and, hence, also $i_Q^G(e)$, we conclude that β is an isomorphism. □

3.6. As an immediate consequence, we obtain the following generalization of the second part of Corollary 2.

COROLLARY. *Suppose $W(L,\sigma) = W^M(L,\sigma)$. Let π be an irreducible object in $\mathfrak{R}_{(L,\sigma)}(M)$. Then $i_Q^G(\pi)$ is irreducible.*

4. An Example and Some Comments

We now show, by an example, that the converse of Corollary 3.6 above is false. Thus, by Theorem 3.1, parabolic induction can fail to be an equivalence of categories but still always take irreducible objects to irreducible objects. When L is maximal in G, Proposition 4.3 below shows that this occurs if and only if the action (induced by conjugation) of $W(L,\sigma)$ on $\mathrm{Irr}(L,\sigma)$, the set of equivalence classes of irreducible objects in $\mathfrak{R}_{(L,\sigma)}(L)$, has no fixed points. Of course, the example shows that Proposition 4.3 is not merely vacuously true; as explained in the introduction, it is, in essence, due to P. Kutzko.

4.1. To construct the supercuspidal σ of the example, we first recall, in the language of [6], a very special case of a construction of certain supercuspidal representations of $\mathrm{GL}_N(F)$ due to Carayol [10]. We refer to [6] for unexplained notation or terminology.

Let E/F be a totally ramified field extension of degree N. For any uniformizer ϖ_E in E, the element $\beta = \varpi_E^{-1}$ is minimal over F (see [6] 1.4.14 or [15]). We identify

$M_N(F)$ with $\mathrm{End}_F(E)$ and thus also $\mathrm{GL}_N(F)$ with $\mathrm{Aut}_F(E)$. Let $\mathfrak{A}$ be the $\mathrm{End}_F(E)$-stabilizer of the $\mathcal{O}_F$-lattice chain $\{\mathcal{P}_E^i : i \in \mathbb{Z}\}$ of E. Then $\mathfrak{A}$ is a minimal hereditary $\mathcal{O}_F$-order in $\mathrm{End}_F(E)$. Write $\mathfrak{P}$ for the Jacobson radical of $\mathfrak{A}$ and set $U^n(\mathfrak{A}) = 1 + \mathfrak{P}^n$ for $n \in \mathbb{N}$. Fixing an additive character ψ_F of F with conductor $\mathcal{P}_F$, we define a (linear) character ψ_β of $U^1(\mathfrak{A})/U^2(\mathfrak{A})$ by

$$\psi_\beta(1+x) = \psi_F(\mathrm{tr}\,(\beta x)), \quad x \in \mathfrak{P}.$$

Then the G-intertwining set $\mathcal{I}_G(\psi_\beta)$ equals $U^1(\mathfrak{A})E^\times$ (by a very special case of Theorem 1.5.8 of [6]). Since $U^1(\mathfrak{A}) \cap E^\times = 1 + \mathcal{P}_E$, ψ_β extends to the group $U^1(\mathfrak{A})E^\times$. Let Λ denote any such extension. Then the compactly induced representation $\pi = \mathrm{ind}\,\Lambda$ is irreducible and, hence, supercuspidal.

We write $\mathfrak{S}(\pi)$ for the set of characters χ of $F^\times$ such that $\pi \cong \pi\chi$ where $\pi\chi$ denotes the representation

$$g \mapsto \pi(g)\chi(\det g), \quad g \in G = \mathrm{GL}_N(F).$$

Clearly, $\chi \in \mathfrak{S}(\pi)$ if and only if Λ and $\Lambda\chi$ intertwine where $\Lambda\chi$ has the analagous meaning. We assume now that the extension E/F is tamely ramified. Then, by the discussion of projective normalizers in [8], Λ and $\Lambda\chi$ intertwine if and only if $\Lambda = \Lambda\chi$ which occurs if and only if

$$\mathrm{Ker}\,\chi \supset \det\,(U^1(\mathfrak{A})E^\times).$$

Since E/F is tame, $N_{E/F}(1 + \mathcal{P}_E) = 1 + \mathcal{P}_F$ and, hence,

$$\det\,(U^1(\mathfrak{A})E^\times) = N_{E/F}(E^\times).$$

Therefore $\mathfrak{S}(\pi) = \{\chi : \chi \circ N_{E/F} = 1\}$.

