ON THE CHARACTERISTIC FUNCTIONAL FOR A REPLACEMENT MODEL

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1. Introduction

J. Gani and G. F. Yeo ([2] and [3]) have recently investigated certain age-distributions associated with a replacement model which serves, in particular, as a model for phage reproduction. In this paper, the characteristic functional for this model will be obtained explicitly.

We suppose that at time $t = 0$ there is a system of n ancestors with ranked ages $0 \leqq x_1 < x_2 < \cdots < x_n$. At successive instants (regeneration points) one individual in the system is replaced by another of age zero, so that the system remains of fixed size n. The probability that the ith ranked individual is replaced at a regeneration point is $p_i > 0$ $(i = 1, \cdots, n)$, where $p_1 + \cdots + p_n = 1$; this approximates to an ideal model in which the probability of an individual being replaced would depend on its age. We shall further suppose that the regeneration times are identically and independently distributed with the arbitrary distribution function $G(t)$. In the case $n = 1$, $p_1 = 1$, we have the ordinary renewal process.

The characteristic functional for the age-distribution of the system at any time $t > 0$ is defined to be

$$C[\theta(u); t] = E\left\{\exp\left[i\int_0^\infty \theta(u)dN(u, t)\right]\right\}, \tag{1.1}$$

where $N(u, t)$ is the number of individuals with ages $\leqq u$ at time t, and $\theta(u)$ is any bounded function which is R-integrable over each finite interval. This may be written

$$C[\theta(u); t] = E\left\{\exp\left[i\sum_{k=1}^{n}\theta(u_k(t))\right]\right\} \tag{1.2}$$

where $u_k(t)$ is the age of the kth ranked individual at time t.

If we take

$$\theta(u) = \begin{cases} \phi, & (0 \leqq u \leqq x) \\ 0, & \text{elsewhere,} \end{cases} \tag{1.3}$$

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$$C(\phi, t) = E \exp [i\phi N(x, t)], \tag{1.4}$$

the characteristic function of $N(x, t)$.

2. Evaluation of the characteristic functional

Following Bartlett and Kendall [1], we write the characteristic functional in the form

$$C[w_1, \cdots, w_n; \theta(u); t] = E\{w_1^{l_1} \cdots w_n^{l_n} \exp [i \sum_k \theta(u_k(t))]\} \tag{2.1}$$

where

$$w_k = \exp [i\theta(t+x_k)], \qquad (k = 1, \cdots, n), \tag{2.2}$$

and the random variable l_k is equal to 1 while the kth ranked ancestor survives, and is equal to 0 after it is replaced. The summation is extended over all descendants alive at time t.

During the arbitrary time interval $(0, t)$, there will either be no regeneration point, or else a first regeneration point occurs at time τ, $0 \leqq \tau \leqq t$. In the latter event, the new individual becomes the first ranking ancestor for the interval (τ, t). Considering the possibilities, we find that the characteristic functional (2.1) satisfies the following integral equation:

$$\begin{aligned} C[w_1, \cdots, w_n; \theta(u); t] &= [1-G(t)]w_1 \cdots w_n \\ &+ \sum_{j=1}^{n} p_j C[e^{i\theta(t)}, w_1, \cdots, w_{j-1}, w_{j+1}, \cdots, w_n; \theta(u); t] * G(t) \end{aligned} \tag{2.3}$$

where

$$F * G(t) = \int_{0-}^{t} F(t-\tau) dG(\tau).$$

Since the characteristic functional is clearly linear in the 2^n products $w_1^{l_1} \cdots w_n^{l_n}$ $(l_k = 0, 1)$, we may conveniently write

$$C[w_1, \cdots, w_n; \theta(u); t] = w_1 \cdots w_n [V_0 + \sum_{(k)} V_{k_1 \cdots k_s} (w_{k_1} \cdots w_{k_s})^{-1}], \tag{2.4}$$

where the summation is extended over all selections of up to n integers $k_1 < k_2 < \cdots < k_s$ from the set $1, 2, \cdots, n$. The V's do not involve the w's, while $V_0 = 1$, $V_{k_1 \cdots k_s} = 0$ at $t = 0$ for all $\theta(u)$.

