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ON NEAR-RINGS OF QUOTIENTS

by A. OSWALD
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1. Introduction

In (2), Holcombe investigated near-rings of zero-preserving mappings of a group Γ which commute with the elements of a semigroup S of endomorphisms of Γ and examined the question: under what conditions do near-rings of this type have near-rings of right quotients which are 2-primitive with minimum condition on right ideals? In the first part of this paper (§2) we investigate further properties of near-rings of this type. The second part of the paper (§3) deals with those near-rings which have semisimple near-rings of right quotients. Our results here are analogous to those of Goldie (1); in particular, with a suitable definition of finite rank we prove that a near-ring which has a semisimple near-ring of right quotients has finite rank.

Basic concepts for near-rings can be found in Pilz (5). All our near-rings will be left near-rings (i.e. $a(b+c)=ab+ac$ for all a, b, $c \in N$) with a two-sided zero.

2. Near-rings of quotients of mapping near-rings

If S is a semigroup, a set A is an *S-set* if it admits the elements of S as right operators with $a(s_1s_2)=(as_1)s_2$ for all $a \in A$, s_1, $s_2 \in S$. A mapping $f: A \to B$, where A and B are *S-sets*, is an *S-homomorphism* if for each $a \in A$ and $s \in S$ we have $(af)s=(as)f$. If $X \subset A$, an S-set, and every mapping $f: X \to B$, where B is an S-set, defines a unique S-homomorphism $f^*: A \to B$ then A is the *free S-set on X*.

If S has a semigroup G of two-sided quotients we define $AG=\{ag : a \in A, g \in G\}$. Denoting by C the set of cancellative elements of S it is obvious that $AG=\{as^{-1}: a \in A, s \in C\}$. In AG we define $as^{-1}=bt^{-1}$ if, and only if, for some u, $v \in C$ we have $su=tv$ and $au=bv$. Then in AG we can define an action by $(ag)h=a(gh)$ whenever $a \in A$, g, $h \in G$ and AG becomes a G-set.

The following result is well known.

Lemma 0. *If a semigroup X has a semigroup of right quotients and if C denotes the subsemigroup of X consisting of the cancellative elements then for each set* $t_1, t_2, \ldots, t_k \in C$ *there exist* $s_1, s_2, \ldots, s_k, t \in C$ *with* $t=s_it_i$, $1 \leq i \leq k$.

Proof. See (6; Lemma 1.4), for example.

A similar result holds if X has a semigroup of left quotients.

Now suppose that A is a group (written additively but not necessarily commutative) and that S is a semigroup of endomorphisms of A in which the cancellative elements of S are monomorphisms of A. We suppose the zero endomorphism is not in S. If S has a semigroup G of two-sided quotients we can define an addition in AG by $(as^{-1}+bt^{-1})=(au+bv)w^{-1}$ where $u, v \in C$ with $w=su=tv$. It is easy to show that AG is a group and that the elements of G act as endomorphisms of AG.

Now we introduce the two sets

$$M=\mathrm{Map}_S(A)=\{f: A \to A: 0f=0, (af)s=(as)f, a \in A, s \in S\}$$

$$M^*=\mathrm{Map}_G(AG)=\{f: AG \to AG: 0f=0, ((ag_1)f)g_2=(ag_1g_2)f, ag_1 \in AG, g_2 \in G\}.$$

It is easy to see that both M and M^* are left near-rings. Furthermore, if $f \in M$ we can define $f^* \in M^*$ by $(as^{-1})f^*=(af)s^{-1}$. Since it is easily seen that the mapping $f \to f^*$ is an injective near-ring homomorphism we see that M is embedded in M^*. We identify f with f^* in M^*. We wish to introduce restrictions on A, S and AG which will lead to M^* being a near-ring of quotients of M.

We say that A is a *torsion-free S-set* if whenever $a \in A$ with $a \neq 0$ and $s, t \in S$ with $s \neq t$ then $as \neq at$. Using Lemma 0 it can be shown that if A is a torsion-free S-set then AG is a torsion-free G-set. Furthermore, we call A *finitely-generated* if for some set $a_1, a_2, \ldots, a_n \in A$ we have $A=a_1S \cup a_2S \cup \ldots \cup a_nS$. Then clearly $AG=a_1G \cup a_2G \cup \ldots \cup a_nG$. Restricting ourselves to the case when AG is a free G-set we can choose $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ with $AG=a_{i_1}G \dot\cup a_{i_2}G \dot\cup \ldots \dot\cup a_{i_k}G$ where distinct terms in this union are disjoint. Since $A \subset AG$, $a_j=a_{\lambda_j}r_j^{-1}s_j$ for some $\lambda_j \in \{i_1, \ldots, i_k\}$ and $r_j \in C$, $s_j \in S$. Using Lemma 0 choose e_j, $d \in C$ with $r_j^{-1}=d^{-1}e_j$. Then $a_j \in a_{\lambda_j}d^{-1}S$ and so $A \subseteq a_{i_1}d^{-1}S \dot\cup a_{i_2}d^{-1}S \dot\cup \ldots \dot\cup a_{i_k}d^{-1}S$ as a disjoint union and also $AG=a_{i_1}d^{-1}G \dot\cup \ldots \dot\cup a_{i_k}d^{-1}G$. For $h \in G$ we define $\mu_h: AG \to AG$ by $(a_{i_j}d^{-1}q)\mu_h=a_{i_j}d^{-1}hq$ where $q \in G$ and $1 \leq j \leq k$. Then $\mu_h \in M^*$ and if $h \in C$ then μ_h has an inverse in M^* and $(\mu_h)^{-1}=\mu_{h^{-1}}$.