We now fix a uniformizer ϖ_F in F and let ζ denote a primitive $(q-1)$-st root of unity in F where q is the order of the residue field k_F. We assume also that 4 divides $q-1$. It follows that there is a unique character $\eta : F^\times \to \mathbb{C}^\times$ that is trivial on ϖ_F and $1 + \mathcal{P}_F$ and satisfies $\eta(\zeta) = \sqrt{-1}$.

We need two special cases of the above construction. First, let E_1 be the splitting field of the polynomial $X^2 + \varpi_F$. Choose a uniformizer ϖ_{E_1} in E_1 such that $\varpi_{E_1}^2 = -\varpi_F$ and set $\beta_1 = \varpi_{E_1}^{-1}$. Let σ_1 denote a supercuspidal representation of $\mathrm{GL}_2(F)$ constructed from β_1 as above. Since $N_{E_1/F}(E_1^\times) = (1 + \mathcal{P}_F)\,\langle\zeta^2\rangle\,\langle\varpi_F\rangle$,

$$\mathfrak{S}(\sigma_1) = \{1, \eta^2\}. \tag{4.1.1}$$

Second, let E_2 be the splitting field of the polynomial $X^4 + \zeta\varpi_F$. Fix a uniformizer ϖ_{E_2} in E_2 with $\varpi_{E_2}^4 = -\zeta\varpi_F$ and set $\beta_2 = \varpi_{E_2}^{-1}$. Let σ_2 be a supercuspidal representation of $\mathrm{GL}_4(F)$ obtained from β_2 as above. Using

$$N_{E_2/F}(E_2^\times) = (1 + \mathcal{P}_F)\,\langle\zeta^4\rangle\,\langle\varpi_F\zeta\rangle,$$

we deduce that

$$\mathfrak{S}(\sigma_2) = \langle\chi_0\eta\rangle \tag{4.1.2}$$

with χ_0 the unramified character of $F^\times$ such that $\chi_0(\varpi_F) = -\sqrt{-1}$.

Let $\tilde{L}$ be the standard Levi subgroup $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_4(F)$ of $\mathrm{GL}_8(F)$. Then $\tilde{\sigma} = \sigma_1 \otimes \sigma_1\eta \otimes \sigma_2$ is an irreducible supercuspidal representation of $\tilde{L}$. We let $L = \tilde{L} \cap \mathrm{SL}_8(F)$. As above, we put

$$\mathfrak{S}(\tilde{\sigma}) = \{\chi: F^\times \to \mathbb{C}^\times : \tilde{\sigma} \cong \tilde{\sigma}\chi\},$$

where $\tilde{\sigma}\chi$ again has the obvious meaning. Since $\tilde{\sigma}$ admits a Whittaker model, the restriction $\tilde{\sigma}|L$ is multiplicity-free (see, for example, [16]). (For our particular $\tilde{\sigma}|L$, this also follows immediately from [8] Corollary 1.6 iv).) Therefore, by, for instance, Section 1 of [8] (see especially Corollary 1.6 and Remark (ii)), the length of the restriction $\tilde{\sigma}|L$ equals $|\mathfrak{S}(\tilde{\sigma})|$.

Let $\sigma = \tilde{\sigma}|L$. Then $\mathfrak{S}(\tilde{\sigma}) = \mathfrak{S}(\sigma_1) \cap \mathfrak{S}(\sigma_2)$ which is trivial by (4.1.1) and (4.1.2). Therefore σ is an irreducible, and supercuspidal, representation of L.

4.2. We have $W(L) = N(L)/L = \{1, w\}$ where the non-trivial element w interchanges (up to conjugacy in L) the two 2×2 blocks of L. Then, using (4.1.1) and (4.1.2),

$$\begin{aligned} {}^w\tilde{\sigma} &\cong \sigma_1\eta \otimes \sigma_1 \otimes \sigma_2 \\ &\cong (\sigma_1 \otimes \sigma_1\eta \otimes \sigma_2\chi_0)\eta \\ &\cong \tilde{\sigma}\tilde{\chi}_0\eta, \end{aligned}$$

where $\tilde{\chi}_0 \in X(\tilde{L})$ is given by

$$(g_1, g_2, g_3) \mapsto \chi_0(\det g_3), \quad g_1, g_2 \in \mathrm{GL}_2(F),\ g_3 \in \mathrm{GL}_4(F).$$