Substituting (2.4) in (2.3),

$$\begin{aligned} V_0 + \sum_{(k)} V_{k_1 \cdots k_s} (w_{k_1} \cdots w_{k_s})^{-1} &= [1-G(t)] \\ &+ \sum_{j=1}^{n} p_j \{ \sum_{(k)} V_{1k_2 \cdots k_s} (w_{k_2-1} \cdots w_{k_J-1} w_j w_{k_{J+1}} \cdots w_{k_s})^{-1} \\ &+ e^{i\theta(t)} [V_0 w_j^{-1} + \sum_{k_1>1} V_{k_1 \cdots k_s} (w_{k_1-1} \cdots w_{k_J-1} w_j w_{k_{J+1}} \cdots w_{k_s})^{-1}]\} * G(t), \end{aligned} \tag{2.4}$$

k_J being the largest member of $(k_1, \cdots, k_s)$ which is $\leqq j$. The following equations for the V's may now be obtained by equating coefficients:

$$V_0 = 1-G(t), \tag{2.5}$$

$$V_{k_1} = p_{k_1}[V_1+e^{i\theta(t)}V_0] * G(t), \tag{2.6}$$

and for $s > 1$,

$$V_{k_1\cdots k_s} = \sum_{r=1}^{s} p_{k_r}[V_{1k_1+1\cdots k_{r-1}+1k_{r+1}\cdots k_s} + e^{i\theta(t)}V_{k_1+1\cdots k_{r-1}+1k_{r+1}\cdots k_s}] * G(t). \tag{2.7}$$

We will now show that if we define

$$q_r = p_{r+1}+\cdots+p_n, \qquad (r = 0, \cdots, n-1), \tag{2.8}$$

$$Q_s = \prod_{r=0}^{s-1} (q_r-q_s), \qquad (s = 1, \cdots, n), \tag{2.9}$$

$$\gamma_s(t) = \sum_{m=0}^{\infty} (1-q_s)^m G^{(m+1)*}(t), \qquad (s = 1, \cdots, n), \tag{2.10}$$

taking $q_n = 0$, $Q_0 = 1$, and $\gamma_0(t) = G(t)$, then the solution of the system of equations (2.6) and (2.7) is given by the recurrence relations:

$$V_{12\cdots s} = \frac{Q_s}{Q_{s-1}}\{e^{i\theta(t)}V_{12\cdots s-1}\} * \gamma_s(t), \qquad (s = 1, \cdots, n), \tag{2.11}$$

$$V_{k_1\cdots k_s} = Q_s^{-1}\prod_{r=1}^{s} (q_{k_r-1}-q_{k_r+s-r})V_{12\cdots s}. \tag{2.12}$$

Clearly, the solutions for V_0 and the $V_{k_1\cdots k_s}$ satisfy the initial conditions at $t = 0$.

To prove (2.11), we observe that on taking $k_1 = 1, \cdots, k_s = s$, equation (2.7) reduces to

$$\begin{aligned} V_{12\cdots s} &= \sum_{r=1}^{s} p_r[V_{12\cdots s}+e^{i\theta(t)}V_{2\cdots s}] * G(t), \\ &= (1-q_s)[e^{i\theta(t)}V_{2\cdots s} * G(t)+V_{12\cdots s} * G(t)]. \end{aligned}$$

The solution to this equation is

$$V_{12\cdots s} = \{e^{i\theta(t)}V_{2\cdots s}\} * \{\sum_{m=1}^{\infty}(1-q_s)^m G^{m*}(t)\},$$

which will yield (2.11) once we have established (2.12), since the latter gives

$$V_{2\cdots s} = (1-q_s)^{-1}\frac{Q_s}{Q_{s-1}}V_{12\cdots s-1}.$$

We shall now indicate how (2.12) may be built up by successive inductions on the suffixes. Considering first the V_{k_1} we must prove that

$$\frac{V_{k_1}}{p_{k_1}} = \frac{V_1}{p_1}. \tag{2.13}$$

But this follows immediately from (2.6). Now assuming that the relations have been proved for the $V_{k_1 \cdots k_{s-1}}$, we proceed to establish them for the $V_{k_1 \cdots k_s}$. From (2.7),

