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A Double Triangle Operator Algebra From $SL_2(\mathbb{R}_+)$

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Abstract. We consider the w*-closed operator algebra $\mathcal{A}_+$ generated by the image of the semigroup $SL_2(\mathbb{R}_+)$ under a unitary representation ρ of $SL_2(\mathbb{R})$ on the Hilbert space $L^2(\mathbb{R})$. We show that $\mathcal{A}_+$ is a reflexive operator algebra and $\mathcal{A}_+ = \operatorname{Alg} \mathcal{D}$ where $\mathcal{D}$ is a double triangle subspace lattice. Surprisingly, $\mathcal{A}_+$ is also generated as a w*-closed algebra by the image under ρ of a strict subsemigroup of $SL_2(\mathbb{R}_+)$.

1 Introduction

Given a set $\mathcal{S}$ of operators on a Hilbert space, let w*-alg $\mathcal{S}$ denote the w*-closed operator algebra generated by $\mathcal{S}$. Write M_λ, D_μ and V_t for the unitary operators on the Hilbert space $L^2(\mathbb{R})$ defined by

$$M_\lambda f(x) = e^{i\lambda x} f(x), \quad D_\mu f(x) = f(x-\mu) \quad \text{and} \quad V_t f(x) = e^{t/2} f(e^t x).$$

Katavolos and Power [3, 4] introduced two nonselfadjoint operator algebras. These are the *Fourier binest algebra*

$$\mathcal{A}_p = \text{w}^*\text{-alg}\{M_\lambda, D_\mu \mid \lambda, \mu \geq 0\} \tag{1}$$

and the *hyperbolic algebra*

$$\mathcal{A}_h = \text{w}^*\text{-alg}\{M_\lambda, V_t \mid \lambda, t \geq 0\}. \tag{2}$$

These algebras have several interesting properties. First, whilst they contain no finite rank operators, the Hilbert–Schmidt operators they contain form a w*-dense set. Secondly, their invariant subspace lattices $\operatorname{Lat} \mathcal{A}$ are naturally topologically isomorphic to Euclidean manifolds; in fact $\operatorname{Lat} \mathcal{A}_p$ is isomorphic to the closed unit disc and $\operatorname{Lat} \mathcal{A}_h$ is a compact connected 4-manifold. Thirdly, $\mathcal{A}_p$ and $\mathcal{A}_h$ are reflexive, that is, $\mathcal{A} = \operatorname{Alg} \operatorname{Lat} \mathcal{A}$, where as usual, $\operatorname{Alg} \mathcal{L}$ is the algebra of operators leaving every element of the subspace lattice $\mathcal{L}$ invariant. The reflexivity of $\mathcal{A}_h$ is proven in [6].

As observed in [4], both $\mathcal{A}_p$ and $\mathcal{A}_h$ are examples of *Lie semigroup algebras*. These are weak operator topology closed operator algebras generated by the image of a Lie semigroup in a unitary representation of the ambient Lie group. It is therefore

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natural to look at examples of Lie semigroup algebras and ask if they share the properties of $\mathcal{A}_p$ and $\mathcal{A}_h$. In this note we consider the Lie group $SL_2(\mathbb{R})$ of 2×2 matrices with determinant +1 and the Lie semigroup $SL_2(\mathbb{R}_+)$ given by

$$SL_2(\mathbb{R}_+) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R}) \mid a, b, c, d \geq 0 \}.$$

This is generated (as a semigroup) by elements of the form

$$r_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad u_\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad l_\gamma = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$$

for $\alpha > 0$ and $\beta, \gamma \geq 0$. If we add the generator $j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ then we get the full group $SL_2(\mathbb{R})$. We will use the standard *principal series* representations $\rho_{h,s}$ of $SL_2(\mathbb{R})$ on $L^2(\mathbb{R})$ given by

$$\rho_{h,s} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} f(x) = \frac{\operatorname{sgn}(\beta x + \delta)^h |\beta x + \delta|^{is}}{|\beta x + \delta|} f\Big(\frac{\alpha x + \gamma}{\beta x + \delta} \Big), \tag{3}$$

where $h \in \{0, 1\}$, $s \in \mathbb{R}$ and $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL_2(\mathbb{R})$. As is well known (see, for example, [8]), $\rho_{h,s}$ is a unitary representation on $L^2(\mathbb{R})$ for each $h \in \{0, 1\}$ and $s \in \mathbb{R}$. It is irreducible, that is, $\operatorname{Lat} \rho_{h,s}(SL_2(\mathbb{R}))$ is trivial, unless $h = 1$ and $s = 0$.