Proposition 1. *The elements of M^* have the form $\theta\mu_t^{-1}$ for some $\theta, \mu_t \in M$.*

Proof. If $f \in M^*$ then f is uniquely determined by its effect upon $a_{i_j}d^{-1}$, $1 \leq j \leq k$. Let

$$a_{i_j}d^{-1}f=a_{\lambda_j}d^{-1}p_j^{-1}s_j,\ p_j \in C,\ s_j \in S,\ i_j, \lambda_j \in \{i_1, i_2, \ldots, i_k\}.$$

Applying the symmetric form of Lemma 0 we can choose $q_1, \ldots, q_k$, $r \in C$ with $p_j^{-1}=r^{-1}q_j$, $1 \leq j \leq k$ and then

$$a_{i_j}d^{-1}f=a_{\lambda_j}d^{-1}r^{-1}q_js_j.$$

Then $a_{i_j}d^{-1}f\mu_{dr}=a_{\lambda_j}q_js_j \in A$ so that $f\mu_{dr}: a_{i_1}d^{-1}S \dot\cup \ldots \dot\cup a_{i_k}d^{-1}S \to A$ and so $f\mu_{dr}: A \to A$ and $f\mu_{dr} \in M$. Also $a_{i_j}d^{-1}\mu_{dr}=a_{i_j}r \in A$ so $\mu_{dr} \in M$ and writing $\theta=f\mu_{dr}$ and $t=dr \in C$ we have $f=\theta\mu_t^{-1}$ as required.

It is easy to see that if $t \in C$ and $\mu_t \in M$ then μ_t is cancellative in M.

Proposition 2. *If $f \in M$ is cancellative then f is cancellative in M^*.*

Proof. Let $\phi, \psi \in M^*$ and $f\phi=f\psi$. Write $a_j^*=a_{i_j}d^{-1}$. If $\phi \neq \psi$ let $a_j^*\phi \neq a_j^*\psi$. If

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$a \in A$ and $s \in S$ with $af = a_j^* s$ then $a_j^* \psi s = a_j^* s\psi = af\psi = af\phi = a_j^* \phi s$ and since AG is torsion-free and $s \in G$ we see that $a_j^* \psi = a_j^* \phi$ which is false. Consequently there is no $a \in A$ with $af = a_j^* s$ for any $s \in S$. Let $\lambda_i : AG \to AG$ be defined by

$$a_i^* uv^{-1}\lambda_j = \begin{cases} 0 & \text{if } i \neq j \\ a_j^* uv^{-1} & \text{if } i = j \end{cases}$$

where $uv^{-1} \in G$. Then $\lambda_j \in M^*$ for $1 \leq j \leq k$. Then for each $a \in A$ $af\lambda_j = 0 = af0$ so that in M we have $f(\lambda_j|_A) = f0$ and $\lambda_j|_A = 0$. But then $a_j^* \lambda_j = (a_{i_j}\lambda_j|_A)d^{-1} = 0$ and so $f\phi = f\psi$ implies $\phi = \psi$. Now suppose $\phi f = \psi f$ and $\phi \neq \psi$. Without loss of generality we can take $a_1^* \phi \neq a_1^* \psi$. Set $a_1^* \phi = a_1^* \alpha\beta^{-1}$, $a_1^* \psi = a_j^* \gamma\delta^{-1}$ where $\alpha\beta^{-1}$, $\gamma\delta^{-1} \in G$. Then

$$a_i^* f\alpha\beta^{-1} = a_i^* \alpha\beta^{-1} f = a_1^* \phi f = a_1^* \psi f = a_j^* \gamma\delta^{-1} f = a_j^* f\gamma\delta^{-1}.$$

If $i = j$ either $\alpha\beta^{-1} = \gamma\delta^{-1}$ or $a_i^* f = 0$ since AG is torsion-free. If $\alpha\beta^{-1} = \gamma\delta^{-1}$ then $a_1^* \phi = a_i^* \alpha\beta^{-1} = a_i^* \gamma\delta^{-1} = a_1^* \psi$ which is false. Hence $a_i^* f = 0$. In this case $a\lambda_i f = 0$ for each $a \in A$ so that in M we have $(\lambda_i|_A)f = 0f$ and $\lambda_i|_A = 0$ which is false. Thus $i \neq j$. In this case let $a_i^* f = (a_p^* t^{-1})s$, $a_j^* f = (a_q^* t^{-1})s_1$ where $s, s_1 \in S$ and we have used Lemma 0 to find $t \in C$. Then

$$a_p^* t^{-1} s\alpha\beta^{-1} = a_i^* f\alpha\beta^{-1} = a_i^* \alpha\beta^{-1} f = a_1^* \phi f$$

$$a_q^* t^{-1} s_1\gamma\delta^{-1} = a_j^* f\gamma\delta^{-1} = a_j^* \gamma\delta^{-1} f = a_j^* \psi f$$

and so $a_p^* t^{-1} s\alpha\beta^{-1} = a_q^* t^{-1} s_1 \gamma\delta^{-1} \in a_p^* G \cap a_q^* G$ and thus $p = q$. Hence $a_i^* f$, $a_j^* f \in a_p^* G$ and so for some r, $1 \leq r \leq k$, $a_r^* G \cap \operatorname{Im} f = \phi$. Again we have the situation where for no $a \in A$ is $af = a_r^* s$ for any $s \in S$ and again we deduce that $\lambda_r = 0$. Hence $\phi = \psi$ is cancellative in M^*.