$$\begin{aligned} V_{12\cdots s-1k_s} = {} & (1-q_{s-1})[V_{12\cdots s-1k_s} + e^{i\theta(t)} V_{2\cdots s-1k_s}] * G(t) \\ & + p_{k_s}[V_{12\cdots s} + e^{i\theta(t)} V_{2\cdots s}] * G(t), \end{aligned}$$

whence, using the induction hypothesis,

$$\begin{aligned} V_{12\cdots s-1k_s} = {} & (1-q_{s-1}) V_{12\cdots s-1k_s} * G(t) \\ & + p_{k_s} \left[V_{12\cdots s} + p_s^{-1} \frac{Q_s}{Q_{s-1}} e^{i\theta(t)} V_{12\cdots s-1} \right] * G(t). \end{aligned}$$

Since $V_{12\cdots s}$ satisfies the same equation with $k_s = s$,

$$\frac{V_{12\cdots s-1k_s}}{p_{k_s}} - \frac{V_{12\cdots s}}{p_s} = (1-q_{s-1}) \left[\frac{V_{12\cdots s-1k_s}}{p_{k_s}} - \frac{V_{12\cdots s}}{p_s} \right] * G(t),$$

so that

$$\frac{V_{12\cdots s-1k_s}}{p_{k_s}} = \frac{V_{12\cdots s}}{p_s},$$

which establishes (2.12) for the $V_{12\cdots s-1k_s}$. Similarly one can deduce the required relations for the $V_{12\cdots s-2k_{s-1}k_s}$, and so on for all the $V_{k_1\cdots k_s}$. We shall omit the details.

Equations (2.2), (2.4), (2.11) and (2.12) provide the required evaluation of the characteristic functional.

3. The distribution of $N(x, t)$

Calculations of product-densities for the process based on the characteristic functional obtained above would clearly involve extensive algebra, and will not be attempted here [1]. However, to give some idea of the type of algebra that would be encountered, we shall outline the evaluation of the characteristic function (1.3) of $N(x, t)$, and deduce the age-distribution of the ith ranked individual at any time t. The explicit formulae for the latter were in fact first obtained from the characteristic functional, although more direct approaches are possible in two particular cases (Gani [3]).

[1] See the author's note in *J. Appl. Prob.* 1, No. 1 (1964).

To find the V's for the particular $\theta(u)$ defined by (1.2), it is convenient to introduce the functions

$$A_s(t) = \begin{cases} V_{12\cdots s}(t), & (0 \leqq t \leqq x), \\ 0, & \text{elsewhere}; \end{cases} \tag{3.1}$$

$$B_s(t) = \begin{cases} V_{12\cdots s}(t+x), & (t > 0), \\ 0, & (t \leqq 0). \end{cases} \tag{3.2}$$

From (2.11), we find that these functions satisfy the recurrence relations

$$A_s(t) = e^{i\phi} \frac{Q_s}{Q_{s-1}} A_{s-1} * \gamma_s(t), \qquad (0 \leqq t \leqq x), \tag{3.3}$$

$$B_s(t) = \frac{Q_s}{Q_{s-1}} \left[e^{i\phi} \int_t^{t+x} A_{s-1}(t+x-\tau)\, d\gamma_s(\tau) + B_{s-1} * \gamma_s(t) \right], \qquad (t > 0). \tag{3.4}$$

