COMMUTATORS IN FACTORS OF TYPE III

ARLEN BROWN AND CARL PEARCY

1. Introduction. Let $\mathfrak{H}$ denote a separable, complex Hilbert space, and let $\mathbf{R}$ be a von Neumann algebra acting on $\mathfrak{H}$. (A von Neumann algebra is a weakly closed, self-adjoint algebra of operators that contains the identity operator on its underlying space.) An element A of $\mathbf{R}$ is a *commutator in* $\mathbf{R}$ if there exist operators B and C in $\mathbf{R}$ such that $A = BC - CB$. The problem of specifying exactly which operators are commutators in $\mathbf{R}$ has been solved in certain special cases; e.g. if $\mathbf{R}$ is an algebra of type I_n $(n < \infty)$ **(2)**, and if $\mathbf{R}$ is a factor of type I_∞ **(1)**. It is the purpose of this note to treat the same problem in case $\mathbf{R}$ is a factor of type III. Our main result is the following theorem.

THEOREM 1. *If* $\mathbf{R}$ *is a factor of type* III *acting on a separable Hilbert space, then the commutators in* $\mathbf{R}$ *consist exactly of the non-scalar operators together with the operator* 0.

The proof splits naturally into two parts, and is obtained by suitably modifying the methods of **(1)** so as to make them applicable to factors of type III. In §§ 2–3 (as in (**1**, §3)) we obtain a matricial standard form for the operators we wish to show are commutators, and in §4 (as in (**1**, §4)) there follows a sequence of constructions showing that every such matrix is a commutator.

2. Non-central elements. Throughout the paper $\mathfrak{H}$ will be a fixed complex, separable, infinite-dimensional Hilbert space. We begin by constructing a standard form under similarity transformations for non-central elements of a factor of type III acting on $\mathfrak{H}$. The construction is of some inherent interest and will, therefore, be carried out initially in an arbitrary von Neumann algebra, and then specialized to the case of a type III factor. The following terminology will be convenient: if E is a projection in a von Neumann algebra $\mathbf{R}$ and the range of E is $\mathfrak{M}$, then $\mathfrak{M}$ will be said to *belong* to $\mathbf{R}$ ($\mathfrak{M} \in \mathbf{R}$); if $\mathfrak{M}$ and $\mathfrak{N}$ are two subspaces belonging to $\mathbf{R}$, then $\mathfrak{M}$ and $\mathfrak{N}$ will be called *equivalent* in $\mathbf{R}$ provided their projections are equivalent in $\mathbf{R}$ (in the sense of Murray–von Neumann). We recall the well-known facts that if $A \in \mathbf{R}$ and $\mathfrak{M} \in \mathbf{R}$, then the closure $\mathfrak{N}$ of $A(\mathfrak{M})$ belongs to $\mathbf{R}$, and that if, in addition, A maps $\mathfrak{M}$ onto $A(\mathfrak{M})$ in one–one fashion, then $\mathfrak{M}$ and $\mathfrak{N}$ are equivalent in $\mathbf{R}$.

If E and F are any two projections on $\mathfrak{H}$, with ranges $\mathfrak{M}$ and $\mathfrak{N}$ respectively, we write $\mathfrak{A}(\mathfrak{M}, \mathfrak{N}) = \|EF\|$. It may readily be verified that

$$\mathfrak{A}(\mathfrak{M}, \mathfrak{N}) = \sup\{|(x, y)|: x \in \mathfrak{M}, y \in \mathfrak{N}, \|x\| = \|y\| = 1\}.$$

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The facts we shall need concerning this function of pairs of subspaces are summarized in the following preparatory lemma, which is largely folklore but for which it seems difficult to give a single reference.