Proposition 3. *If $f \in M^*$ and f is cancellative then f has an inverse in M^*.*

Proof. If f is both $1-1$ and onto then the inverse mapping $f^{-1} : AG \to AG$ exists and a simple calculation shows that $f^{-1} \in M^*$. Suppose f is cancellative but not onto. Let $b \in AG$ with $b \notin \operatorname{Im} f$. For some j, $b = a_j^* g$ with $g \in G$ and so $a_j^* \notin \operatorname{Im} f$. But then, for $1 \leq i \leq k$, $a_i^* f\lambda_j = 0$ and so $f\lambda_j = 0$ and $\lambda_j = 0$ since f is cancellative. Hence f is onto. Now suppose $a, b \in AG$ with $af = bf$, $a \neq b$. For some i, j, $a = a_i^* g_1$, $b = a_j^* g_2$ with $g_1, g_2 \in G$. If $i \neq j$ then $a_i^* g_1 f = a_j^* g_2 f$ and so $a_i^* f g_1 = a_j^* f g_2$. Also for some h, r we have $a_i^* f \in a_h^* G$, $a_j^* f \in a_r^* G$ and $a_i^* f g_1 \in a_h^* G \cap a_r^* G \neq \phi$. Since AG is free we have $h = r$ and thus both a_i^* and a_j^* map into $a_h^* G$. But then f cannot be onto. This contradiction establishes the result.

These results now lead to

Theorem 1. *Let S be a semigroup with a semigroup G of two-sided quotients and let A be a group for which S is a semigroup of endomorphisms such that the cancellative elements of S are monomorphisms. If A is torsion-free and finitely generated as an S-set and if AG is free as a G-set then M has a near-ring of two-sided quotients which is isomorphic to M^*.*

Proof. It remains only to prove that M^* is a near-ring of left quotients of M. Let $f \in M^*$. The for some α_i, $\bar{\alpha}_i \in S$, β_i, $d_i \in C a_{\lambda_i} d^{-1} f = a_{\lambda_{j(i)}} d^{-1} \alpha_i \beta_i^{-1} = a_{\lambda_{j(i)}} \bar{\alpha}_i (\beta_i d_i)^{-1}$ since G is a semigroup of two-sided quotients of S. Using Lemma 0 we have, for $1 \leq i \leq k$, $(\beta_i d_i)^{-1} = e_i r^{-1}$ and $d^{-1} = e r^{-1}$ for some e_i, e, $r \in C$. Then $a_{\lambda_i} d^{-1} f r = a_{\lambda_{j(i)}} \bar{\alpha}_i e_i \in A$. But $a_{\lambda_i} d^{-1} f r = a_{\lambda_i} d^{-1} r f = a_{\lambda_i} d^{-1} \mu_r f$ and so $\mu_r f \in M$. Also $a_{\lambda_i} d^{-1} \mu_r = a_{\lambda_i} d^{-1} r = a_{\lambda_i} e \in A$ and so $\mu_r \in M$. Since $r \in C$ we have $f = (\mu_r)^{-1} \theta$ for some $\theta \in M$ as required.

We now turn to the structure of M^*. Since AG is free, $AG = a_{\lambda_1} d^{-1} G \dot{\cup} \ldots \dot{\cup} a_{\lambda_k} d^{-1} G$. Writing $a_j^* = a_{\lambda_j} d^{-1}$ we have defined $\lambda_i : AG \to AG$ by $a_j^* g \lambda_i = 0$ if $j \neq i$ and $a_i^* g \lambda_i = a_i^* g$ where $g \in G$. Clearly $\lambda_i \in M^*$ and it is an easy calculation to show that, for each i, $\lambda_i M^*$ is a right ideal of M^* and that $M^* = \lambda_1 M^* \oplus \ldots \oplus \lambda_k M^*$.

For the remainder of this section we will suppose that G is a group. If $X \subset N$, where N is a near-ring, we denote by $r(X)$ the set $r(X) = \{n \in N : xn = 0$ for all $x \in X\}$; by $r(x)$ we mean $r(\{x\})$.

Theorem 2. *M^* has no proper two-sided ideals and has the descending chain condition on M^*-subgroups.*

Proof. If $\lambda_i g$, $\lambda_i h \in \lambda_i M^*$ with $\lambda_i g \neq 0 \neq \lambda_i h$ then $a_i^* \lambda_i h = a_m^* \alpha \beta^{-1}$ for some m and $\alpha \beta^{-1} \in G$ and $a_j^* \lambda_i h = 0$ for $j \neq i$. Similarly for some p and $\gamma \delta^{-1} \in G$, $a_i^* \lambda_i g = a_p^* \gamma \delta^{-1}$, $a_j^* \lambda_i g = 0$ if $j \neq i$. Defining $q : AG \to AG$ by $a_p^* q = a_m^* \alpha \beta^{-1} \gamma \delta^{-1}$ and $a_j^* q = 0$ if $j \neq p$ we can extend q to $q^* \in M^*$ to get $\lambda_i h = \lambda_i g q^* \in \lambda_i g M^*$ and so $\lambda_i h M^* = \lambda_i g M^*$. Hence each $\lambda_i M^*$ is minimal. Thus M^* is completely reducible with identity and by (3) M^* has no nilpotent M^*-subgroups and the descending chain condition on M^*-subgroups. Next we let $f \in r(\lambda_1 M^*)$. Defining $\lambda_{1j} \in M^*$ by $a_i^* \lambda_{1j} = 0$ if $i \neq 1$ and $a_1^* \lambda_{1j} = a_j^*$ we see that $f = 0$. From (4; Thm 3) we can write M^* as a direct sum of two-sided ideals each of which is simple as a near-ring. If U, V are two such with $U \cap V = (0)$ then $UV = 0$. If $U \neq 0$, $\lambda_1 M^* U \neq 0$ so for some $f \in M^*$, $\lambda_1 f U = \lambda_1 M^*$. But then $\lambda_1 M^* V = 0$ so $V = 0$. It follows that M^* is simple as a near-ring.