Solving, and using the readily-proved formula

$$\gamma_{r_1} * \cdots * \gamma_{r_s}(t) = \sum_{k=1}^{s} \frac{\gamma_{r_k}(t)}{\prod_{\substack{j=1 \\ j \neq k}}^{s} (q_{r_j} - q_{r_k})}, \qquad (0 \leqq r_1 < r_2 < \cdots < r_s \leqq n), \tag{3.5}$$

for convolutions of the γ's, it eventually follows that, for $s = 1, \cdots, n$,

$$A_s(t) = -e^{is\phi} Q_s \sum_{k=0}^{s} \frac{q_k \gamma_k(t)}{\prod_{\substack{j=0 \\ j \neq k}}^{s} (q_j - q_k)}, \tag{3.6}$$

and

$$\begin{aligned} B_s(t) = e^{i\phi} Q_s \Bigg\{ & \int_t^{t+x} \frac{A_{s-1}(t+x-\tau)}{Q_{s-1}}\, d\gamma_s(\tau) \\ & + \sum_{r=2}^{s} \sum_{k=r}^{s} \frac{1}{\prod_{\substack{j=r \\ j \neq k}}^{s} (q_j - q_k)} \int_0^t d\gamma_k(\tau) \int_{t-\tau}^{t+x-\tau} \frac{A_{r-2}(t+x-\tau-\sigma)}{Q_{r-2}}\, d\gamma_{r-1}(\sigma) \Bigg\} \\ & + Q_s \sum_{k=1}^{s} \frac{1}{\prod_{\substack{j=1 \\ j \neq k}}^{s} (q_j - q_k)} \int_0^t V_0(t+x-\tau)\, d\gamma_k(\tau). \end{aligned} \tag{3.7}$$

Hence, for $0 \leqq x \leqq t$,

$$V_{12\cdots s}(t) = -e^{is\phi} Q_s \sum_{k=0}^{s} \frac{q_k \gamma_k(t)}{\prod_{\substack{j=0 \\ j \neq k}}^{s} (q_j - q_k)}, \qquad (s = 1, \cdots, n), \tag{3.8}$$

while for $t > x$,

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$$V_{12\cdots s}(t) = e^{i\phi} Q_s \left\{ \int_{t-x}^{t} \frac{A_{s-1}(t-\tau)}{Q_{s-1}}\, d\gamma_s(\tau) \right.$$

$$(3.9) \qquad + \sum_{r=2}^{s} \sum_{k=r}^{s} \frac{1}{\prod\limits_{\substack{j=r\\ j\neq k}}^{s} (q_j - q_k)} \int_0^{t-x} d\gamma_k(\tau) \int_{t-x-\tau}^{t-\tau} \frac{A_{r-2}(t-\tau-\sigma)}{Q_{r-2}}\, d\gamma_{r-1}(\sigma) \Bigg\}$$

$$+ Q_s \sum_{k=1}^{s} \frac{1}{\prod\limits_{\substack{j=1\\ j\neq k}}^{s} (q_j - q_k)} \int_0^{t-x} V_0(t-\tau) d\gamma_k(\tau), \qquad (s = 1, \cdots, n),$$

remembering that V_0 is given by (2.5).

We may now calculate the characteristic function for $N(x, t)$. From (2.2) and (2.4) we have, for $0 < x < t$,

$$C(\phi, t) = V_0(t) + \sum_{(k)} V_{k_1 \cdots k_s}(t).$$

After some algebra, it follows from (2.12) that

$$(3.10) \qquad C(\phi, t) = \sum_{s=0}^{r} \left(\prod_{h=0}^{s-1} q_h \right) Q_s^{-1} V_{12\cdots s}(t), \qquad (0 < x < t).$$

Taking $x_0 = 0$, $x_{r+1} = +\infty$, we find similarly that for $t + x_m \leqq x < t + x_{m+1}$ $(m = 0, \cdots, n)$

$$C(\phi, t) = e^{im\phi} \left[V_0(t) + \sum_{s=1}^{n} \sum_{k_1 > m} V_{k_1 \cdots k_s}(t) \right.$$

$$\left. + \sum_{l=1}^{m} e^{-il\phi} \sum_{s=l}^{n+l-m} \sum_{k_1 \cdots k_l = 1}^{m} \sum_{k_{l+1} > m} V_{k_1 \cdots k_s}(t) \right],$$

which reduces to

$$(3.11) \qquad C(\phi, t) = \sum_{l=0}^{m} e^{il\phi} \sum_{s=m-l}^{n-l} \left(\prod_{h=m}^{l+s-1} q_h \right)$$

$$\cdot \left\{ Q_s^{-1} V_{12\cdots s}(t) \sum_{k_1, \cdots, k_{m-l}=1}^{m} \left[\prod_{r=1}^{m-l} (q_{k_r - 1} - q_{k_r + s - r}) \right] \right\}.$$

