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# THE COLLINEATION GROUP OF THE VEBLEN-WEDDERBURN PLANE OF ORDER NINE

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**1. Introduction.** In this paper we prove that the order of the collineation group of the Veblen-Wedderburn plane of order nine is 311,040. This result was stated by Hall [**3**] in 1943 and proved by Pierce [**9**] in 1964. Hall assumed that there were $10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 3840$ collineations which permute points on the ideal line $L$ and 81 collineations which leave $L$ pointwise fixed. In 1955 André [**1**] verified this assumption. When it was realized that a harmonic homology with axis $L$ had been overlooked, the number of central collineations with axis $L$ doubled and hence the order of the collineation group became $3840 \cdot 162 = 622{,}080$. This latter figure has been assumed to be correct as recently as 1965 ([**6**]).

Here it is proved that there are 1920 collineations which move points on $L$ and 162 collineations which leave $L$ pointwise fixed, thus giving the figure 311,040. Pierce's proof of this fact is established from a different viewpoint.

**2. The Veblen-Wedderburn plane of order nine.** We may represent the Veblen-Wedderburn plane of order nine as follows:

The *points* are of three types: $[x, y, 1]$, $[1, x, 0]$, and $[0, 1, 0]$ ,where $x$ and $y$ are elements of the nearfield $N = (R, +, \cdot)$ of order 9.

Similarly, *lines* are of three types: $\langle m, 1, k\rangle$, $\langle 1, 0, k\rangle$, and $\langle 0, 0, 1\rangle$, where $m, k \in R$. The ideal line $L = \langle 0, 0, 1\rangle$.

*Incidence* is defined by: $[x, y, z] \in \langle m, n, k\rangle$ if and only if $xm + yn + zk = 0$.

The nearfield $N$ is the system $R$ of Hall [**3**, p. 273]. We shall use Hall's notation here. It should be noted that $N$ satisfies the usual properties of a finite nearfield and one important additional property:

$$x^2 = -1 \text{ for all } x \in R \text{ such that } x \neq 0, 1, -1.$$

We shall denote the plane above by $\Pi$, the intersection of two lines $\langle m, n, k\rangle$ and $\langle m', n', k'\rangle$ by $\langle m, n, k\rangle \cap \langle m', n', k'\rangle$, and the line joining the points $[x, y, z]$ and $[x', y', z']$ will be denoted by $[x, y, z] \cdot [x', y', z']$.

**3. Collineations on $\Pi$.** We may define a collineation on a projective plane as a pair of functions $(f, F)$, where $f$ is a one-to-one correspondence from the set of points onto itself and $F$ is a one-to-one correspondence from the set of lines onto itself such that $p \in L$ if and only if $f(p) \in F(L)$ for any point $p$ and line $L$.

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We describe five types of collineations on $\Pi$ below by stating the correspondences for non-invariant elements:

$$
\begin{aligned}
(1)\quad & f_{s,t}\colon [x, y, 1] \to [x+s, y+t, 1]\colon s, t \in R,\\
& F_{s,t}\colon \langle m, 1, k\rangle \to \langle m, 1, k - sm - t\rangle\\
& \qquad \langle 1, 0, k\rangle \to \langle 1, 0, k - s\rangle;
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad & g\colon [x, y, 1] \to [-x, -y, 1],\\
& G\colon \langle m, 1, k\rangle \to \langle m, 1, -k\rangle\\
& \qquad \langle 1, 0, k\rangle \to \langle 1, 0, -k\rangle;
\end{aligned}
$$

$$
\begin{aligned}
(3)\quad & h_{s,t}\colon [x, y, 1] \to [xs, ty, 1]\colon s, t \in R,\ s, t \neq 0.\\
& \qquad [1, x, 0] \to [1, s^{-1}xt, 0],\\
& \mathrm{H}_{s,t}\colon \langle m, 1, k\rangle \to \langle s^{-1}mt, 1, kt\rangle\\
& \qquad \langle 1, 0, k\rangle \to \langle 1, 0, ks\rangle;
\end{aligned}
$$