Suppose X is a subset of the near-ring N. Set $X_0 = NX \cup X$ and denote by X_0^+ the normal subgroup of N generated by X_0. Let $[X_0^+] = \{(a + x)c - ac : x \in X_0^+, a, c \in N\}$. Define $X_1 = [X_0^+] \cup X_0^+$ and observe that $NX_1 \subseteq X_1$, We now suppose that X_k has been constructed and we construct successively X_k^+, $[X_k^+]$ and $X_{k+1} = X_k^+ \cup [X_k^+]$. This leads to a chain $X_0 \subseteq X_1 \subseteq \ldots$ and we then define $I(X) = \cup X_k$. A simple calculation shows that $I(X)$ is the ideal generated by X.

If N has a near-ring Q of two-sided quotients and B is an N-subgroup of N then we may construct the ideal B^* of Q generated by QBQ. In this case $B^* = \cup B_k$ where $B_0 = QBQ$, etc.

Lemma 1. *If A is an N-subgroup with $AB = (0)$ and $A \neq (0)$ then for each k the elements of B_k have the form $c^{-1}t$ where c is a cancellative element in N and $At = (0)$.*

Proof. If $u \in QBQ$ then $u = c^{-1} vbq$ where $v \in N$, $b \in B$, $q \in Q$ and c is a cancellative element of N. Then $avbq \in ABq = (0)$ and $u = c^{-1}t$ as required. Now suppose the result is true for B_k. If $u \in B_{k+1}$ then $u \in B_k^+$ or $u \in [B_k^+]$. In the first case $u = \Sigma(d_i^{-1} x_i + c_i^{-1} t_i - d_i^{-1} x_i) = c^{-1} \Sigma(y_i + r_i t_i - y_i)$, on applying Lemma 0, where y_i, $r_i \in N$ and $At_i = (0)$. But then $A r_i t_i = (0)$ so with $t = \Sigma(y_i + r_i t_i - y_i)$ we have $u = c^{-1}t$ and $At = (0)$. A similar argument applies to $[B_k^+]$ and the lemma follows.

Theorem 3. *If N has a near-ring Q of two-sided quotients which is simple then N is strictly prime.*

Proof. Let A, B be N-subgroups of N with $AB=(0)$ but $A\neq(0)$. Let B^* be the ideal of Q generated by QBQ. Since Q is simple either $B^*=(0)$ or $B^*=Q$. If $B^*=(0)$ then trivally $B=(0)$. If $B^*=Q$ then $1\in B^*$ so $1\in B_k$ for some k. From Lemma 1, $1=c^{-1}t$ with c cancellative and $At=(0)$. Thus $Ac=(0)$ contrary to $A\neq(0)$.

Corollary 1. *If G is a group M is strictly prime.*

3. Semisimple Near-rings of Right Quotients

We now turn to an investigation of the properties of those near-rings which have near-rings of right quotients which are semisimple. If Q is a near–ring then Q is semisimple if it has no nilpotent Q-subgroups and the minimum condition on Q-subgroups. To fix our terminology, an N-subgroup A is nilpotent if $A\neq(0)$ but for some $n\geqq 2$, $A^n=\{a_1a_2\ldots a_n : a_i\in A\}=(0)$.

Most of our results are the analogues of those of Goldie. We will determine necessary conditions for the near-ring Q of right quotients of N to be semisimple. It seems most unlikely, however, that these conditions will be sufficient. Throughout this section Q will denote a near-ring of right quotients of N.

Theorem 4. *If N has a near-ring Q of right quotients which is semisimple then N has no nilpotent N-subgroups.*

Proof. Let A be a nilpotent N-subgroup of A with $A^n=(0)$. Set $P=\ell(NA)=\{x\in N : xNA=(0)\}$. If $u\in N(u\neq 0)$ with $uN\neq(0)$ and $P\cap uN=(0)$ then $NA\cap uN\neq(0)$ so we can take $ux\in NA\cap uN$ with $ux\neq 0$. For some $k>1$, $(NA)^k=(0)$, $(NA)^{k-1}\neq(0)$. Hence $ux(NA)^{k-1}=(0)$. It follows that $(ux)^{k-1}NA=(0)$ and $(ux)^{k-1}\in P\cap uN=(0)$. Thus $(ux)^{k-1}=0$. If $(ux)^{k-2}NA\neq(0)$ then for some $v\in NA$, $(ux)^{k-2}v\neq 0$ but $(ux)^{k-2}vNA=(0)$ so that $(ux)^{k-2}v\in P\cap uN=(0)$ which is a contradiction. Hence $P\cap uN\neq(0)$ and P has non-empty intersection with every non-zero N-subgroup. Now consider PQ. Clearly if $q\in Q$ then $PQ\cap qQ\neq(0)$ unless $q=0$. In particular $PQ\cap e_iQ\neq(0)$ where e_iQ is a minimal right ideal of Q and $Q=e_1Q\oplus e_2Q\oplus\ldots\oplus e_nQ$. Hence $e_iQ=p_iQ$ for some $p_i\in P$. Notice that $p_1N+p_2N+\cdots+p_nN$ is an N-subgroup of N and that $(p_1a_1+p_2a_2+\cdots+p_na_n)b=p_1a_1b+p_2a_2b+\cdots+p_na_nb$ for $b\in N$. Thus $p_1N+p_2N+\cdots+p_nN\subseteq P$. However, $1=p_1a_1c_1^{-1}+\cdots+p_na_nc_n^{-1}=(p_1a_1b_1+\cdots+p_na_nb_n)c^{-1}$ where a_n, $b_n\in N$ and we have used Lemma 0. It follows that $c\in P$ and P contains a cancellative element. Hence $NA=(0)$ and $A=(0)$ as required.