Let us write $\mathcal{A}_+$ for the w*-closed algebra generated by $\rho_{h,s}(SL_2(\mathbb{R}_+))$. Then

$$\mathcal{A}_+ = \text{w}^*\text{-alg}\{ \rho_{h,s}(r_\alpha), \rho_{h,s}(l_\gamma), \rho_{h,s}(u_\beta) \mid \alpha > 0 \text{ and } \beta, \gamma \geq 0 \}.$$

A computation reveals that for $\alpha > 0$ and $\gamma \geq 0$,

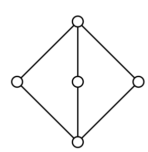
$$\rho_{h,s}(r_\alpha) = \alpha^{-is} V_{2 \log \alpha} \quad \text{and} \quad \rho_{h,s}(l_\gamma) = D_{-\gamma}, \tag{4}$$

but the expression for $\rho_{h,s}(u_\beta)$ looks unpleasantly complicated. However, since $u_\beta = j l_{-\beta} j^{-1}$,

$$\rho_{h,s}(u_\beta) = \rho_{h,s}(j) \rho_{h,s}(l_{-\beta}) \rho_{h,s}(j)^{-1} = Y D_\beta Y^*$$

where $Y = Y_{h,s} = \rho_{h,s}(j)$.

In Sections 2 and 3, we fix $h = 1$, $s = 0$ and write $\rho = \rho_{1,0}$ and $Y = Y_{1,0}$. We will show that, in this exceptional case, $\mathcal{A}_+$ is in fact an example of a known class of reflexive operator algebras [5, 7]. These are algebras of the form $\operatorname{Alg} \mathcal{D}$ where $\mathcal{D}$ is a double triangle lattice, *i.e.,* a 5-element subspace lattice with the following Hasse diagram.



This analysis also gives the unexpected result that $\mathcal{A}_+$ is generated as a w*-closed algebra by $\rho(\mathcal{S})$ where $\mathcal{S}$ is the strict subsemigroup of $SL_2(\mathbb{R}_+)$ which is generated by $\{ r_\alpha, l_\gamma \mid \alpha > 0,\ \gamma \geq 0 \}$. In contrast, the corresponding norm-closed algebras generated by $\rho(\mathcal{S})$ and $\rho(SL_2(\mathbb{R}_+))$ are distinct.

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2 Invariant Subspace Lattices

In [4], the authors examine the w*-closed algebra $\mathcal{A}_h$ defined by (2). They show that the invariant subspace lattice of $\mathcal{A}_h$ is

$$\operatorname{Lat} \mathcal{A}_h = \{K_{\alpha,\lambda,\mu} \mid \alpha \in \mathbb{C}^*,\ \lambda, \mu \geq 0\} \cup \{L^2([-a, b]) \mid a, b \in [0, \infty]\}$$

where

$$\mathbb{C}^* = \mathbb{C} \setminus \{0\}, \quad \varphi_\alpha(x) = \begin{cases} 1 & x \geq 0, \\ \alpha & x < 0, \end{cases}$$

and $K_{\alpha,\lambda,\mu}$ is the closed subspace

$$K_{\alpha,\lambda,\mu} = \varphi_\alpha(x) e^{i(\lambda x + \mu x^{-1})} H^2(\mathbb{R}).$$

We also use the notation $L^2(S)$ for the subspace of functions in $L^2(\mathbb{R})$ vanishing off the closed subset S of $\mathbb{R}$.

Let $\mathcal{A}_\ell$ be the "lower triangular" subalgebra of $\mathcal{A}_+$

$$\mathcal{A}_\ell = \text{w}^*\text{-alg}\{\rho(r_\alpha), \rho(l_\gamma) \mid \alpha > 0, \gamma \geq 0\}.$$

Armed with knowledge of $\operatorname{Lat} \mathcal{A}_h$, an expression for $\operatorname{Lat} \mathcal{A}_\ell$ is fairly easy to come by. As in [5], a *double triangle lattice* of subspaces of $\mathcal{H}$ is a five-element subspace lattice $\mathcal{L} = \{(0), K, L, M, \mathcal{H}\}$ such that $K \cap L = L \cap M = M \cap K = (0)$ and $K \vee L = L \vee M = M \vee K = \mathcal{H}$.