LEMMA 2.1. *Let E and F be projections with ranges $\mathfrak{M}$ and $\mathfrak{N}$, and suppose $\mathfrak{A}(\mathfrak{M}, \mathfrak{N}) < 1$. Then $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ and $\mathfrak{K} = \mathfrak{M} + \mathfrak{N}$ is closed. Moreover, there exists a partial isometry W with initial space $\mathfrak{K} \ominus \mathfrak{M}$ and final space $\mathfrak{N}$, and for any such W the operator $E + W$ maps $\mathfrak{K}$ invertibly onto itself. Finally, if* **R** *is any von Neumann algebra containing E and F, then W, and hence $E + W$, can be chosen so as to belong to* **R**.

Proof. That $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ is clear. Let $\mathfrak{A}(\mathfrak{M}, \mathfrak{N}) = \sigma$. Then for $x \in \mathfrak{M}$, $y \in \mathfrak{N}$ we have $|(x, y)| \leqslant \sigma\|x\| \, \|y\|$ and consequently

$$\begin{aligned}\|x + y\|^2 &\geqslant \|x\|^2 - 2\sigma\|x\| \, \|y\| + \|y\|^2 \\ &= (1 - \sigma)(\|x\|^2 + \|y\|^2) + \sigma(\|x\| - \|y\|)^2.\end{aligned}$$

It follows that the mappings $x + y \to x$ and $x + y \to y$ of $\mathfrak{K}$ into itself are bounded, and hence that $\mathfrak{K}$ is closed.

Let $\mathfrak{L} = F(\mathfrak{K} \ominus \mathfrak{M})$. Clearly $\mathfrak{K}$ and $\mathfrak{K} \ominus \mathfrak{M}$, and hence cl $\mathfrak{L}$, all belong to any von Neumann algebra **R** that contains E and F. Moreover, F maps $\mathfrak{K} \ominus \mathfrak{M}$ onto $\mathfrak{L}$ in one–one fashion. Hence in order to show that $\mathfrak{K} \ominus \mathfrak{M}$ and $\mathfrak{N}$ are equivalent in **R**, it suffices to verify that cl $\mathfrak{L} = \mathfrak{N}$. But now, if $y \in \mathfrak{N}$ and $y \perp \mathfrak{L}$, then $y \perp F(y - Ey)$ so that

$$0 = (y, F(y - Ey)) = (y, y - Ey) = \|(1 - E)y\|^2.$$

Thus $y \in \mathfrak{M}$ and therefore $y = 0$.

It remains to show that if W is any partial isometry with initial space $\mathfrak{K} \ominus \mathfrak{M}$ and final space $\mathfrak{N}$, then $E + W$ is invertible when regarded as a mapping of $\mathfrak{K}$ into $\mathfrak{K}$. Since $\mathfrak{K}$ has been shown to be closed, the closed graph theorem applies, and it suffices to verify that $E + W$ maps $\mathfrak{K}$ onto itself in one–one fashion, a fact that is easily established by direct calculation.

LEMMA 2.2. *Let* **R** *be a von Neumann algebra acting on $\mathfrak{H}$, and let A be a non-central element of* **R**. *Then there exist non-trivial subspaces $\mathfrak{M}$ and $\mathfrak{N}$ belonging to* **R** *such that $\mathfrak{A}(\mathfrak{M}, \mathfrak{N}) < 1$ and such that A maps $\mathfrak{M}$ onto $\mathfrak{N}$ in one–one fashion.*

Proof. Since A is not in the centre of **R**, there exists a projection $E \in$ **R** such that $AE \neq EAE$. We choose one such E and keep it fixed for the duration of the proof. Then $T = (1 - E)AE$ is a non-zero element of **R** and the same is true of $P = (T^* T)^{1/2}$. Let $0 < \epsilon < \|P\| = \|T\|$, let F denote the spectral projection of P associated with the interval $[\epsilon, \|P\|]$, and let $\mathfrak{L} = F(\mathfrak{H})$. Note that $F \in$ **R** and $F \neq 0$. It follows from the spectral theorem that $\|Px\| \geqslant \epsilon\|x\|$ for all $x \in \mathfrak{L}$, and therefore, for such x,

$$\epsilon\|x\| \leqslant \|Px\| = \|Tx\| = \|(1 - E)AEx\| \leqslant \|A\| \, \|Ex\|.$$