A near-ring N is *strictly semiprime* if it has no nilpotent N-subgroups. Thus

Corollary 2. *Q semisimple implies N strictly semiprime.*

We say that an N-subgroup A of N is *module-essential* if, whenever X is non-zero right ideal of N, then $A\cap X\neq(0)$; A is *N-essential* if whenever X is a non-zero N-subgroup of N then $A\cap X\neq(0)$. Certainly if A is N-essential then A is

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module essential. However, it is not generally true that if A is module-essential then A is N-essential.

Theorem 5. *If Q is semisimple then module essential N-subgroups of N are N-essential.*

Proof. Let X be module-essential and consider XA. If eQ is a minimal right ideal of Q then $X \cap eQ \cap N \neq (0)$ so $XQ \cap eQ \neq (0)$. As in the proof of Theorem 4 we now deduce that $XQ = Q$. If $a \in N(a \neq 0)$ then $a \in Q = XQ$ so $a = xc^{-1}$ where $x \in X$ and c is cancellative. It follows that $ac \in X \cap aN \neq (0)$ and X is N-essential.

We denote by $Z(N)$ the set $\{x \in N : r(x)$ is N-essential in $N\}$. The proof of Theorem 5 can be modified to show that each N-essential N-subgroup contains a cancellative element when Q is semisimple and we deduce

Corollary 3. *If Q is semisimple then $Z(N) = (0)$.*

In the usual way we prove

Theorem 6. *If Q is semisimple then N has the maximum condition on right annihilators.*

We say that a near-ring N has *finite rank* if each chain $A_1 \subset A_2 \subset \ldots$ of N-subgroups in which for each $i \geqq 2$ there is a non-zero N-subgroup $B_i \subset A_i$ with $B_i \cap A_{i-1} = (0)$ terminates finitely. If N is a ring this reduces to the usual definition.

Lemma 2. *If Q is semisimple and $A_1 \subset A_2 \subset \ldots$ is a chain of N-subgroups of N such that for each $i \geqslant 2$ there is a non-zero N-subgroup $B_i \subset A_i$ with $B_i \cap A_{i-1} = (0)$ then B_i can be chosen to be a submodule of the N-module A_i.*

Proof. Let X be a right ideal of N maximal subject to the condition $A_{i-1} \cap X = (0)$. Then $A_{i-1} + X$ is module-essential in N so by Theorem 5, $A_{i-1} + X$ is N-essential. Now $A_i \cap (A_{i-1} + X) = A_{i-1} + (A \cap X)$ and $B_i \cap A_i \cap (A_{i-1} + X) \neq (0)$ so that $A_i \cap X \neq (0)$. Clearly $A_i \cap X$ is a submodule of A_i with $A_i \cap X \cap A_{i-1} = (0)$.

We wish to prove that a near-ring with a semisimple near-ring of right quotients has finite rank. Because the form of a Q-subgroup generated by an N-subgroup of N does not have a simple representation in the form IQ, say, we have to adopt a more indirect approach. We begin by showing that there are essentially two different problems to be solved.

From (**4**; Thm 3) a semisimple near-ring Q with identity is the direct sum of finitely many two-sided ideals each of which is simple as a near-ring. We divide these direct summands into two classes, those which are rings and those which are *non-rings* i.e. near-rings which are not rings. Write $Q = R \oplus T$ where R, T are ideals of Q, R is the sum of those direct summands which are rings and T the sum of those which are non-rings. Clearly R is a ring. Define $N_1 = N \cap R$, $N_2 = N \cap T$. Then N_1, N_2 are ideals of N and N_1 is a ring. Clearly

Lemma 3. *As a right ideal $N_1 \oplus N_2$ is N-essential in N.*

Lemma 4. *As near-rings R and T are each semisimple with an identity.*

We now prove the following result.

Theorem 7. *Let Q be a semisimple near-ring of right quotients of the near-ring N and suppose that I is an ideal of Q. Then I is a semisimple near-ring of right quotients of the near-ring $J = I \cap N$.*