Hence, ${}^w\sigma \cong \sigma\nu_0$, where $\nu_0 = \tilde{\chi}_0|L \in X(L)$, since $\eta|L = 1$. We conclude that $W(L, \sigma) = \{1, w\}$. In particular, $i_P^G: \mathfrak{R}_{(L,\sigma)}(L) \to \mathfrak{R}_{(L,\sigma)}(G)$ is *not* an equivalence of categories for any parabolic subgroup P of $G = \mathrm{SL}_8(F)$ with Levi factor L. We now show, however, that $i_P^G(\sigma\nu)$ is irreducible for all $\nu \in X(L)$.

Recall that $\mathrm{Irr}(L, \sigma)$ denotes the set of equivalence classes of irreducible objects in $\mathfrak{R}_{(L,\sigma)}(L)$. Thus each element of $\mathrm{Irr}(L, \sigma)$ is represented by an unramified twist of σ. The action of $N(L, \sigma)$ on L by conjugation induces an action of $W(L, \sigma)$ on $\mathrm{Irr}(L, \sigma)$. Suppose now that $\sigma\nu$ belongs to the fixed-point set $\mathrm{Irr}(L, \sigma)^{W(L,\sigma)}$ for some $\nu \in X(L)$, i.e., ${}^w(\sigma\nu) \cong \sigma\nu$. Then ${}^w(\tilde{\sigma}\tilde{\nu})$ and $\tilde{\sigma}\tilde{\nu}$ share an irreducible component on restriction to L where $\tilde{\nu}$ is any element of $X(\tilde{L})$ such that $\tilde{\nu}|L = \nu$. Using (the proof of) Proposition 1.17 (ii) [7], we see there exists a character χ of $F^\times$ such that ${}^w(\tilde{\sigma}\tilde{\nu}) \cong \tilde{\sigma}\tilde{\nu}\chi$. Hence, there are unramified characters ν_1, ν_1' of $\mathrm{GL}_2(F)$ and ν_2 of $\mathrm{GL}_4(F)$ such that

$$\sigma_1\eta\nu_1' \cong \sigma_1\nu_1\chi, \qquad \sigma_1\nu_1 \cong \sigma_1\eta\nu_1'\chi, \qquad \sigma_2\nu_2 \cong \sigma_2\nu_2\chi.$$

The final equivalence implies that $\chi \in \langle\chi_0\,\eta\rangle$ (by (4.1.2)). Tensoring the first equivalence with χ and comparing with the second, we obtain $\sigma_1\chi^2 \cong \sigma_1$ (i.e., $\chi^2 \in \mathfrak{S}(\sigma_1)$). Hence, by (4.1.1), $\chi^2 = \eta^2$ or $\chi^2 = 1$. Using $\chi \in \langle\chi_0\,\eta\rangle$, we see that $\chi^2 = \eta^2$ is impossible. Therefore we must have $\chi^2 = 1$, whence $\chi = 1$ or $\chi = \chi_0^2\eta^2$. However this is inconsistent with the first equivalence by (4.1.1). This contradiction proves that $\mathrm{Irr}(L, \sigma)^{W(L,\sigma)}$ is empty.

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Since $W(\tilde{L}, \tilde{\sigma}) = \{1\}$, $i_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}\tilde{\nu})$ is irreducible for all $\tilde{\nu} \in X(\tilde{L})$ where $\tilde{P}$ is any parabolic subgroup of $\tilde{G} = \mathrm{GL}_8(F)$ with Levi factor $\tilde{L}$. Let χ be a character of $F^\times$ such that $i_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}\tilde{\nu}) \cong i_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}\tilde{\nu})\,\chi$ for some $\tilde{\nu} \in X(\tilde{L})$. Then $\tilde{\sigma} \cong \tilde{\sigma}\chi$ or ${}^{w}(\tilde{\sigma}\tilde{\nu}) \cong \tilde{\sigma}\tilde{\nu}\chi$. We have just seen that the latter is impossible. Therefore $\chi \in \mathfrak{S}(\tilde{\sigma}) = \{1\}$ and hence $\mathfrak{S}(i_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}\tilde{\nu})) = \{1\}$. Since, by [16], the restriction $i_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}\tilde{\nu})|\,G$ is multiplicity-free, it follows as above that $i_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}\tilde{\nu})|\,G$ is irreducible. Finally, it is easy to verify that $i_{\tilde{P}}^{\tilde{G}}(\tilde{\sigma}\tilde{\nu})|\,G \cong i_P^G(\sigma\nu)$, where $\nu = \tilde{\nu}|L$ and $P = \tilde{P} \cap G$ (or see 9.2 of [8]). We conclude that $i_P^G(\sigma\nu)$ is irreducible for all $\nu \in X(L)$.