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Thus

$$\|Ex\| \geqslant (\epsilon/\|A\|)\|x\|, \qquad x \in \mathfrak{L},$$

whence it follows that $\mathfrak{M} = E(\mathfrak{L})$ is closed and belongs to **R**. Moreover, if $y \in \mathfrak{M}$, then $y = Ex$ for some $x \in \mathfrak{L}$, and we have

$$\|Ay\| = \|AEx\| \geqslant \|Tx\| \geqslant \epsilon\|x\| \geqslant \epsilon\|y\|$$

so that A is bounded below on $\mathfrak{M}$. It follows at once that $\mathfrak{N} = A(\mathfrak{M})$ is closed and belongs to **R** and also, of course, that A maps $\mathfrak{M}$ onto $\mathfrak{N}$ in one–one fashion.

It remains only to show that $\mathfrak{A}(\mathfrak{M}, \mathfrak{N}) < 1$. To this end choose $\delta > 0$ such that $\delta\|EAE\| \leqslant \epsilon$. Then for $x \in \mathfrak{L}$ we have

$$\|Tx\| \geqslant \epsilon\|x\| \geqslant \delta\|EAE\|\,\|x\| \geqslant \delta\|EAEx\|.$$

Next, let σ denote the positive root of the equation

$$\sigma^2 = 1/(\delta^2 + 1)$$

so that

$$\delta^2 = (1 - \sigma^2)/\sigma^2.$$

(Note that $0 < \sigma < 1$.) Then for $x \in \mathfrak{L}$ we have

$$\sigma^2\|Tx\|^2 \geqslant \sigma^2\delta^2\|EAEx\|^2 = (1 - \sigma^2)\|EAEx\|^2.$$

But also

$$\|AEx\|^2 = \|EAEx\|^2 + \|(1 - E)AEx\|^2 = \|EAEx\|^2 + \|Tx\|^2,$$

so that for all $x \in \mathfrak{L}$

$$\sigma^2\|AEx\|^2 \geqslant \sigma^2\|EAEx\|^2 + (1 - \sigma^2)\|EAEx\|^2 = \|EAEx\|^2.$$

Finally, let u and v denote unit vectors in $\mathfrak{M}$ and $\mathfrak{N}$ respectively, let $v = Aw$, $w \in \mathfrak{M}$, and select x and y in $\mathfrak{L}$ such that $Ex = u$ and $Ey = w$. Then

$$|(u, v)| = |(Ex, AEy)| = |(u, EAEy)| \leqslant \|EAEy\| \leqslant \sigma\|AEy\| = \sigma\|w\| = \sigma < 1,$$

and the proof is complete.

PROPOSITION 2.3. *Let* **R** *be a von Neumann algebra acting on* $\mathfrak{H}$ *and let* A *be any non-central element of* **R**. *Then there exist in* **R** (1) *three orthogonal projections* E_1, E_2, E_3 $(E_1 \neq 0)$ *having sum equal to* 1 *and* (2) *an invertible operator* T *such that* $T^{-1}AT$ *maps the range of* E_1 *isometrically onto the range of* E_2, *i.e., such that* $T^{-1}ATE_1$ *is a partial isometry with* E_1 *and* E_2 *for initial and final projections, respectively.*

Proof. Let $\mathfrak{M}$ and $\mathfrak{N}$ be any pair of subspaces satisfying the conditions of Lemma 2.2, and let $\mathfrak{K} = \mathfrak{M} + \mathfrak{N}$. According to Lemma 2.1, $\mathfrak{K}$ is closed. We define $\mathfrak{M}_1 = \mathfrak{M}$, $\mathfrak{M}_2 = \mathfrak{K} \ominus \mathfrak{M}_1$, $\mathfrak{M}_3 = \mathfrak{K}^\perp$, and take E_i to be the projection on $\mathfrak{M}_i$, $i = 1, 2, 3$. Clearly the E_i are mutually orthogonal and have sum 1.