Proof. From (4; Corollary 6) there is an ideal X of Q with $I \cap X = (0)$, $I \oplus X = Q$. As before $J \oplus X \cap N$ is essential as a right ideal in N. Clearly $J \subseteq I$ and I has an identity. Write $Y = X \cap N$. Let $d \in J$ be cancellative in J and $g \in Y$ be cancellative in Y. Suppose $u \in N$ with $(d+g)u = 0$. There is a cancellative $c \in N$ with $uc \in J \oplus Y$ (unless $u = 0$). Also $uc = n_1 + n_2$; $n_1 \in J$, $n_2 \in Y$ and $0 = (d+g)(n_1+n_2) = dn_1 + gn_2$ so $dn_1 = 0 = gn_2$ and $n_1 = n_2 = 0$ from which we deduce $uc = 0$ and $u = 0$. It follows that $d+g$ is right cancellative in N and $r_Q(d+g) = 0$. Write $p = d+g$. Then $pQ \supseteq p^2Q \supseteq \dots$ and thus $p^nQ = p^{n+1}Q$ for some n. But then $p^n = p^{n+1}q$ for some $q \in Q$ and $p^{n-1} - p^nq \in r_Q(p) = (0)$. Continuing in this way we get $1 = pq$ for some $q \in Q$. If u_1, $u_2 \in N$ with $u_1(d+g) = u_2(d+g)$ then, in Q, $u_1p = u_2p$ and $u_1pq = u_1 = u_2pq = u_2$. We see that $d+g$ is cancellative in N. In Q, $(d+g)^{-1} = \alpha + \beta$ where $\alpha \in I$, $\beta \in X$ and $1 = (d+g)(\alpha+\beta) = d\alpha + g\beta$. Thus $d\alpha$ is the identity of I and $d^{-1} = \alpha \in I$. Now let $r \in I$. For some a, $b \in N$ with b cancellative $r = ab^{-1}$ and $rb = a \in I \cap N = J$. For some cancellative $c_1 \in N$, $bc_1 \in J \oplus Y$ so $bc_1 = u_1 + v_1$, $u_1 \in J$, $v_1 \in Y$. Since bc_1 is cancellative in N, u_1 is cancellative in J and $c_1^{-1}b^{-1} = u_1^{-1} + v_1^{-1}$, $u_1^{-1} \in I$, $v_1^{-1} \in X$. But then $ac_1c_1^{-1}b^{-1} = ac_1u_1^{-1} + ac_1v_1^{-1}$ and $ac_1 \in I$ so $ac_1v_1^{-1} = (0)$. Thus $r = ac_1u_1^{-1}$ where ac_1, $u_1 \in J$ and u_1 is cancellative in J. Hence I is a near-ring of right quotients of J and clearly I is semisimple with identity.

Corollary 4. *N_1 has R as a semisimple ring of right quotients and N_2 has T as semisimple near-ring of right quotients.*

Now let $A_1 \subset A_2 \subset \dots$ be a chain of N-subgroups in which, for each $i \geqq 2$, there is a non-zero N-subgroup $B_i \subset A_i$ with $A_{i-1} \cap B_i = (0)$. Since $N_1 \oplus N_2$ is N-essential in N as a right ideal we can suppose that $A_i \subset N_1 \oplus N_2$ for each i and for convenience we take $N = N_1 \oplus N_2$. There are then the following possibilities:–

(i) infinitely many $A_i \subset N_2$;

(ii) only finitely many $A_i \subset N_2$;

  (a) for infinitely many i, B_i can be chosen with $B_i \cap N_2 \neq (0)$,

  (b) for only finitely many i can we choose B_i with $B_i \cap N_2 \neq (0)$.

In the case (ii) (a) we can consider the chain $A_1 \cap N_2 \subset A_2 \cap N_2 \subset \dots$ and we are back with case (i).

Lemma 5. *If case (ii) (b) arises then the chain $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ is finite.*

Proof. For convenience we suppose that for each possible choice of B_i we have

$B_i \cap N_2 = (0)$. Then $A_i \cap N_2$ is N-essential in $A_{i+1} \cap N_2$ for each i. Pass to $\bar{N} = N/N_2 \cong N_1$. Then $\bar{A}_i \subset \bar{A}_{i+1}$, $\bar{B}_{i+1} \subset \bar{A}_{i+1}$ and if $\bar{u} \in \bar{A}_i \cap \bar{B}_{i+1}$ then $u \in (A_i + N_2) \cap (B_{i+1} + N_2)$ and so $u = a + n = b + n'$ where $a \in A_i$, $b \in B_{i+1}$, n, $n' \in N_2$. Now $-b + a = n' - n \in N_2 \cap A_{i+1}$ so for some cancellative $c \in N$, $(-b + a)c \in A_i \cap N_2$. In view of Lemma 2, we can choose B_{i+1} to be a submodule of A_{i+1} and then $(-b + a)c - ac \in B_{i+1} \cap A_i \cap N_2 = (0)$. Hence $b = 0$ and $\bar{u} = \bar{0}$. But then $\bar{B}_1 \oplus \bar{B}_2 \oplus \dots$ is a direct sum of right ideals of a ring with a semisimple ring of right quotients so by Goldie's Theorem the sum is finite. It follows that the chain $A_1 \subset A_2 \subset \dots$ terminates finitely.

For case (i) we will need the following results.