Lemma 2.1 *The invariant subspace lattice of $\mathcal{A}_\ell$ is*

$$\text{(5)} \qquad \operatorname{Lat} \mathcal{A}_\ell = \{F^*(\varphi_\alpha H^2(\mathbb{R})) \mid \alpha \in \mathbb{C}^*\} \cup \{(0), H^2(\mathbb{R}), \overline{H^2(\mathbb{R})}, L^2(\mathbb{R})\}.$$

In particular, the double triangle lattice

$$\mathcal{E} = \{(0), H^2(\mathbb{R}), L^2(\mathbb{R}_-), \overline{H^2(\mathbb{R})}, L^2(\mathbb{R})\}$$

is contained in $\operatorname{Lat} \mathcal{A}_\ell$.

Proof Recall from (4) that $\rho(r_\alpha) = V_{2 \log \alpha}$ and $\rho(l_\gamma) = D_{-\gamma}$. Thus

$$\mathcal{A}_\ell = \text{w}^*\text{-alg}\{D_{-\lambda}, V_t \mid \lambda \geq 0,\ t \in \mathbb{R}\}.$$

Let F be the unitary operator on $L^2(\mathbb{R})$ given by $Ff = \hat{f}$, the Fourier–Plancherel transform. Since $FV_tF^* = V_{-t}$ and $FD_{-\lambda}F^* = M_\lambda$,

$$F\mathcal{A}_\ell F^* = \text{w}^*\text{-alg}\{V_t, M_\lambda \mid \lambda \geq 0,\ t \in \mathbb{R}\}.$$

Comparing this to the generator description (2) of the hyperbolic algebra $\mathcal{A}_h$, we see that the algebra $F\mathcal{A}_\ell F^*$ contains $\mathcal{A}_h$ and that

$$\operatorname{Lat} F\mathcal{A}_\ell F^* = \{K \in \operatorname{Lat} \mathcal{A}_h \mid V_t K \subseteq K \text{ for each } t < 0\}.$$

Now $V_t K_{\alpha,\lambda,\mu} = K_{\alpha,e^t\lambda,e^{-t}\mu}$, and for $t < 0$ and $\lambda, \mu \geq 0$ this is contained in $K_{\alpha,\lambda,\mu}$ only if $\lambda = \mu = 0$. Similarly, when $t < 0$, $V_t L^2([-a,b]) \subseteq L^2([-a,b])$ only if $a, b \in \{0, \infty\}$. Thus

$$\operatorname{Lat} F\mathcal{A}_\ell F^* = \{\varphi_\alpha H^2(\mathbb{R}) \mid \alpha \in \mathbb{C}^*\} \cup \{(0), L^2(\mathbb{R}_+), L^2(\mathbb{R}_-), L^2(\mathbb{R})\}.$$

Since $\operatorname{Lat} F\mathcal{A}_\ell F^* = F \operatorname{Lat} \mathcal{A}_\ell$, we can apply F^* to either side of this equation to obtain (5).

To see that $\mathcal{E} \subseteq \operatorname{Lat} \mathcal{A}_\ell$, observe that $F^*(\varphi_1 H^2(\mathbb{R})) = F^* H^2(\mathbb{R}) = L^2(\mathbb{R}_-)$. ■

In fact, $\mathcal{E}$ is a sublattice not only of $\operatorname{Lat} \mathcal{A}_\ell$ but also of the smaller lattice $\operatorname{Lat} \mathcal{A}_+$.

Lemma 2.2 $\mathcal{E} \subseteq \operatorname{Lat} \mathcal{A}_+$.

Proof Since $\mathcal{A}_+ = \text{w}^*\text{-alg}(\mathcal{A}_\ell \cup \mathcal{A}_1)$, we have $\operatorname{Lat} \mathcal{A}_+ = \operatorname{Lat} \mathcal{A}_\ell \cap \operatorname{Lat} \mathcal{A}_1$ where the algebra $\mathcal{A}_1$ is generated by the one-parameter semigroup $\{\rho(u_\beta)\}_{\beta \geq 0}$. Let $\beta \geq 0$. Recall that $\rho(u_\beta) = Y D_\beta Y^*$. Since $Y^* = -Y$ and

$$Y f(x) = x^{-1} f(-x^{-1}),$$

$H^2(\mathbb{R})$ reduces Y and so $H^2(\mathbb{R})$ and $\overline{H^2(\mathbb{R})}$ are invariant under $\rho(u_\beta)$. Moreover,

$$\rho(u_\beta) L^2(\mathbb{R}_-) = Y D_\beta Y^* L^2(\mathbb{R}_-) = Y D_\beta L^2(\mathbb{R}_+) \subseteq Y L^2(\mathbb{R}_+) = L^2(\mathbb{R}_-).$$