Note that $E_1 \neq 0$ and that E_1 and E_2 are equivalent in $\mathbf{R}$, while all we can say of E_3 in general is that $0 \leqslant E_3 < 1$. By Lemma 2.1 there is a partial isometry W in $\mathbf{R}$ having initial space $\mathfrak{M}_2$ and final space $\mathfrak{N}$, and $E_1 + W$ maps $\mathfrak{K}$ onto itself invertibly. It follows that $R = E_1 + W + E_3$ is an invertible element of $\mathbf{R}$ satisfying $R^{-1}(\mathfrak{N}) = \mathfrak{M}_2$. Hence $Z = R^{-1}AE_1$ is an element of $\mathbf{R}$ that maps $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ in a one-one fashion and annihilates $\mathfrak{M}_1^{\perp}$. If $Z = VP$ is the polar decomposition of Z, then the positive operator P maps $\mathfrak{M}_1$ onto itself and V is a partial isometry of $\mathfrak{M}_1$ onto $\mathfrak{M}_2$. It follows that $Q = P + E_2 + E_3$ is also an invertible element of $\mathbf{R}$, and if $S = Q^{-1}$, then $ZS = V$. Now define $T = RS \in \mathbf{R}$. Then $TE_1 = RSE_1 = SE_1 = E_1 SE_1$, so that

$$T^{-1}ATE_1 = S^{-1}(R^{-1}AE_1)SE_1 = S^{-1}ZSE_1 = S^{-1}VE_1 = V,$$

and the proof is complete.

Proposition 2.3 admits an obvious matricial interpretation. If we write $\tilde{V} = V|\mathfrak{M}_1$ for the isometry of $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ induced by V, then $T^{-1}AT$ has the matrix

$$\begin{pmatrix} 0 & * & * \\ \tilde{V} & * & * \\ 0 & * & * \end{pmatrix}$$

with respect to the direct resolution $\mathfrak{H} = \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \mathfrak{M}_3$. Indeed, if we allow $\tilde{V}$ to identify $\mathfrak{M}_1$ and $\mathfrak{M}_2$ we may write $T^{-1}AT$ as the matrix

$$\begin{pmatrix} 0 & * & * \\ 1 & * & * \\ 0 & * & * \end{pmatrix}.$$

The trouble with this interpretation, of course, is that while E_1 and E_2 are equivalent in $\mathbf{R}$, so that we may identify $\mathfrak{M}_1$ and $\mathfrak{M}_2$, we have virtually no control over $\mathfrak{M}_3$. In particular, the above matrix cannot, in general, be regarded as a matrix with entries taken from a fixed ring of operators, but can only be viewed as a block matrix with entries consisting of mappings from one subspace to another.

We note at once, however, that if $\mathbf{R}$ contains a non-trivial subspace $\mathfrak{M}'$ properly smaller than $\mathfrak{M}$, then $\mathfrak{M}'$ and $\mathfrak{N}' = A(\mathfrak{M}')$ also satisfy the conditions of Lemma 2.2, so that we may replace $\mathfrak{M}$ and $\mathfrak{N}$ by $\mathfrak{M}'$ and $\mathfrak{N}'$, respectively, in the proof of Proposition 2.3. If this is done, then $\mathfrak{K}$ is replaced by the properly smaller subspace $\mathfrak{K}' = \mathfrak{M}' + \mathfrak{N}'$, which ensures that $\mathfrak{M}_3$ is not trivial. We express the consequences of this idea in the following corollary.