Theorem 8. *If R is a semisimple near-ring with identity and A is a maximal right annihilator then $A = r(e)$ for some idempotent e with eR a minimal R-subgroup of R.*

Proof. Since $A \neq R$ we let U be a right ideal of R with $A \cap U = (0)$, $A + U = R$. If $\ell(A) \cap U = (0)$ then $U\ell(A) \subseteq U \cap \ell(A) = (0)$ so if $m \in \ell(A)$, $Um = (0)$ and hence $(mU)^2 = (0)$. It follows that $mU = (0)$ and $U \subseteq r\ell(A) = A$. Since $A \cap U = (0)$ this contradiction establishes that $\ell(A) \cap U \neq (0)$. Let $0 \neq u \in \ell(A) \cap U$. Now $A \subseteq r(u) \neq R$ so $A = r(u)$. Clearly $u^k x = 0$ yields $u^{k-1}x \in r(u) \cap U = A \cap U = (0)$, for any positive integer k, and hence $ux = 0$. Hence $r(u) = r(u^2) = \cdots$. Now let I be a right ideal of R with $I \cap (uR + r(u)) = (0)$. For each k set $T_k = uI + u^2I + \cdots + u^kI$ and observe that T_k is an N-subgroup. By (3; Thm 6) R has maximum condition on R-subgroups so for some n, $T_n = T_{n+1}$. Then $u^{n+1}I \subseteq uI + u^2I + \cdots + u^nI$ and $x \in I$ implies $u^{n+1}x = ut_1 + u^2t_2 + \cdots u^nt_n$ where $t_i \in I$. We see that $t_1 + ut_2 + \cdots + u^{n-1}t_n \in r(u) + uR$ and hence $t_1 \in I \cap (r(u) + uR) = (0)$. Similarly $t_2 \in (r(u^2) + uR) \cap I = (r(u) + uR) \cap I = (0)$ and $t_1 = t_2 = \cdots = t_n = 0$. Thus $u^{n+1}I = (0)$ and $I \subseteq r(u^{n+1}) = r(u)$ from which it follows that $I = (0)$. We then have $r(u) + uR$ module-essential in N and hence $r(u) + uR = R$. Write $1 = ut + v$, $ut \in uR$, $v \in r(u)$. Then $u = (ut + v)u = utu + v'$ for some $v' \in r(u)$ and so $-utu + u \in uR \cap r(u) = (0)$. Hence $u(1 - tu) = 0$ and $1 - tu \in r(u)$. We see that $R = tuR + r(u)$ and $1 = tu + w$, $w \in r(u)$. Writing $e = tu$ we let $s \in eR \cap r(u)$. Then $s = ey = tuy$ so $us = utuy = 0$. Using $u = utu$ we see that $uy = 0$ and hence $s = 0$ so that $eR \cap r(u) = (0)$. Also if $ua = 0$ then $tua = ea = 0$ and $r(u) \subseteq r(e) \neq R$. We see, therefore, that $A = r(e)$. Next we have $e - e^2 = (e + w)e - e^2 \in eR \cap r(u) = (0)$ and $e = e^2 \neq 0$. To prove that eR is minimal we observe that, as an R-module, eR is completely reducible so if eR is not minimal we can take non-zero submodules B, C of eR with $eR = B \oplus C$. Then $e = b + c$. If $ex = 0$ then $(b + c)x = bx + cx = 0$ and thus $bx = 0$. Then $r(e) \subseteq r(b) \neq N$ so that $r(e) = r(b)$. If we take B to be minimal then $B = bR$ and $b^2 = b \neq 0$. Also, $bR \cap r(b) = (0)$ so $bR + r(b) = R$. For $c \in C$ we have $c = bn + z$ where $z \in r(b)$ and $-bn + c \in eR \cap r(b) = eR \cap r(e) = (0)$. Thus $c = bn \in B \cap C = (0)$ and $C = (0)$. Hence we have eR minimal as required.

Lemma 6. *If R is semisimple with identity and A is a maximal right annihilator and if I is an R-subgroup with $A \neq A + I$ then $A + I = R$.*

Proof. From Theorem 8, $A = r(e)$ with eR minimal. Then $R = r(e) + eR = r(e) + I + eR$. Thus if $x \in r(e) + I$ then $x = v + eu (v \in r(e))$ and $eu = -v + x \in eR \cap (A + I)$. If $eR \cap (A + I) = (0)$ then $eu = 0$ so $x \in A$ and $A + I \subseteq A$ which is false. Hence $eR \subseteq (A + I)$ so $A + I = R$.

Theorem 9. *Let R be a semisimple near-ring with identity which is the direct sum of ideals each of which is a simple non-ring. Then R has only finitely many maximal right annihilators.*

Proof. Let A be a maximal right annihilator and write $R = I_1 \oplus I_2 \oplus \ldots \oplus I_k$ where each I_j is an ideal of R and a simple non-ring. Furthermore, each I_j is a direct sum of minimal right ideals of R. For one of these minimal right ideals, say e_1R, we must have $A \cap e_1R = (0)$. Without loss of generality we can suppose $e_1R \subseteq I_1$. If fR is a minimal right ideal contained in I_j for some $j > 1$ then $A \cap (e_1R \oplus fR) \neq (0)$ so $0 \neq e_1u + fv \in A$ and $fv \neq 0$. Also, since $I_pI_q = (0)$ if $p \neq q$ we see that for each $j > 1$, $(e_1u + fv)I_j = fvI_j \subseteq A$ and since $fvI_1 = (0) \subseteq A$ we have $fR \subseteq A$. Thus for $j \geqslant 1$, $I_j \subseteq A$. Next we see from Lemma 6 that $A + e_1R = R$ so $(A \cap I_1) + e_1I_1 = I_1$, $A \cap I_1 \cap e_1I_1 = (0)$. From (4; Lemma 8), $A \cap I_1 \subseteq r_{I_1}(e_1)$. It follows that $A \subseteq r(e_1) \neq R$ and thus $A = r(e_1)$.