$$
\begin{aligned}
(4)\quad & j\colon [x, y, 1] \to [x+y, -x+y, 1]\\
& \qquad [0, 1, 0] \to [1, 1, 0]\\
& \qquad [1, 1, 0] \to [1, 0, 0]\\
& \qquad [1, 0, 0] \to [1, -1, 0]\\
& \qquad [1, -1, 0] \to [0, 1, 0],\\
& J\colon \langle m, 1, k\rangle \to \langle m, 1, km + k\rangle, \quad m \neq 0, \pm 1,\\
& \qquad \langle 1, 0, k\rangle \to \langle -1, 1, k\rangle\\
& \qquad \langle -1, 1, k\rangle \to \langle 0, 1, k\rangle\\
& \qquad \langle 0, 1, k\rangle \to \langle 1, 1, k\rangle\\
& \qquad \langle 1, 1, k\rangle \to \langle 1, 0, k\rangle;
\end{aligned}
$$

(5) $r_{s,t}\colon [x, y, z] \to [\alpha_{s,t}(x), \alpha_{s,t}(y), \alpha_{s,t}(z)]\colon s = 0, \pm 1, t = \pm 1.$
$R_{s,t}\colon \langle m, n, k\rangle \to \langle \alpha_{s,t}(m), \alpha_{s,t}(n), \alpha_{s,t}(k)\rangle$, where $\alpha_{s,t}\colon R \to R$ is defined as follows: $\alpha_{s,t}\colon x + ya \to (x + sy) + (ty)a$. These mappings constitute the six automorphisms on $N$ (see Hughes [**5**] or André [**1**]).

For simplicity we shall denote these collineations by $f_{s,t}$, $g$, $h_{s,t}$, $j$, and $r_{s,t}$, respectively.

It should be noted that the collineations above leave the ideal line fixed. Hall [**3**; **4**] showed that this must be true for every collineation on $\Pi$. In fact, this is a property that the collineations on any plane defined over a nearfield must share.

It is also true of the five collineations above that each fixes [0, 1, 0] if and only if it fixes [1, 0, 0]. This too is true for general non-Desarguesian planes defined over nearfields. Following the notation, definitions, and theorems of Dembowski [**2**, pp. 123, 129, 130] we may prove this fact as follows.

THEOREM 3.1. *If $f$ is a collineation on* $\Pi$ *such that $f$ fixes* [0, 1, 0], *then $f$ must fix* [1, 0, 0].

*Proof.* Using Dembowski's notation we have $v = [0, 1, 0]$, $u = [1, 0, 0]$, and $o = [0, 0, 1]$. Suppose that there exists a collineation $f$ such that

$$f: u, v \to p, v,$$

where $p \neq u$. Now $\Pi$ is $(u, v)$-transitive, and so it follows that it must be $(p, v)$-transitive. Thus, in particular, $\Pi$ is $(u, ov)$- and $(p, ov)$-transitive. It follows [**2**, p. 123, theorem 18] that $\Pi$ is $(pu, ov)$-transitive and since $p \in uv$ we obtain that $\Pi$ is $(v, ov)$-transitive. But $\Pi$ is also $(v, uv)$-transitive and so (by [**2**, p. 123, theorem 18]) $\Pi$ must be $(v, v)$-transitive. Thus (by [**2**, p. 130, theorem 22 (f)]) $N$ must be semifield. But $N$ does not satisfy the law of left distributivity and so we have a contradiction. Therefore such a collineation $f$ does not exist.

Labeling the points on $L$ as follows: $u = [1, 0, 0]$, $m = [1, 1, 0]$, $n = [1, a, 0]$, $q = [1, b, 0]$, $r = [1, c, 0]$, where $b = 1 + a$, and $c = -1 + a$, we may define a unary operation "$-$" on $L$ by $-u = [0, 1, 0]$, $-m = [1, -1, 0]$, $-n = [1, -a, 0]$, $-q = [1, -b, 0]$, $-r = [1, -c, 0]$, and $-(-p) = p$ for $p \in L$. Then Theorem 3.1 implies the following theorem.