4.3. Let G again be a general (reductive p-adic) group. Let L be a maximal Levi subgroup of G. From the example, we know that the functor $i_P^G\colon \mathfrak{R}_{(L,\sigma)}(L) \to \mathfrak{R}_{(L,\sigma)}(G)$ can fail to be an equivalence of categories but still always take irreducible objects to irreducible objects. (Simply take L and σ as in 4.2 and let G be the intersection of $\mathrm{SL}_8(F)$ and the standard Levi subgroup $\mathrm{GL}_4(F) \times \mathrm{GL}_4(F)$ of $\mathrm{GL}_8(F)$.)

We now show that a feature we noted in this example, that the fixed-point set $\mathrm{Irr}(L, \sigma)^{W(L,\sigma)}$ is empty, in fact characterizes such functors.

PROPOSITION. *Let L be a maximal Levi subgroup of G. The functor $i_P^G\colon \mathfrak{R}_{(L,\sigma)}(L) \to \mathfrak{R}_{(L,\sigma)}(G)$ is not an equivalence of categories but takes irreducible objects to irreducible objects if and only if* $\mathrm{Irr}(L, \sigma)^{W(L,\sigma)}$ *is empty.*

Proof. If $W(L, \sigma) = \{1\}$, then $i_P^G\colon \mathfrak{R}_{(L,\sigma)}(L) \to \mathfrak{R}_{(L,\sigma)}(G)$ is an equivalence of categories and, clearly, $\mathrm{Irr}(L, \sigma)^{W(L,\sigma)}$ is nonempty. We assume therefore that $W(L, \sigma) = \{1, w\}$ with $w \neq 1$.

Suppose first that $\mathrm{Irr}(L, \sigma)^{W(L,\sigma)}$ is nonempty. Adjusting σ if necessary, we have ${}^{w}\sigma \cong \sigma$. We write $\sigma = \sigma_0\,\eta$ where σ_0 is unitary and $\eta \in X_r(L) = \mathrm{Hom}(L, \mathbb{R}_{>0})$. Therefore, ${}^{w}\sigma_0 \cong \sigma_0\zeta$ where $\zeta = \eta\,({}^{w}\eta)^{-1} \in X_r(L)$. Examining central characters, we see that $\zeta = 1$, whence ${}^{w}\sigma_0 \cong \sigma_0$. By results of Harish-Chandra and Silberger (see, for example, Lemma 2.2 of [18]), it follows that one of two possibilities must occur: either $i_P^G\,\sigma_0$ is itself reducible or $i_P^G(\sigma_0\chi)$ is reducible for some (nontrivial) $\chi \in X_r(L)$. In particular, $i_P^G(\sigma\nu)$ is reducible for some $\nu \in X(L)$.

Finally, we show that if reducibility occurs then $\mathrm{Irr}(L, \sigma)^{W(L,\sigma)}$ is nonempty. We might as well assume that $i_P^G(\sigma)$ is itself reducible. Then $i_P^G(\sigma)$ has length two and so we have an exact sequence

$$0 \to \pi_1 \to i_P^G(\sigma) \to \pi_2 \to 0$$

with π_i irreducible for $i = 1, 2$.