Now we can prove

Theorem 10. *If N has a semisimple near-ring of right quotients then N has finite rank.*

Proof. There remains the case where $A_i \subset N_2$ for an infinite number of i. Clearly we may assume that $A_i \subset N_2$ for all i. Furthermore, T is a semisimple near-ring of right quotients of N_2 and is the direct sum of ideals each of which is a simple non-ring. Apply Lemma 2 to choose B_i to be a submodule of A_i and consider A_1T. Since A_1T contains T-subgroups it contains minimal T-subgroups and these have the form eT where e is a non-zero idempotent. Let e_1T be a minimal T-subgroup contained in A_1T and $e_1 = p_1v_1^{-1}$ where $p_1 \in A_1$ and $v_1 \in N_2$ is cancellative. Since $e_1T + r_T(e_1) = T$ we see that $p_1N_2 + r(e_1)$ is N_2-essential in N_2 (here and for the remainder of the proof $r(x)$ denotes the right annihilator of x in N_2 i.e. $r(x) = r_T(x) \cap N_2$). Furthermore, $A_2T \cap r_T(e_1) \neq (0)$. Let e_2T be a minimal T-subgroup of T contained in $A_2T \cap r_T(e_1)$. Suppose now that we have constructed $e_1 = p_1v_1^{-1} \in A_1T$, $e_2 = p_2v_2^{-1} \in A_2T \cap r_T(e_1), \ldots, e_k = p_kv_k^{-1} \in A_kT \cap r_T(e_1) \ldots \cap r_T(e_{k-1})$ where each e_i is a non-zero idempotent and e_iT is a minimal T-subgroup. Suppose $A_{k+1} \cap r(e_1) \cap \ldots \cap r(e_k) = (0)$. Since $p_kN_2 + r(e_k)$ is N_2-essential in N_2 we see that $(A_k \cap r(e_1) \cap \ldots \cap r(e_{k-1})) + r(e_k)$ is N_2-essential in N_2 and thus $A_k \cap r(e_1) \cap \ldots \cap r(e_{k-1})$ is N_2-essential in $A_{k+1} \cap r(e_1) \cap \ldots \cap r(e_{k-1})$. Let $0 \neq x \in B_{k+1}$. There is a cancellative element $c \in N_2$ with $xc \in (p_1N_2 + r(e_1)) \cap A_{k+1} = p_1N_2 + A_{k+1} \cap r(e_1)$. Hence for some $s_1 \in N_2$ we have $0 \neq p_1s_1 + xc \in A_{k+1} \cap r(e_1)$. Then $(p_1s_1 + xc)d_1 - p_1s_1d_1 + p_1s_1d_1 \in A_{k+1} \cap r(e_1) \cap (p_2N_2 + r(e_2)) = p_2N_2 + A_{k+1} \cap r(e_1) \cap r(e_2)$ for some cancellative $d_1 \in N_2$. Let $x_1 = (p_1s_1 + xc)d_1 - p_1s_1d_1 \in B_{k+1}$. Then $x_1 \neq 0$ and for some $s_2 \in N_2$ we have $0 = p_2s_2 + p_1s_1 + x_1 \in A_{k+1} \cap r(e_1) \cap r(e_2)$ so with $a_1 = p_1s_1d_1 \in A_1$ and $a_2 = p_2s_2 + p_1s_1d_1 \in A_2$ we have, for some cancellative $d_2 \in N_2$, $(a_2 + x_1)d_2 - a_2d_2 + a_2d_2 \in A_{k+1} \cap r(e_1) \cap r(e_2) \cap (p_3N_2 + r(e_3))$ and we find $x_2 = (a_2 + x_1)d_2 - a_2d_2 \in B_{k+1}$, $x_2 \neq 0$ and $a_3 \in A_3$ with $a_3 + x_2 \in A_{k+1} \cap r(e_1) \cap r(e_2) \cap r(e_3)$. Continuing in this way we arrive at $a_{k-1} \in A_{k-1}$, $x_{k-2} \in B_{k+1}$ and $a_{k-1} + x_{k-2} \in A_{k+1} \cap r(e_1) \cap \ldots \cap r(e_{k-1})$ and at each stage $x_i \neq 0$. For some cancellative $d' \in N_2$ we now have $(a_{k-1} + x_{k-2})d'_{k-1} \in A_k \cap r(e_1) \cap \ldots \cap r(e_{k-1}) \subseteq A_k$. Hence $(a_{k-1} + x_{k-2})d' - a_{k-1}d' \in A_k \cap B_{k+1} = (0)$. But then $x_{k-2} = 0$ which is false. Thus if the chain $A_1 \subset A_2 \subset \ldots$ does not terminate finitely then for each m, $A_m \cap r(e_1) \cap \ldots \cap r(e_{m-1}) \neq (0)$. However, $r_T(e_i)$ is a maximal right annihilator in T, T has only finitely many such and their intersection is zero. This contradiction establishes the result.

REFERENCES

(1) A. W. Goldie, Some aspects of ring theory, *Bull. London Math. Soc.* (2) **1** (1969), 129–154.

(2) W. M. L. Holcombe, Near-rings of quotients of endomorphism near-rings, *Proc. Edinburgh Math. Soc.* **19**, (1975), 345–352.

(3) A. Oswald, Semisimple near-rings have maximum condition on N-subgroups, *J. London Math. Soc.* (2) **11** (1975), 408–412.

(4) A. Oswald, Completely reducible near-rings, *Proc. Edinburgh Math. Soc.* **20** (1976), 187–197.

(5) G. Pilz, *Near-rings* (North-Holland, 1977).

(6) L. W. Small, Orders in Artinian Rings, *J. Algebra* **4** (1966), 13–41.

Department of Mathematics
Teesside Polytechnic
Middlesbrough
Cleveland
England