THEOREM 3.2. *If $f$ is a collineation on* $\Pi$, *then $f: p \to p'$ if and only if $-p \to -p'$.*

*Proof.* If $f = h_{s,t}$ or $j$, it is easily checked that $f: p \to p'$ if and only if $f: -p \to -p'$. Thus compositions of $h_{s,t}$ and $j$ satisfy this property. Now suppose that there exists a collineation $g': p \to p'$ and $-p \to x \neq -p'$. Then let $h: p \to v$ and $h': p' \to v$, where $h$ and $h'$ are compositions of the mappings $h_{s,t}$ and $j$. It is easily seen that such mappings exist. Now $h' \circ g' \circ h^{-1}$ fixes $v$ and maps $u \to h(x) \neq u$. This contradicts Theorem 3.1. Therefore such a collineation $g'$ does not exist.

## 4. The collineation group of $\Pi$.

Since all collineations on $\Pi$ fix line $L$ we may divide our study into two parts: those collineations which fix $L$ pointwise and those which do not. We begin by showing that there are exactly 162 central collineations with axis $L$.

LEMMA 4.1. *If $f$ is a homology with centre* $[0, 0, 1]$ *and axis $L$ and* $f: [1, 0, 1] \to [t, 0, 1]$, *then* $f: [x, y, 1] \to [tx, ty, 1]$.

*Proof.* Since

$$f: [1, 0, 1] \to [t, 0, 1]$$
$$[0, 1, 0] \to [0, 1, 0]$$

we have $f: [1, 0, 1] \cdot [0, 1, 0] = \langle 1, 0, -1\rangle \to [t, 0, 1] \cdot [0, 1, 0] = \langle 1, 0, -t\rangle$. Also $f$ fixes $\langle y, 1, 0\rangle$ for any $y$, and so

$$\langle 1, 0, -1\rangle \cap \langle -y, 1, 0\rangle = [1, y, 1] \to \langle 1, 0, -t\rangle \cap \langle -y, 1, 0\rangle = [t, ty, 1].$$

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It follows that:

$$[1, y, 1] \cdot [1, 0, 0] = \langle 0, 1, -y\rangle \rightarrow [t, ty, 1] \cdot [1, 0, 0] = \langle 0, 1, -ty\rangle.$$

Hence

$$\langle 0, 1, -y\rangle \cap \langle -(x^{-1})y, 1, 0\rangle = [x, y, 1] \rightarrow \langle 0, 1, -ty\rangle \cap \langle -(x^{-1})y, 1, 0\rangle = [tx, ty, 1].$$

Notice that this lemma applies to any plane coordinatized by a nearfield. The next lemma, as it is proved here, applies to the specific nearfield, $N$.

LEMMA 4.2. *If $f$ is a homology with centre* $[0, 0, 1]$ *and axis $L$ and* $F$: $[1, 0, 1] \rightarrow [t, 0, 1]$, *then* $t = \pm 1$.

*Proof.* Suppose that $t \neq \pm 1$. First we notice that because of Lemma 4.1,

$$f: [t, t+1, 1] \rightarrow [t^2, t(t+1), 1] = [-1, -t+1, 1].$$

This last equality follows because $t^2 = -1$ since $t \neq \pm 1$ and

$$t(t+1) = -(t+1)t = -(t^2+t) = -(-1+t) = -t+1.$$

Also $f$: $[0, t-1, 1] \rightarrow [0, t(t-1), 1] = [0, t+1, 1]$. This equality is established in the same way as the one above.

Thus:

$$[t, t+1, 1] \cdot [0, t-1, 1] = \langle -t, 1, -t+1\rangle \rightarrow [-1, -t+1, 1] \cdot [0, t+1, 1] = \langle t, 1, -t-1\rangle.$$

The first equality is true since $t(-t) + t + 1 - t + 1 = 1 + t + 1 - t + 1 = 0$. The second equality follows similarly. Finally we have

$$\langle -t, 1, -t+1\rangle \cap \langle 0, 0, 1\rangle = [-t, 1, 0] \rightarrow \langle t, 1, -t-1\rangle \cap \langle 0, 0, 1\rangle = [t, 1, 0].$$

But $[-t, 1, 0]$ must be held fixed by $f$, and so we have a contradiction. Thus $t = \pm 1$.