If this sequence splits, then a simple Jacquet module calculation, as in the initial part of the proof of Proposition 1.5, shows that ${}^{w}\sigma \cong \sigma$. We assume therefore that the sequence is nonsplit. In this case, we can use Casselman's square-integrability criterion exactly as in the remainder of the proof of Proposition 1.5 to show that ${}^{w}\sigma_0 \cong \sigma_0$ for a certain (unitary) representation in $\mathrm{Irr}(L, \sigma)$. Hence, in either case, $\mathrm{Irr}(L, \sigma)^{W(L,\sigma)}$ is nonempty. □

Remark. The phenomenon described in the proposition is quite rare. In particular, it is not hard to show that $\mathrm{Irr}(L,\sigma)^{W(L,\sigma)}$ is always nonempty when G is semisimple. Indeed, suppose $W(L)=\{1,w\}$ with $w\neq 1$. (Of course, the case where $W(L)=\{1\}$ is utterly trivial.) It is straightforward to check that w acts nontrivially on $X(L)$ (or see the proof of (6) on page 48 of [17]). Hence, the homomorphism

$$\nu\mapsto\nu\,({}^{w}\nu)^{-1}\colon\ X(L)\to X(L)$$

has nontrivial image. When G is semisimple, the complex torus $X(L)$ is one-dimensional. It follows that, in this case, the above homomorphism is surjective (and coincides with the map $\nu\mapsto\nu^2\colon X(L)\to X(L)$). Therefore, if ${}^{w}\sigma\cong\sigma\zeta$ for some $\zeta\in X(L)$, we may write $\zeta=\eta\,({}^{w}\eta)^{-1}$ for some $\eta\in X(L)$, whence ${}^{w}(\sigma\eta)\cong\sigma\eta$. (A variant of this simple argument was pointed out to us by M. Reeder.)

Remark. The proposition is no longer valid, as stated, when the Levi subgroup L is not maximal.

5. Induction and Restriction

We return to the system of notation of Section 3. Thus L and M are Levi subgroups of G with L contained in M, P is a parabolic subgroup of G with Levi component L and Q is the unique parabolic subgroup of G containing P and having Levi component M. Of course, $M\cap P$ is then a parabolic subgroup of M with Levi component L. As always, σ is an irreducible supercuspidal representation of L.

Let $\mathcal{A}=\mathrm{End}_G(i_P^G\Sigma)$, $\mathcal{B}=\mathrm{End}_M(i_{M\cap P}^M\Sigma)$. By Proposition 1.2 and Corollary 1.6, the functors

$$\begin{aligned}&W\mapsto\mathrm{Hom}_M(i_{M\cap P}^M\Sigma,W)\colon\ \mathfrak{R}_{(L,\sigma)}(M)\to\mathfrak{mod}-\mathcal{B},\\&V\mapsto\mathrm{Hom}_G(i_P^G\Sigma,V)\colon\ \mathfrak{R}_{(L,\sigma)}(G)\to\mathfrak{mod}-\mathcal{A}\end{aligned}\tag{5.0.1}$$

are equivalences of categories.

Parabolic induction corresponds, under these equivalences, to a functor $t\colon\mathfrak{mod}-\mathcal{B}\to\mathfrak{mod}-\mathcal{A}$, i.e., there is a commutative diagram of functors

$$\begin{array}{ccc}\mathfrak{R}_{(L,\sigma)}(G)&\xrightarrow{\cong}&\mathfrak{mod}-\mathcal{A}\\ i_Q^G\uparrow&&\uparrow t\\ \mathfrak{R}_{(L,\sigma)}(M)&\xrightarrow{\cong}&\mathfrak{mod}-\mathcal{B}.\end{array}\tag{5.0.2}$$

We describe t below (up to natural equivalence).

5.1. Let $b\in\mathcal{B}=\mathrm{End}_M(i_{M\cap P}^M\Sigma)$. Since induction is an additive functor, the process $b\mapsto i_Q^G(b)$ defines a ring homomorphism from $\mathrm{End}_M(i_{M\cap P}^M\Sigma)$ to $\mathrm{End}_G(i_Q^G\,i_{M\cap P}^M\Sigma)$. Explicitly, if we realise (the space of) $i_Q^G(i_{M\cap P}^M\Sigma)$ as a space of functions from G to (the space of) $i_{M\cap P}^M\Sigma$ in the usual way, then

$$(i_Q^G(b)F)(g)=b(F(g)),\tag{5.1.1}$$

for all $b\in\mathrm{End}_M(i_{M\cap P}^M\Sigma)$, $F\in i_Q^G(i_{M\cap P}^M\Sigma)$, $g\in G$.