This shows that $\mathcal{E} \subseteq \operatorname{Lat} \mathcal{A}_1$ and we have already seen in Lemma 2.1 that $\mathcal{E}$ is a sublattice of $\operatorname{Lat} \mathcal{A}_\ell$. Hence $\mathcal{E} \subseteq \operatorname{Lat} \mathcal{A}_\ell \cap \operatorname{Lat} \mathcal{A}_1 = \operatorname{Lat} \mathcal{A}_+$. ■

The next theorem is an immediate consequence of a result of Lambrou and Longstaff [5, Corollary 2.1], which they prove in a Banach space setting. The Hilbert space version which we use is attributed in [5] to an earlier result of H. K. Middleton.

Theorem 2.3 *Let $\mathcal{D} = \{(0), K, K^\perp, M, \mathcal{H}\}$ be a double triangle lattice of subspaces of a Hilbert space $\mathcal{H}$. Then*

$$\operatorname{Lat} \operatorname{Alg} \mathcal{D} = \{N_\alpha \mid \alpha \in \mathbb{C}^*\} \cup \{(0), K, K^\perp, \mathcal{H}\},$$

where if $[J]$ denotes the orthogonal projection onto the subspace J of $\mathcal{H}$,

$$N_\alpha = \big([K] + \alpha[K^\perp]\big) M \quad \textit{for } \alpha \in \mathbb{C}^*.$$

Moreover, the infimum and supremum of any two distinct elements of $\operatorname{Lat} \operatorname{Alg} \mathcal{D}$ *are the zero subspace and $\mathcal{H}$ respectively.*

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Remark For our purposes it would suffice to know that $\operatorname{Lat}\operatorname{Alg}\mathcal{D}$ contains the set $\{N_\alpha \mid \alpha \in \mathbb{C}^*\} \cup \{(0), K, K^\perp, \mathcal{H}\}$. This can be established with an attractive argument using techniques of Halmos [1, 2] which makes use of the fact that the subspaces K and M in $\mathcal{D}$ are in generic position and that N_α is the graph of the unbounded closed operator $[K]M \to K^\perp$, $[K]g \mapsto \alpha[K^\perp]g$ for $g \in M$ and $\alpha \in \mathbb{C}^*$. There is also a very short proof of this fact in [7].

It is natural to define $N_0 = K^\perp$ and $N_\infty = K$. Indeed, if we do so then when viewed as a set of projections endowed with the strong operator topology, $\operatorname{Lat}\operatorname{Alg}\mathcal{D}$ becomes the union of a topological sphere $\{N_\alpha \mid \alpha \in \mathbb{C}\cup\{\infty\}\}$ with the two disjoint points $\{(0), \mathcal{H}\}$. Let us henceforth write N_α for the subspaces so obtained in the case $\mathcal{D} = \mathcal{E}$, $K = H^2(\mathbb{R})$, $K^\perp = \overline{H^2(\mathbb{R})}$, $M = L^2(\mathbb{R}_-)$, $\mathcal{H} = L^2(\mathbb{R})$; that is,

$$N_\alpha = \begin{cases} \big([H^2(\mathbb{R})] + \alpha[\overline{H^2(\mathbb{R})}]\big)\, L^2(\mathbb{R}_-) & \text{for } \alpha \in \mathbb{C}^*, \\ H^2(\mathbb{R}) & \alpha = 0, \\ \overline{H^2(\mathbb{R})} & \alpha = \infty. \end{cases}$$

We will also write $\mathcal{B}$ for the "ball lattice"

$$\mathcal{B} = \operatorname{Lat}\operatorname{Alg}\mathcal{E} = \{N_\alpha \mid \alpha \in \mathbb{C} \cup \{\infty\}\} \cup \{(0), L^2(\mathbb{R})\}.$$

Lemma 2.4 *For each $\alpha \in \mathbb{C}^*$, $F^*(\varphi_\alpha H^2(\mathbb{R})) = N_\alpha$. Thus $\operatorname{Lat}\mathcal{A}_\ell = \mathcal{B}$.*

Proof Let $\alpha \in \mathbb{C}^*$. Since $\varphi_\alpha = \chi_{\mathbb{R}_+} + \alpha\chi_{\mathbb{R}_-}$,

$$\varphi_\alpha H^2(\mathbb{R}) = \big([L^2(\mathbb{R}_+)] + \alpha[L^2(\mathbb{R}_-)]\big)\, H^2(\mathbb{R}).$$