THEOREM 4.3. *There are* 162 *central collineations with axis L. The set of central collineations is*

$$\{f_{s,t}: s,t \in R\} \cup \{f_{s,t} \circ g \circ f_{s,t}^{-1}: s, t \in R\}.$$

*Proof.* The proof is straightforward but we shall include it for completeness.

Since an elation with a given axis is uniquely determined by a point off the axis and its image, there are no elations with axis $L$ other than those of the form $f_{s,t}$.

Similarly, a homology with a given centre and axis is determined by a point off the axis distinct from the centre and its image. Thus it follows from Lemma 4.2 that the only homologies with centre [0, 0, 1] are the identity and $g$. Now collineations of the form $f_{s,t} \circ g \circ f_{s,t}^{-1}$ are easily shown to be homologies with centre [$s$, $t$, 1]. Any other homology, $h$, with centre [$s$, $t$, 1], would yield a homology $f_{s,t}^{-1} \circ h \circ f_{s,t}$ with centre [0, 0, 1]. Since this collineation must be the identity, so must $h$ be the identity. Thus every homology is of the form $f_{s,t} \circ g \circ f_{s,t}^{-1}$.

Now we show that there are 1920 collineations which move points on $L$. $u$, $m$, $n$, $q$, $r$, ... are as previously defined.

THEOREM 4.4. *If a collineation $f$ on $\Pi$ fixes $\pm p$ for $p = m, n, q, r$, then $f$ fixes $\pm u$ (i.e., $u$ and $v$).*

*Proof*. Suppose that $f$: $[1, x, 0] \to [1, x, 0]$, $x = \pm 1, \pm a, \pm b, \pm c$, and also $f$: $u \to v$. Now $f$: $[0, 0, 1] \to [s, t, 1]$ for some $s$, $t$. Thus, letting $h = f_{s,t}^{-1} \circ f$, we have $h$: $o \to o$, $p \to p$ for $p = m, n, q, r$ and $u \to v$. Let $y$ be a fixed non-zero element of $R$. Now

$$h: \langle 1, 0, 0\rangle = [0, 1, 0] \cdot [0, 0, 1] \to [1, 0, 0] \cdot [0, 0, 1] = \langle 0, 1, 0\rangle,$$

and so we have $h$: $[0, y, 1] \to [x(y), 0, 1]$. We shall denote $x(y)$ by $x$. Notice that $x \neq 0$. Now $h$: $\langle 0, 1, -y\rangle = [1, 0, 0] \cdot [0, y, 1] \to [0, 1, 0] \cdot [x, 0, 1] = \langle 1, 0, -x\rangle$. Also $h$: $\langle t, 1, 0\rangle \to \langle t, 1, 0\rangle$ because $h$ holds fixed [0, 0, 1] and $[1, t^{-1}, 0]$. Thus

$$[x, y, 1] = \langle 0, 1, -y\rangle \cap \langle -x^{-1}y, 1, 0\rangle \to \langle 1, 0, -x\rangle \cap \langle -x^{-1}y, 1, 0\rangle = [x, y, 1];$$

hence $[x, y, 1]$ is held fixed by $h$.

Let $z \in R$ such that $z \neq 0, yx^{-1}$. Now $\langle -z, 1, xz - y\rangle = [x, y, 1] \cdot [1, z, 0]$ is held fixed; hence

$$\begin{aligned}[0, y - xz, 1] &= \langle -z, 1, xz - y\rangle \cap \langle 1, 0, 0\rangle \to \langle -z, 1, xz - y\rangle \cap \langle 0, 1, 0\rangle \\ &= [(xz - y)z^{-1}, 0, 1].\end{aligned}$$

By the argument above, $h$ fixes $[(xz - y)z^{-1}, y - xz, 1]$. We shall denote this point by $p_z$.