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Transitivity of parabolic induction gives a canonical isomorphism

$$i_Q^G\, i_{M\cap P}^M\, \Sigma \cong i_P^G\, \Sigma. \tag{5.1.2}$$

We therefore obtain a ring homomorphism $t_Q\colon \mathcal{B} \to \mathcal{A}$. The functor i_Q^G is exact and clearly takes nonzero objects to nonzero objects. It is therefore faithful. In particular, the homomorphism t_Q is injective. (Of course, injectivity of t_Q also follows directly from (5.1.1).)

We view $\mathcal{A}$ as a left $\mathcal{B}$-module via t_Q (i.e., $b.a = t_Q(b)a$ for $b \in \mathcal{B}$ and $a \in \mathcal{A}$) and as a right $\mathcal{A}$-module via multiplication in $\mathcal{A}$. This defines a $(\mathcal{B}, \mathcal{A})$-bimodule structure on $\mathcal{A}$. We show in 5.3 that t is equivalent to the resulting functor

$$- \otimes_{\mathcal{B}} \mathcal{A}\colon\ \mathfrak{mod} - \mathcal{B} \to \mathfrak{mod} - \mathcal{A}.$$

5.2. We will need the following general observation on adjoint isomorphisms. Let $\underline{C}$ and $\underline{D}$ be additive categories and $F\colon \underline{D} \to \underline{C}$ and $G\colon \underline{C} \to \underline{D}$ a pair of adjoint functors. It follows easily that F and G are also additive functors (Corollary 7.2 of [1]) and thus, for each object M in $\underline{D}$ and N in $\underline{C}$, there is a natural isomorphism of Abelian groups

$$\alpha\colon \mathrm{Hom}_{\underline{C}}(FM, N) \xrightarrow{\simeq} \mathrm{Hom}_{\underline{D}}(M, GN). \tag{5.2.1}$$

The abelian group $\mathrm{Hom}_{\underline{C}}(FM, N)$ is a right $\mathrm{End}_{\underline{C}}\, FM$-module via composition (i.e., $x.y = x \circ y$ for $x \in \mathrm{Hom}_{\underline{C}}(FM, N)$, $y \in \mathrm{End}_{\underline{C}}(FM)$). It is therefore also a right $\mathrm{End}_{\underline{D}}\, M$-module via the ring homomorphism $\mathrm{End}_{\underline{D}}\, M \to \mathrm{End}_{\underline{C}}\, FM$ induced by F. Similarly, $\mathrm{Hom}_{\underline{D}}(M, GN)$ is a right $\mathrm{End}_{\underline{D}} M$-module via composition.

Since the isomorphism (5.2.1) is natural, we have

$$\alpha(f Fg) = \alpha(f)\, g, \quad f \in \mathrm{Hom}_{\underline{C}}(FM, N), \quad g \in \mathrm{End}_{\underline{D}}\, M;$$

that is, α is an isomorphism of right $\mathrm{End}_{\underline{D}}\, M$-modules.

5.3. We now apply this observation in the context of the second adjoint theorem and the equivalences (5.0.1).

Recall that N denotes the unipotent radical of Q. Write $\bar{N}$ for the unipotent radical of the M-opposite $\bar{Q}$ of Q. Let $r_{\bar{N}}^{(L,\sigma)} = p_{(L,\sigma)} \circ r_{\bar{N}}$ where $r_{\bar{N}}\colon\ \mathfrak{R}_{(L,\sigma)}(G) \to \mathfrak{R}(M)$ is the normalized Jacquet functor (restricted to $\mathfrak{R}_{(L,\sigma)}$ (G)) and $p_{(L,\sigma)}\colon\ \mathfrak{R}(M) \to \mathfrak{R}_{(L,\sigma)}(M)$ projects from $\mathfrak{R}(M)$ to $\mathfrak{R}_{(L,\sigma)}(M)$. By the second adjoint theorem and the Bernstein decomposition of $\mathfrak{R}(M)$,

$$r_{\bar{N}}^{(L,\sigma)}\colon\ \mathfrak{R}_{(L,\sigma)}(G) \to \mathfrak{R}_{(L,\sigma)}(M)$$

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is right adjoint to $i_Q^G\colon \mathfrak{R}_{(L,\sigma)}(M) \to \mathfrak{R}_{(L,\sigma)}(G)$. In particular, for each object V in $\mathfrak{R}_{(L,\sigma)}(G)$, there is a natural isomorphism

$$\mathrm{Hom}_G(i_Q^G\, i_{M\cap P}^M\,\Sigma,\, V) \xrightarrow{\simeq} \mathrm{Hom}_M(i_{M\cap P}^M\,\Sigma,\, r_{\bar N}^{(L,\sigma)}V).$$

By the discussion in Subsection 5.2, this is an isomorphism of $\mathcal{B}$-modules (viewing the left-hand side as a $\mathcal{B}$ module via the embedding $t_Q\colon\mathcal{B}\to\mathcal{A}$ and the isomorphism (5.1.2)).