Substituting (3.9) in (3.10), it may be shown that if $0 < x < t$, then

$$C(\phi, t) = V_0(t) + \int_0^{t-x} V_0(t-\tau) d\gamma_n(\tau)$$

$$+ e^{i\phi} \left[\int_{t-x}^{t} V_0(t-\tau) d\gamma_1(\tau) + q_1 \int_0^{t-x} d\gamma_n(\tau) \int_{t-x-\tau}^{t-\tau} V_0(t-\tau-\sigma) d\gamma_1(\sigma) \right]$$

$$+ \sum_{s=2}^{n} \left(\prod_{h=0}^{s-1} q_h \right) \left[\int_{t-x}^{t} \frac{A_{s-1}(t-\tau)}{Q_{s-1}} d\gamma_s(\tau) \right.$$

$$\left. + q_s \int_0^{t-x} d\gamma_n(\tau) \int_{t-x-\tau}^{t-\tau} \frac{A_{s-1}(t-\tau-\sigma)}{Q_{s-1}}\, d\gamma_s(\sigma) \right],$$

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which from (3.5) and (3.6) reduces to

$$
(3.12) \quad \begin{aligned} C(\phi, t) &= V_0(t) + \int_0^{t-x} V_0(t-\tau) d\gamma_n(\tau) \\ &\quad - \sum_{s=1}^{n} e^{is\phi} \left(\prod_{h=0}^{s-1} q_h \right) \sum_{\kappa=0}^{s} \frac{q_k}{\prod_{\substack{j=0 \\ j \neq k}}^{s} (q_j - q_k)} \int_{t-x}^{t} \{1 - q_k \gamma_k(t-\tau)\} d\gamma_n(\tau). \end{aligned}
$$

On the other hand, if $t+x_m \leqq x < t+x_{m+1}$ $(m = 0, \cdots, n)$,

$$
(3.13) \quad C(\phi, t) = e^{im\phi} - \sum_{s=m}^{n} e^{is\phi} \left(\prod_{h=m}^{s-1} q_h \right) \sum_{k=m}^{s} \frac{q_k \gamma_k(t)}{\prod_{\substack{j=m \\ j \neq k}}^{s} (q_j - q_k)}.
$$

Hence, writing

$$
(3.14) \quad p_s(x, t) = Pr\{N(x, t) = s\},
$$

the probability that at time t there are s individuals with ages $\leqq x$, it follows from (3.12) and (3.13) that

$$
(3.15) \quad p_0(x, t) = \begin{cases} 1 - \int_{t-x}^{t} \{1 - q_0 \gamma_0(t-\tau)\} d\gamma_n(\tau), & (0 < x < t), \\ 1 - q_0 \gamma_0(t), & (t \leqq x < t+x_1), \\ 0, & (x \geqq t+x_1), \end{cases}
$$

while, for $s = 1, \cdots, n$,

$$
(3.16) \quad p_s(x, t) = \begin{cases} -\left(\prod_{h=0}^{s-1} q_h \right) \sum_{k=0}^{s} \frac{q_k}{\prod_{\substack{j=0 \\ j \neq k}}^{s} (q_j - q_k)} \int_{t-x}^{t} \{1 - q_k \gamma_k(t-\tau)\} d\gamma_n(\tau), & (0 < x < t), \\ -\left(\prod_{h=m}^{s-1} q_h \right) \sum_{k=m}^{s} \frac{q_k \gamma_k(t)}{\prod_{\substack{j=m \\ j \neq k}}^{s} (q_j - q_k)}, & (t+x_m \leqq x < t+x_{m+1}; \ m = 0, \cdots, s-1), \\ 1 - q_s \gamma_s(t), & (t+x_s \leqq x < t+x_{s+1}), \\ 0, & (x \geqq t+x_{s+1}), \end{cases}
$$

noting that $p_n(x, t) = 1$ for $x \geqq t+x_n$.