COROLLARY 2.4. *If the von Neumann algebra* $\mathbf{R}$ *of Proposition* 2.3 *is not a factor of type* I_2, *then the projection* E_3 *can always be taken to be non-zero.*

Proof. What must be shown is that if $E_1 + E_2 = 1$ and E_1 is a minimal projection in $\mathbf{R}$, then $\mathbf{R}$ is a factor of type I_2. This is an easy exercise whose proof we omit.

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3. Standard form. Suppose now that $\mathbf{R}$ is a type III factor acting on $\mathfrak{H}$. Then all the non-zero projections in $\mathbf{R}$ are equivalent to one another. In particular, each is equivalent to 1. Hence if $\{E_1, \ldots, E_n\}$ is any (finite) system of orthogonal non-zero projections in $\mathbf{R}$ satisfying

$$E_1 + \ldots + E_n = 1,$$

then the rings $E_i \mathbf{R} E_i$ are all spatially isomorphic with one another and with $\mathbf{R}$ itself. On the other hand, it is standard algebra that the system $\{E_1, \ldots, E_n\}$ can be used to obtain a spatial isomorphism between $\mathbf{R}$ and the $n \times n$ matrix ring over, say, $E_1 \mathbf{R} E_1$. Putting these two remarks together, we conclude that the system $\{E_1, \ldots, E_n\}$ yields a spatial isomorphism between $\mathbf{R}$ and the von Neumann algebra $\mathbf{M}_n$ of all $n \times n$ matrices over $\mathbf{R}$. (The ring $\mathbf{M}_n$ is viewed as acting in the usual fashion on the direct sum of n copies of $\mathfrak{H}$. It will be noted that the determination of a particular isomorphism between $\mathbf{R}$ and $\mathbf{M}_n$ requires not only the specification of the system of projections $\{E_1, \ldots, E_n\}$ but also the choice of a particular system of isometries in $\mathbf{R}$ having the E_i as final projections. Nevertheless we shall speak of any one such isomorphism as *effected by* $\{E_1, \ldots, E_n\}$.)

The applications we shall make of these remarks are limited to the cases $n = 2$ and $n = 3$. In particular, taking $n = 3$, we obtain the following improvement of Proposition 2.3.

PROPOSITION 3.1. *Let* $\mathbf{R}$ *be a type* III *factor acting on (a separable Hilbert space)* $\mathfrak{H}$, *and let* A *be any non-scalar element of* $\mathbf{R}$. *Then there exist in* $\mathbf{R}$ *an invertible operator* T *and a system* $\{E_1, E_2, E_3\}$ *of projections effecting a spatial isomorphism of* $\mathbf{R}$ *onto* $\mathfrak{M}_3$ *under which* $T^{-1}AT$ *is carried onto a matrix of the form*

$$(\pi) \qquad \begin{pmatrix} 0 & * & * \\ 1 & * & * \\ 0 & * & * \end{pmatrix}.$$

With this result the way is paved for the exploitation of the same sort of matrix calculations employed in **(1)**. Indeed, the balance of the program for proving Theorem 1 is so closely parallel to the proof in **(1**, §4**)** that the operators of "class (F)" are commutators, that some of the individual arguments can and will be curtailed.

4. Some matrix commutators. The separable complex Hilbert space $\mathfrak{H}$ and the type III factor $\mathbf{R}$ will remain fixed as before. The class of all non-scalar elements of $\mathbf{R}$ will be denoted by (Π) both for the sake of brevity and also to emphasize the fact that, from this point on, the main property of these operators will be the one embodied in Proposition 3.1. The program begins with the following definition.

If $A, B \in \mathbf{R}$, then an $\mathbf{R}$-*generalized sum* of A and B is any operator of the form $S^{-1}AS + T^{-1}BT$ where S and T are invertible elements of $\mathbf{R}$.

LEMMA 4.1. *If A, $B \in (\Pi)$, then some* **R**-*generalized sum of A and B is a commutator in* **R**.

Proof. It is known (**4**, Theorem 4) that any operator in a type III factor with non-trivial null space is a commutator in the factor. Hence the present lemma will follow if we exhibit an **R**-generalized sum of A and B having a non-trivial null space.