Let $L_z = \langle z(xz - y)^{-1}xz, 1, -y\rangle = [0, y, 1] \cdot p_z$. Now

$$L_z \cap uv = [1, -z(xz - y)^{-1}xz, 0]$$

and this is a fixed point since $-z(xz - y)^{-1}xz \neq 0$. Hence $L_z$ is a fixed line since it contains two fixed points ($o$ and the one above). Since

$$h: [0, y, 1] \to [x, 0, 1],$$

it follows that $[x, 0, 1] \in L_z$. However, this is not necessarily the case as the following example shows: suppose that $y \neq \pm 1$ and let $z = x^{-1}$; then $L_z = \langle x^{-1}(1 - y)^{-1}, 1, -y\rangle$. Now if $[x, 0, 1] \in L_z$, we would have

$$xx^{-1}(1 - y)^{-1} - y = (1 - y)^{-1} - y = (y - 1) - y = -1 = 0,$$

a contradiction. Thus $h$ and, therefore, $f$ do not exist.

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THEOREM 4.5. *There exists a collineation $f$ on $\Pi$ which maps $v \to p$ for any $p \in uv$.*

*Proof.* We may map $v \to m$ by $j$ and $v \to -v$ by $j^2$. Also we may map $m \to p = [1, x, 0]$ by $h_{1,x}$ for any $x = \pm 1, \pm a, \pm b, \pm c$. Thus with compositions of the mappings $j$ and $h_{1,x}$, we may map $v \to p$ for any $p \in uv$.

THEOREM 4.6. *There exists a collineation $f$ on $\Pi$, such that $f$ fixes $u$ and $v$ and $f$: $m \to p$ for any $p \neq u, v$.*

*Proof.* As mentioned in Theorem 4.5,

$$h_{1,x}: m \to [1, x, 0] \text{ for } x = \pm 1, \pm a, \pm b, \pm c.$$

Also $h_{1,x}$: $u, v \to u, v$.

THEOREM 4.7. *There exists a collineation $f$ on $\Pi$ such that $f$ fixes $u$, $v$, $m$, $-m$, and $f$: $n \to p$ for $p \neq u, v, \pm m$.*

*Proof.* It may be easily observed that $h_{b,b}$, $r_{1,1}$, and $h_{a,a}$ hold $u$ and $m$ fixed and $h_{b,b}$: $n \to -n$, $r_{1,1}$: $n \to q$, $r_{1,1}{}^2$: $n \to r$, $h_{a,a} \circ r_{1,1}$: $n \to -q$, $h_{a,a} \circ r_{1,1}{}^2$:$n \to -r$. The identity maps $n \to n$.

THEOREM 4.8. *There exists a collineation $f$ on $\Pi$ such that $f$ fixes $u$, $v$, $\pm m$, $\pm n$ and $f$: $q \to p$, where $p = \pm q, \pm r$.*

*Proof.* Notice that $r_{0,-1}$, $h_{a,a}$, and $h_{b,b}$ hold $u$ and $m$ fixed. Now

$$r_{0,-1}: n \to -n,\ -n \to n,\ q \to -r;$$

hence $h_{b,b} \circ r_{0,-1}$: $n \to n$, $q \to r$. Also $h_{a,a}$: $n \to n$, $q \to -q$, and so $h_{a,a} \circ h_{b,b} \circ r_{0,-1}$: $n \to n$, $q \to -r$. The identity maps $q \to q$.

THEOREM 4.9. *There exist 1920 collineations which permute points on $uv$.*

*Proof.* There exist 10 collineations which map $u \to p$ where $p \in uv$; 8 collineations which fix $\pm u$ and map $m \to p$, $p \neq \pm u$; 6 collineations which fix $\pm u$, $\pm m$, and map $n \to p$, $p \neq \pm u$, $\pm m$; and 4 collineations which fix $\pm u$, $\pm m$, $\pm n$ and map $q \to p$ where $p = \pm q, \pm r$. If $f$ fixes $\pm u$, $\pm m$, $\pm n$, and $\pm q$, it follows easily from Theorem 4.4 that $f$ fixes $\pm r$ and hence is the identity. Thus there are $10 \cdot 8 \cdot 6 \cdot 4$ in all.

We conclude by noting that the order of the group $G$ of collineations on $\Pi$ is $162 \cdot 1920 = 311{,}040$. This follows because the set of central collineations with axis $L$ is a normal subgroup of $G$.

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