The age-distribution of the ith ranked individual at time t is given by

$$
(3.17) \quad \begin{aligned} f_i(x, t) &= Pr\,\{i\text{th ranked individual has age} \leqq x \text{ at time } t\} \\ &= \sum_{s=i}^{n} p_s(x, t). \end{aligned}
$$

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This distribution is found to be

$$(3.18)\qquad f_i(x,t) = \begin{cases} \displaystyle\sum_{k=0}^{i-1} \frac{q_k}{\prod_{\substack{j=0\\ j\neq k}}^{i-1}(1-q_k/q_j)} \int_{t-x}^{t} \{1-q_k\gamma_k(t-\tau)\}d\gamma_n(\tau), & (0 < x < t),\\ \displaystyle\sum_{k=m}^{i-1} \frac{q_k\gamma_k(t)}{\prod_{\substack{j=m\\ j\neq k}}^{i-1}(1-q_k/q_j)}, & (t+x_m \leqq x < t+x_{m+1};\ m = 0,\cdots, i-1),\\ 1, & (x \geqq t+x_i), \end{cases}$$

for $i = 1, \cdots, n$.

In particular, if $n = 1$ and $p_1 = 1$ we have a renewal process in which the initial component has age x at $t = 0$, and its residual useful life has distribution $G(t)$. Also, $\gamma_0(t) = G(t)$, while $\gamma_1(t) = \sum_{m=1}^{\infty} G^{m*}(t) = H(t)$ is the renewal function. From (3.18), the age-distribution of the article in use at time t is

$$(3.19)\qquad f_1(x,t) = \begin{cases} \int_{t-x}^{t} \{1-G(t-\tau)\}dH(t), & (0 < x < t),\\ G(t), & (t \leqq x < t+x_1),\\ 1, & (x \geqq t+x_1), \end{cases}$$

which agrees with Smith [4] equation (5.2).

4. The Poisson case

If the replacement process is Poisson, with $G(t) = 1-e^{-\lambda t}$ $(t \geqq 0)$, we readily find that if $k = 0, \cdots, n-1$,

$$(4.1)\qquad \begin{aligned} \gamma_k(t) &= q_k^{-1}(1-e^{-\lambda q_k t})\\ 1-q_k\gamma_k(t) &= e^{-\lambda q_k t}, \end{aligned}$$

for $t \geqq 0$, these functions vanishing for $t < 0$.

On the other hand, $\gamma_n(t) = \sum_{m=1}^{\infty} G^{m*}(t)$, the renewal function for the replacements, is given by

$$(4.2)\qquad \gamma_n(t) = \begin{cases} \lambda t, & t \geqq 0,\\ 0, & t < 0. \end{cases}$$

From (4.1) and (4.2), the system of probabilities (3.15) and (3.16) reduce to

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$$p_s(x,t) = \begin{cases} \left(\prod_{h=0}^{s-1} q_h\right) \sum_{k=0}^{s} \dfrac{e^{-\lambda q_k x}}{\prod_{\substack{j=0\\ j\neq k}}^{s} (q_j - q_k)}, & (0 < x < t), \\ \left(\prod_{h=m}^{s-1} q_h\right) \sum_{k=m}^{s} \dfrac{e^{-\lambda q_k t}}{\prod_{\substack{j=m\\ j\neq k}}^{s} (q_j - q_k)}, & (t+x_m \leqq x < t+x_{m+1};\ m = 0, \cdots, s), \\ 0, & (x \geqq t+x_{s+1}), \end{cases} \tag{4.3}$$

for $s = 0, \cdots, n$, with $p_n(x, t) = 1$ for $x \geqq t+x_n$.

The age distribution of the ith ranked individual at time t is found to be:

$$f_i(x,t) = \begin{cases} 1 - \sum_{k=0}^{i-1} \dfrac{e^{-\lambda q_k x}}{\prod_{\substack{j=0\\ j\neq k}}^{i-1} (1 - q_k/q_j)}, & (0 < x < t), \\ 1 - \sum_{k=m}^{i-1} \dfrac{e^{-\lambda q_k t}}{\prod_{\substack{j=m\\ j\neq k}}^{i-1} (1 - q_k/q_j)}, & (t+x_m \leqq x < t+x_{m+1};\ m = 0, \cdots, i-1), \\ 1, & (x \geqq t+x_i). \end{cases} \tag{4.4}$$

This result has also been obtained by Gani ([3], equation 1.10), using a more direct argument.