Let T and $\{E_1, E_2, E_3\}$ be chosen, according to Proposition 3.1, so that $\{E_1, E_2, E_3\}$ effects a spatial isomorphism ϕ of **R** onto $\mathbf{M}_3$ such that $\phi(T^{-1}AT)$ is a matrix of the form (π). Since a generalized sum of $A_1 = T^{-1}AT$ and B is also a generalized sum of A and B, we may clearly assume that $A_1 = A$ or, in other words, that $\phi(A)$ has the form (π). Similarly it is no loss of generality to assume there exist projections $\{F_1, F_2, F_3\}$ effecting a spatial isomorphism of **R** onto $\mathbf{M}_3$ which carries B onto a matrix of the form (π). For $i = 1, 2, 3$, let W_i be a partial isometry in **R** having initial projection E_i and final projection F_i. Then $U = W_1 + W_2 + W_3$ is a unitary element of **R** and, as a simple calculation shows, $B_1 = U^*BU$ is carried by ϕ onto a matrix of the form

$$\begin{pmatrix} 0 & * & * \\ Z & * & * \\ 0 & * & * \end{pmatrix}$$

where Z is invertible. Finally, let $R \in \mathbf{R}$ satisfy

$$\phi(R) = \begin{pmatrix} -Z^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\phi(R^{-1}B_1R) = \begin{pmatrix} 0 & * & * \\ -1 & * & * \\ 0 & * & * \end{pmatrix},$$

$$\phi(A + R^{-1}B_1R) = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

and it follows at once that $A + (UR)^{-1}B(UR)$ annihilates the range of E_1.

LEMMA 4.2. *If C is a commutator in* **R**, *then all operators of the form*

$$\begin{pmatrix} C & * \\ * & 0 \end{pmatrix}$$

are commutators in $\mathbf{M}_2$.

The proof of this lemma is virtually indistinguishable from that of (**1**, Lemma 4.1) and will be omitted.

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LEMMA 4.3. *Suppose that K, L, and N are elements of* $\mathbf{R}$, *that L is invertible, and that some* $\mathbf{R}$*-generalized sum of K and N is a commutator in* $\mathbf{R}$. *Then every operator in* $\mathbf{M}_2$ *of the form*

$$\begin{pmatrix} K & L \\ * & N \end{pmatrix}$$

is a commutator in $\mathbf{M}_2$.

Proof. If A and B are operators in $\mathbf{R}$ with disjoint spectra, and $Y \in \mathbf{R}$, then there is an $X \in \mathbf{R}$ satisfying $AX - XB = Y$. For by **(3**, Theorem 10**)** there exists a unique bounded operator X on $\mathfrak{H}$ satisfying the above equation, and the uniqueness of X implies that X commutes with every unitary operator in the commutant of $\mathbf{R}$, i.e., that $X \in \mathbf{R}$.

The remainder of this proof is simply a duplication of the proof of **(1**, Lemma 4.2**)** and is omitted.

COROLLARY 4.4. *If K, N are in* (II) *and $L \in$* $\mathbf{R}$ *is invertible, then every operator in* $\mathbf{M}_2$ *of the form*

$$\begin{pmatrix} K & L \\ * & N \end{pmatrix}$$

is a commutator in $\mathbf{M}_2$.

LEMMA 4.5. *Let A_i, $i = 1, 2, 3, 4$, be any operators in* $\mathbf{R}$, *and suppose that S is an invertible operator of class* (II). *Then for all sufficiently large $\lambda > 0$,*

$$\begin{pmatrix} A_1 & A_3 + \lambda S \\ A_2 & A_4 \end{pmatrix}$$

is a commutator in $\mathbf{M}_2$.