But

$$\begin{aligned} FN_\alpha &= F\big([H^2(\mathbb{R})] + \alpha[\overline{H^2(\mathbb{R})}]\big)\, L^2(\mathbb{R}_-) \\ &= \big([FH^2(\mathbb{R})] + \alpha[F\overline{H^2(\mathbb{R})}]\big)\, FL^2(\mathbb{R}_-) \\ &= \big([L^2(\mathbb{R}_+)] + \alpha[L^2(\mathbb{R}_-)]\big)\, H^2(\mathbb{R}) \\ &= \varphi_\alpha H^2(\mathbb{R}). \end{aligned}$$

So $N_\alpha = F^*(\varphi_\alpha H^2(\mathbb{R}))$, and by Lemma 2.1,

$$\operatorname{Lat}\mathcal{A}_\ell = \{N_\alpha \mid \alpha \in \mathbb{C}^*\} \cup \{(0), H^2(\mathbb{R}), \overline{H^2(\mathbb{R})}, L^2(\mathbb{R})\} = \mathcal{B}. \qquad \blacksquare$$

Remark In [4], the subspaces $\varphi_\alpha H^2(\mathbb{R})$ are introduced and are then shown to be invariant under $\mathcal{A}_h$. On the other hand, Theorem 2.3 and Lemma 2.4 together show that the subspaces $\varphi_\alpha H^2(\mathbb{R})$ lie in the reflexive closure $\operatorname{Lat}\operatorname{Alg}F\mathcal{E}$ of the double triangle lattice

$$F\mathcal{E} = \{(0), L^2(\mathbb{R}_+), L^2(\mathbb{R}_-), H^2(\mathbb{R}), L^2(\mathbb{R})\}.$$

It is easy to see that $F\mathcal{E} \subseteq \operatorname{Lat}\mathcal{A}_h$, so we also have $\operatorname{Lat}\operatorname{Alg}F\mathcal{E} \subseteq \operatorname{Lat}\mathcal{A}_h$. Thus we obtain a transparent argument showing that each subspace $\varphi_\alpha H^2(\mathbb{R})$ lies in $\operatorname{Lat}\mathcal{A}_h$.

Corollary 2.5 $\operatorname{Lat}\mathcal{A}_+ = \operatorname{Lat}\mathcal{A}_\ell = \mathcal{B}$.

Proof Since $\mathcal{A}_\ell \subseteq \mathcal{A}_+$, it follows that $\operatorname{Lat}\mathcal{A}_+ \subseteq \operatorname{Lat}\mathcal{A}_\ell$. By Lemma 2.2, $\mathcal{E} \subseteq \mathcal{A}_+$, so by Lemma 2.4 we have

$$\mathcal{B} = \operatorname{Lat}\operatorname{Alg}\mathcal{E} \subseteq \operatorname{Lat}\operatorname{Alg}(\operatorname{Lat}\mathcal{A}_+) = \operatorname{Lat}\mathcal{A}_+ \subseteq \operatorname{Lat}\mathcal{A}_\ell = \mathcal{B}. \qquad \blacksquare$$

3 Reflexivity

We show that $\mathcal{A}_+$ is a reflexive operator algebra. Our method is somewhat surprising: we identify $\mathcal{A}_+$ with what appears at first sight to be the proper subalgebra $\mathcal{A}_\ell$. Let $\mathcal{A}_B = \operatorname{Alg}\mathcal{B}$. Since $\operatorname{Lat}\mathcal{A}_+ = \mathcal{B}$, it follows that

$$\mathcal{A}_\ell \subseteq \mathcal{A}_+ \subseteq \operatorname{Alg}\operatorname{Lat}\mathcal{A}_+ \subseteq \mathcal{A}_B.$$

We will show that all of these inclusions are actually equalities.

Lemma 3.1 *The Hilbert–Schmidt operators in each of the algebras $\mathcal{A}_\ell$ and $\mathcal{A}_B$ are w*-dense.*

Proof As shown in [6], there is a sequence X_n of Hilbert–Schmidt contractions in $\mathcal{A}_h$ which converge in the strong operator topology to the identity. Since the Hilbert–Schmidt operators $\mathcal{C}_2$ form an ideal in $\mathcal{L}(L^2(\mathbb{R}))$, for any operator algebra $\mathcal{A}$ we have $F(\mathcal{A} \cap \mathcal{C}_2)F^* = F\mathcal{A}F^* \cap \mathcal{C}_2$. Now, since $\mathcal{A}_h \subseteq F\mathcal{A}_\ell F^*$, the sequence X_n lies in $F\mathcal{A}_\ell F^* \cap \mathcal{C}_2 = F(\mathcal{A}_\ell \cap \mathcal{C}_2)F^* \subseteq F(\mathcal{A}_B \cap \mathcal{C}_2)F^*$.