5. The limiting distributions

It is easily seen from (2.10) that the $\gamma_k(t)$ are non-decreasing functions of t. Furthermore, as $t \to \infty$,

$$\gamma_k(t) \to q_k^{-1}, \qquad (k = 0, \cdots, n-1), \tag{5.1}$$

while, by the elementary renewal theorem,

$$\gamma_n(t) \sim t/\mu, \tag{5.2}$$

where μ is the mean time between replacements.

Hence if we define

$$\Gamma_k(u) = \begin{cases} 1 - q_k \gamma_k(u), & (0 \leqq u < x) \\ 0, & \text{elsewhere}, \end{cases} \tag{5.3}$$

$\Gamma_k(u)$ is certainly of bounded variation in the finite interval $0 \leqq u < x$, and so, by Corollary (1.1) of Smith's paper [4],

$$\lim_{t\to\infty}\int_{t-x}^{t}\{1-q_k\gamma_k(t-\tau)\}d\gamma_n(\tau)$$

(5.4)
$$\begin{aligned}&= \lim_{t\to\infty}\Gamma_k * \gamma_n(t)\\ &= \mu^{-1}\int_0^\infty \Gamma_k(u)du\\ &= \mu^{-1}\int_0^x \{1-q_k\gamma_k(u)\}du.\end{aligned}$$

For convenience, we have assumed that $G(t)$ is a nonlattice distribution.

Hence, the limiting distribution of $N(x, t)$ is, from (3.15) and (3.16),

(5.5)
$$\begin{aligned}p_0(x, \infty) &= 1-\mu^{-1}\int_0^x \{1-q_0\gamma_0(u)\}du,\\ p_s(x, \infty) &= -\mu^{-1}\left(\prod_{h=0}^{s-1} q_h\right)\sum_{k=0}^{s}\frac{q_k}{\prod\limits_{\substack{j=0\\ j\neq k}}^{s}(q_j-q_k)}\int_0^x\{1-q_k\gamma_k(u)\}du,\end{aligned}$$
$$(s = 1, \cdots, n).$$

The limiting age-distribution of the ith ranked individual follows from (3.18):

(5.6)
$$f_i(x, \infty) = \mu^{-1}\sum_{k=0}^{i-1}\frac{q_k}{\prod\limits_{\substack{j=0\\ j\neq k}}^{i-1}(1-q_k/q_j)}\int_0^x\{1-q_k\gamma_k(u)\}du.$$

In the Poisson case, this becomes

(5.7)
$$f_i(x, \infty) = 1-\sum_{k=0}^{i-1}\frac{e^{-\lambda q_k x}}{\prod\limits_{\substack{j=0\\ j\neq k}}^{i-1}(1-q_k/q_j)}.$$

Finally, we shall relate the limiting distribution (5.7) to the result obtained by Gani and Yeo ([2], p. 59) for the stationary age-distribution $F_i(x)$ immediately after a regeneration point. Using equations (13) of [2] and (1.2) of [3], Mr. Yeo has shown that

(5.8)
$$f_i(x, \infty) = F_i(x) * (1-e^{-\lambda x}),$$

a relation which is intuitively obvious. Taking $F_i(x)$ in the form [1]

[1] The gamma-type terms appearing in Gani and Yeo's expression for $F_i(x)$ are redundant, as may be seen by substituting $\varphi(\theta) = \mu/(\mu+\theta)$ in their equation (14) for the Laplace transform of $F_i(x)$.

$$F_i(x) = 1 - \sum_{r=1}^{i-1} \frac{e^{-\lambda q_r x}}{\prod_{\substack{j=1 \\ j \neq r}}^{i-1} (1-q_r/q_j)}, \tag{5.9}$$

(noting that q_i has been redefined), we may readily deduce (5.7) from (5.8).

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