Proof. Since (II) is the complement in $\mathbf{R}$ of a closed subspace, it follows that (II) is open in $\mathbf{R}$ (in the uniform topology) and closed with respect to multiplication by non-zero scalars. Hence for any $Y \in \mathbf{R}$ the operators

$$Y + \lambda S = \lambda(S + (1/\lambda)Y) \qquad \text{and} \qquad Y - \lambda S = -\lambda(S - (1/\lambda)Y)$$

are invertible and in (II) along with S for all sufficiently large λ. The result now follows from Corollary 4.4 by making a similarity transformation via the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

since

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} A_1 & A_3 + \lambda S \\ A_2 & A_4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} A_1 + A_3 + \lambda S & A_3 + \lambda S \\ * & A_1 - A_3 - \lambda S \end{pmatrix}.$$

LEMMA 4.6. *If $A \in$* $\mathbf{R}$ *and V is any isometry in* $\mathbf{R}$ *such that $1 - VV^*$ is equivalent to* 1, *then there exists $X \in$* $\mathbf{R}$ *such that $XV = 0$ while $A + VX$ is a commutator in* $\mathbf{R}$.

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Proof. Use the equivalent projections $E = VV^*$ and $1 - E$ to effect a spatial isomorphism ϕ of $\mathbf{R}$ onto $\mathbf{M}_2$, and suppose that E and $1 - E$ are mapped by ϕ onto the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

respectively. Clearly, then,

$$\phi(V) = \begin{pmatrix} V_1 & V_2 \\ 0 & 0 \end{pmatrix}$$

where V_1, V_2 are isometries in $\mathbf{R}$ with final projections E_1, E_2 satisfying $E_1 E_2 = 0$ and $E_1 + E_2 = 1$. Define $S = E_1 + 2E_2$ and observe that S is an invertible operator of class (Π). For $\lambda > 0$ define $Y_\lambda = \lambda V_1{}^*E_1$ and $Z_\lambda = 2\lambda\, V_2{}^*E_2$, and let X_λ be such that

$$\phi(X_\lambda) = \begin{pmatrix} 0 & Y_\lambda \\ 0 & Z_\lambda \end{pmatrix}.$$

Calculation shows that

$$\phi(VX_\lambda) = \begin{pmatrix} 0 & \lambda S \\ 0 & 0 \end{pmatrix}$$

and that $X_\lambda V = 0$. Thus Lemma 4.5 is applicable and the desired X may be obtained by taking λ sufficiently large.

The end is now in sight.

Proof of Theorem 1. It is known that non-zero scalars cannot be commutators (in $\mathbf{R}$ or elsewhere) and obviously 0 is a commutator in $\mathbf{R}$. Thus all that is required is to show that every $A \in$ (Π) is a commutator in $\mathbf{R}$. Let T and $\{E_1, E_2, E_3\}$ be the operators (corresponding to A) obtained from Proposition 3.1, and define $F_2 = E_1$, $F_1 = E_2 + E_3$. Then F_1 and F_2 are equivalent projections in $\mathbf{R}$ with sum equal to 1, and if these projections are used to effect a spatial isomorphism between $\mathbf{R}$ and $\mathbf{M}_2$, it is easy to verify that $T^{-1}AT$ is carried onto a matrix of the form

$$\begin{pmatrix} A_1 & V \\ B_1 & 0 \end{pmatrix}$$

where V is an isometry in $\mathbf{R}$ such that $1 - VV^*$ is equivalent to 1. Apply Lemma 4.6 to obtain an element X such that $XV = 0$ and $A_1 + VX$ is a commutator in $\mathbf{R}$. Then

$$\begin{pmatrix} 1 & 0 \\ -X & 1 \end{pmatrix}\begin{pmatrix} A_1 & V \\ B_1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} = \begin{pmatrix} A_1 + VX & V \\ * & 0 \end{pmatrix}$$

and, since the matrix on the right is a commutator in $\mathbf{M}_2$ by Lemma 4.2, the proof is complete.

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The University of Michigan,
Ann Arbor, Michigan