Let $\mathcal{A}$ be either $F\mathcal{A}_\ell F^*$ or $F\mathcal{A}_B F^*$ and let $T \in \mathcal{A}$. Then the sequence $X_n T$ is a bounded sequence of Hilbert–Schmidt operators which tends to T in the SOT. Since the SOT and the w*-topology agree on bounded sets and $\mathcal{A}$ is w*-closed, this shows that the Hilbert–Schmidt operators are dense in $\mathcal{A}$. So the Hilbert–Schmidt operators are also dense in $F^*\mathcal{A}F$ and the proof is complete. $\blacksquare$

We introduce some notation which will help us pin down the Hilbert–Schmidt operators in $\mathcal{A}_\ell$ and $\mathcal{A}_B$. Let q be the function defined on $\mathbb{R} \setminus \{0\}$ by

$$q(x) = \begin{cases} x^{-1/2} & x > 0, \\ -i|x|^{-1/2} & x < 0. \end{cases}$$

Then q is the restriction to $\mathbb{R} \setminus \{0\}$ of a branch of the analytic function $z \mapsto z^{-1/2}$ defined on $\mathbb{C} \setminus \mathbb{R}_-$. Observe that the map $M_{\overline{q}}\colon L^2(\mathbb{R}) \to L^2(|x|\,dx)$ is a unitary isomorphism onto the Hilbert space $L^2(|x|\,dx)$. As in [6], we work with the space $V = M_{\overline{q}}H^2(\mathbb{R})$. Let $W' = L^2(e^t\,dt)$. Given a function $k \in L^2(\mathbb{R}^2)$ supported on $Q = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}$, let $\tilde{k}$ be "k with a change of variables," defined by

$$\tilde{k}(x, t) = k(x, e^t x).$$

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A calculation reveals that $\tilde{k} \in L^2(|x|\,dx) \otimes W'$ and that the map $k \mapsto \tilde{k}$, is an isometry $L^2(Q) \to L^2(|x|\,dx) \otimes W'$.

For $k \in L^2(\mathbb{R}^2)$, we define the Hilbert–Schmidt operator $\operatorname{Int} k$ on $L^2(\mathbb{R})$ by

$$(\operatorname{Int} k) f\,(x) = \int_{\mathbb{R}} k(x,y) f(y)\,dy.$$

The following lemma shows that it is natural for us to consider functions supported on Q. Its proof is routine and we omit it.

Lemma 3.2 *Let* $\operatorname{Int} k$ *be a Hilbert–Schmidt operator leaving* $L^2(\mathbb{R}_+)$ *and* $L^2(\mathbb{R}_-)$ *invariant. Then* $\operatorname{supp} k \subseteq Q$.

Proposition 3.3 *Let* $\operatorname{Int} k$ *be a Hilbert–Schmidt operator leaving invariant* $L^2(\mathbb{R}_+)$, $L^2(\mathbb{R}_-)$ *and* $\varphi_a H^2(\mathbb{R})$ *for* $a > 0$. *Then* $\tilde{k} \in V \otimes W'$. *In particular,*

$$F(\mathcal{A}_B \cap \mathcal{C}_2)F^* \subseteq \{\operatorname{Int} k \mid \tilde{k} \in V \otimes W'\}.$$

Outline of proof As observed in [4], when $a > 0$ we have

$$\varphi_a H^2(\mathbb{R}) = |x|^{i\pi^{-1}\log a} H^2(\mathbb{R}). \tag{6}$$

Having made this identification, the proof proceeds almost exactly as the proof of [6, Proposition 2.4]. In short, we consider the equation

$$\langle (\operatorname{Int} k)|x|^{i\sigma} h_1, |x|^{i\sigma}\overline{h_2}\rangle = 0$$

which holds for every $\sigma \in \mathbb{R}$ and each $h_1, h_2 \in H^2(\mathbb{R})$ by virtue of our hypotheses and (6). After a calculation we see that this implies that for almost every t, the function $x \mapsto \tilde{k}(x,t)$ lies in V. It follows from Lemma 3.2 that for almost every x, the function $t \mapsto \tilde{k}(x,t)$ lies in W'. Hence $\tilde{k} \in V \otimes W'$.