We rephrase this as follows: Write $\mathrm{res}_{\mathcal{B}}^{\mathcal{A}}\colon \mathfrak{mod}-\mathcal{B}\to\mathfrak{mod}-\mathcal{A}$ for the functor defined by restriction along the embedding t_Q. Thus, if N is an object in $\mathfrak{mod}-\mathcal{A}$, $\mathrm{res}_{\mathcal{B}}^{\mathcal{A}}(N)=N$ and $n.b=nt_Q(b)$ for $n\in N$, $b\in\mathcal{B}$. Then the following diagram of functors commutes:

$$\begin{array}{ccc}\mathfrak{R}_{(L,\sigma)}(G) & \xrightarrow{\simeq} & \mathfrak{mod}-\mathcal{A}\\ r_{\bar N}^{(L,\sigma)}\downarrow & & \downarrow \mathrm{res}_{\mathcal{B}}^{\mathcal{A}}\\ \mathfrak{R}_{(L,\sigma)}(M) & \xrightarrow{\simeq} & \mathfrak{mod}-\mathcal{B}.\end{array}$$

Using uniqueness of left adjoints, it is now a simple matter to describe the functor t in (5.0.2) (up to natural equivalence). Indeed, for objects M in $\mathfrak{mod}-\mathcal{B}$ and N in $\mathfrak{mod}-\mathcal{A}$, the process

$$f\mapsto(m\mapsto f(m\otimes 1))\colon \mathrm{Hom}_{\mathcal{A}}(M\otimes_{\mathcal{B}}\mathcal{A},\, N)\xrightarrow{\simeq}\mathrm{Hom}_{\mathcal{B}}(M,\, \mathrm{res}_{\mathcal{B}}^{\mathcal{A}}\, N),$$

defines a natural isomorphism. Thus $(-\otimes_{\mathcal{B}}\mathcal{A}, \mathrm{res}_{\mathcal{B}}^{\mathcal{A}})$ is an adjoint pair and hence $t\cong -\otimes_{\mathcal{B}}\mathcal{A}$. We have proved the following:

THEOREM. *With notation as above, the following diagrams of functors commute up to natural equivalence:*

$$\begin{array}{ccc}\mathfrak{R}_{(L,\sigma)}(G) & \xrightarrow{\simeq} & \mathfrak{mod}-\mathcal{A}\\ r_{\bar N}^{(L,\sigma)}\downarrow & & \downarrow \mathrm{res}_{\mathcal{B}}^{\mathcal{A}}\\ \mathfrak{R}_{(L,\sigma)}(M) & \xrightarrow{\simeq} & \mathfrak{mod}-\mathcal{B},\\ \mathfrak{R}_{(L,\sigma)}(G) & \xrightarrow{\simeq} & \mathfrak{mod}-\mathcal{A}\\ i_Q^G\uparrow & & \uparrow -\otimes_{\mathcal{B}}\mathcal{A}\\ \mathfrak{R}_{(L,\sigma)}(M) & \xrightarrow{\simeq} & \mathfrak{mod}-\mathcal{B}.\end{array}$$

Remark. The second diagram yields another proof that parabolic induction takes finitely generated objects to finitely generated objects (cf. Remark 1.3). It suffices to show that if W is finitely generated in $\mathfrak{R}_{(L,\sigma)}(M)$, then $i_Q^G W$ is finitely generated in $\mathfrak{R}_{(L,\sigma)}(G)$. Now, in an abelian category with direct sums, the notion of finite generation can be expressed in purely categorical terms and is preserved under equivalences of categories. Therefore, by the second diagram, we have only to prove that if M is a finitely generated right $\mathcal{B}$-module, then $M\otimes_{\mathcal{B}}\mathcal{A}$ is a finitely generated right $\mathcal{A}$-module. This is obvious.

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