The result follows upon observing that every Hilbert–Schmidt operator $\operatorname{Int} k$ in $F\mathcal{A}_B F^* \cap \mathcal{C}_2 = F(\mathcal{A}_B \cap \mathcal{C}_2)F^*$ satisfies the hypotheses. ■

Proposition 3.4 *If* $\tilde{k} \in V \otimes W'$, *then* $\operatorname{Int} k \in F(\mathcal{A}_\ell \cap \mathcal{C}_2)F^*$. *That is,*

$$F(\mathcal{A}_\ell \cap \mathcal{C}_2)F^* \supseteq \{\operatorname{Int} k \mid \tilde{k} \in V \otimes W'\}.$$

Outline of proof The proof follows [6, Section 3] exactly when we replace the space $W = L^2(\mathbb{R}_+, |x|\,dx)$ there with W' here and recall that $\mathcal{A}_h \subseteq F\mathcal{A}_\ell F^*$. We refer the reader to [6] for the details. ■

Theorem 3.5 $\mathcal{A}_\ell = \mathcal{A}_+ = \mathcal{A}_B$. *In particular,* $\mathcal{A}_+$ *is reflexive.*

Proof We know that $\mathcal{A}_\ell \subseteq \mathcal{A}_+ \subseteq \mathcal{A}_B$. Hence by Propositions 3.3 and 3.4, $\mathcal{A}_\ell \cap \mathcal{C}_2 = \mathcal{A}_B \cap \mathcal{C}_2$. By Lemma 3.1, this set of Hilbert–Schmidt operators is w*-dense in each of the w*-closed algebras $\mathcal{A}_\ell$ and $\mathcal{A}_B$, so $\mathcal{A}_\ell = \mathcal{A}_B = \mathcal{A}_+$. Since $\mathcal{A}_B = \operatorname{Alg}\mathcal{B}$ is plainly reflexive, the proof is complete. ■

Question 3.1 It is shown in [5] that $\mathcal{A}_B$ contains operators of every even rank and their ranges are dense in $L^2(\mathbb{R})$. Is there an alternative proof of Theorem 3.5 in which these finite rank operators take the place of the Hilbert–Schmidt operators?

Remark Let $\mathcal{A}_u$ be the "upper triangular" algebra

$$\mathcal{A}_u = \text{w}^*\text{-alg}\{\rho(r_\alpha), \rho(u_\beta) \mid \alpha > 0,\ \beta \geq 0\}.$$

Then by Theorem 3.5 we also have $\mathcal{A}_+ = \mathcal{A}_u$; indeed, let Z be the unitary on $L^2(\mathbb{R})$ given by $Zf(x) = x^{-1}f(x^{-1})$. Then for $\alpha > 0$ and $\beta, \gamma \geq 0$,

$$Z\rho(r_\alpha)Z^* = \rho(r_{\alpha^{-1}}), \quad Z\rho(u_\beta)Z^* = \rho(l_\beta) \quad \text{and} \quad Z\rho(l_\gamma)Z^* = \rho(u_\gamma).$$

So $\mathcal{A}_u = Z\mathcal{A}_\ell Z^* = Z\mathcal{A}_+Z^* = \mathcal{A}_+ = \mathcal{A}_\ell$.

Theorem 3.5 exhibits a curious collapse phenomenon:

$$\begin{aligned}\mathcal{A}_+ &:= \text{w}^*\text{-alg}\{\rho(r_\alpha), \rho(l_\gamma), \rho(u_\beta) \mid \alpha > 0,\ \beta, \gamma \geq 0\} \\ &= \text{w}^*\text{-alg}\{\rho(r_\alpha), \rho(l_\gamma) \mid \alpha > 0,\ \gamma \geq 0\} =: \mathcal{A}_\ell,\end{aligned}$$

although it is not clear at first sight why $\rho(u_\beta) \in \mathcal{A}_\ell$. It is interesting to ask in which topologies this collapse occurs. We show that the norm-closed algebras do not coincide.

Proposition 3.6 *Let $\mathcal{A}_\ell^{\text{n}}$ and $\mathcal{A}_+^{\text{n}}$ denote the norm-closed operator algebras generated by*

$$\mathcal{S}_\ell = \{\rho(r_\alpha), \rho(l_\gamma) \mid \alpha > 0, \gamma \geq 0\}$$

and

$$\mathcal{S}_+ = \{\rho(r_\alpha), \rho(l_\gamma), \rho(u_\beta) \mid \alpha > 0 \text{ and } \beta, \gamma \geq 0\},$$

respectively. Then $\mathcal{A}_\ell^{\text{n}} \subsetneq \mathcal{A}_+^{\text{n}}$.

Proof Fix $\beta > 0$. Intuitively, elements of $\mathcal{S}_\ell$ "fix ∞" whereas $\rho(u_\beta)$ is a "shift through ∞". We exploit this perspective to show that $\rho(u_\beta) \notin \mathcal{A}_\ell^{\text{n}}$.

Given $t > 0$, let $J_t = (-\infty, -t] \cup [t, \infty)$. Let $\mathcal{A}_\ell^\circ$ denote the algebra generated by $\mathcal{S}_\ell$, so that $\mathcal{A}_\ell^\circ$ is the set of finite sums of finite products of elements of $\mathcal{S}_\ell$.

First, we claim that for any $t > 0$ and each $T \in \mathcal{A}_\ell^\circ$, there is an $s \in \mathbb{R}$ such that whenever $g \in L^2(\mathbb{R})$ and $\operatorname{supp} g \subseteq J_s$, we have $\operatorname{supp} Tg \subseteq J_t$. If $\alpha > 0$ and $T = \rho(r_\alpha)$, then $T = V_{2\log\alpha}$, so $s = t\alpha^{-2}$ suffices. If $\gamma \geq 0$ and $T = \rho(l_\gamma)$ then $T = D_{-\gamma}$, so $s = t + \gamma$ suffices. A simple induction argument establishes the claim for $T = \rho(a_1 a_2 \cdots a_n)$ where $a_i \in \mathcal{S}_\ell$ and another induction shows that the claim holds for a finite sum of such operators.

Our second claim is that we can find a $t > 0$ such that for $f \in L^2(\mathbb{R})$ with $\operatorname{supp} f \subseteq J_t$, we have $\operatorname{supp} \rho(u_\beta)f \cap J_t = \varnothing$. In fact, $t = 4\beta^{-1}$ will do, as a simple but slightly tedious calculation will confirm.

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Fix $t = 4\beta^{-1}$ and $T \in \mathcal{A}_\ell^\circ$. Compute a value of s for T and t and let $g \in L^2(\mathbb{R})$ with $\|g\| = 1$ and $\operatorname{supp} g \subseteq J_s$. Since Tg and $\rho(u_\beta)g$ are orthogonal and $\rho(u_\beta)$ is unitary,

$$\|T - \rho(u_\beta)\|^2 \geq \|Tg - \rho(u_\beta)g\|^2 = \|Tg\|^2 + \|\rho(u_\beta)g\|^2 \geq \|\rho(u_\beta)g\|^2 = 1.$$

The algebra $\mathcal{A}_\ell^\circ$ is norm-dense in $\mathcal{A}_\ell^{\mathrm{n}}$, so this shows that $\operatorname{dist}(\rho(u_\beta), \mathcal{A}_\ell^{\mathrm{n}}) \geq 1$. Thus $\rho(u_\beta) \notin \mathcal{A}_\ell^{\mathrm{n}}$. ■

4 Questions

Fix $(h, s) \neq (1, 0)$. Let $\rho_{h,s}$ be the irreducible representation in the principal series given by (3) and let $\mathcal{A}_+$ be the w*-closed operator algebra generated by $\rho_{h,s}(SL_2(\mathbb{R}_+))$. Now Lemma 2.1 still holds for $\mathcal{A}_+$; indeed, the subalgebra $\mathcal{A}_\ell$ is independent of our choice of h and s. However, the author has been unable to find an analogue of Lemma 2.2, since $Y_{h,s} = \rho_{h,s}(j)$ is no longer reduced by $H^2(\mathbb{R})$ and the only proper subspace obviously invariant for $\mathcal{A}_+$ is $L^2(\mathbb{R}_-)$. This prompts the following two questions in the irreducible case:

Question 4.1 Is $\operatorname{Lat} \mathcal{A}_+ = \{(0), L^2(\mathbb{R}_-), L^2(\mathbb{R})\}$?

Question 4.2 Is $\mathcal{A}_+$ reflexive?

On a more general theme, we pose the following. Recall that when $(h, s) = (1, 0)$, the lattice $\operatorname{Lat} \mathcal{A}_+$ with the strong operator topology is the union of a Euclidean manifold with a finite number of discrete points. We call such a lattice a nearly Euclidean lattice. Of the three Lie semigroup algebras $\mathcal{A}_p$, $\mathcal{A}_h$ and $\mathcal{A}_+$ that we have seen, all are reflexive and all have nearly Euclidean invariant subspace lattices.

Question 4.3 Which operator algebras do other unitary representations of $SL_2(\mathbb{R}_+)$ lead to? Are they reflexive, and are their invariant subspace lattices nearly Euclidean?

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