Canad. Math. Bull. Vol. 65 (4), 2022, pp. 976–993
http://dx.doi.org/10.4153/S0008439522000054

Stieltjes interlacing of the zeros of j_n

William Frendreiss, Jennifer Gao, Austin Lei, Amy Woodall, Hui Xue, and Daozhou Zhu

Abstract. Let j_n be the modular function obtained by applying the nth Hecke operator on the classical j-invariant. For $n > m \geq 2$, we prove that between any two zeros of j_m on the unit circle of the fundamental domain, there is a zero of j_n.

1 Introduction

Let $j(z)$ be the classical elliptic modular invariant that can be defined as

$$j(z) = 1728 \frac{E_4^3(z)}{E_4^3(z) - E_6^2(z)},$$

where z is in the upper half plane, and $E_k(z)$ denotes the normalized Eisenstein series of weight k for the modular group $\mathrm{SL}_2(\mathbb{Z})$. It is well known that $j(z)$ is holomorphic on the upper half plane with a simple pole at infinity and is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$. For each $n \geq 1$ recall that the nth Hecke operator T_n of weight 0 acts on a modular function $f(z)$ through the formula

$$T_n(f)(z) := n^{-1} \sum_{\substack{ad=n,\\ 0 \leq b \leq d-1}} f\left(\frac{az+b}{d}\right).$$

We define

$$j_n(z) := nT_n(j(z) - 744).$$

See [1, 3] for more details and applications of the modular functions $j_n(z)$. One interesting aspect of $j_n(z)$ is that each function can be written as a monic polynomial in $j(z)$ of degree n, denoted φ_n. That is,

$$j_n(z) = \varphi_n(j(z)).$$

The first three φ_n are

$$\begin{aligned}
\varphi_1(j) &= j - 744,\\
\varphi_2(j) &= j^2 - 1488j + 159768,\\
\varphi_3(j) &= j^3 - 2232j^2 + 1069956j - 36866976.
\end{aligned}$$

Received by the editors August 3, 2021; revised December 21, 2021; accepted January 6, 2022.
Published online on Cambridge Core January 13, 2022.
AMS subject classification: **11F11**, 11F03.
Keywords: j-invariant, zeros of j_n, Stieltjes interlacing.

The zeros of modular functions in the fundamental domain of $SL_2(\mathbb{Z})$ have been studied by various authors. Rankin and Swinnerton-Dyer proved in [13] that all the zeros of Eisenstein series E_k in the fundamental domain of $SL_2(\mathbb{Z})$ lie on the unit circle. By extending the work of [13] Nozaki [11] showed that the zeros of E_k and E_{k+12} interlace on the unit circle, which was first observed by Gekeler [6]. Griffin et al. [8] further refined the method of [11] to present a necessary and sufficient condition for the zeros of E_k and E_{k+a} to interlace.

In a different scenario, Asai et al. [1] showed that all the zeros of the modular function $j_n(z)$ in the fundamental domain are simple and lie on the unit circle. By improving their work, Jermann [10] proved that for $n \geq 1$ the zeros of $j_n(z)$ and $j_{n+1}(z)$ interlace on the unit circle of the fundamental domain.

On the other hand, inspired by the analogy between the zeros of Eisenstein series and orthogonal polynomials, the authors [5] proved the following Stieltjes interlacing property between the zeros of Eisenstein series.

Theorem 1.1 ([5, Theorem 1.2]) *Let $n > m \geq 24$ and $m \neq 26$, then between every two zeros of $E_m(z)$ on the unit circle of the fundamental domain there is a zero of $E_n(z)$.*

This interlacing property was first observed by Stieltjes [14, Theorem 3.3.3] for the zeros of orthogonal polynomials and has received continuous attention in the area of numerical analysis and approximation theory, see for example [2, 4, 7]. Our goal is to investigate the Stieltjes interlacing property in the context of modular functions. With this goal and Theorem 1.1 in mind, it is natural to ask whether the Stieltjes interlacing property holds for the zeros of j_n. In this paper, we will adopt the method and results from [5] to show that the Stieltjes interlacing property also holds in this situation. More precisely, we will prove the following result.

Theorem 1.2 *Let $n > m \geq 2$ be positive integers. Then on the unit circle in the fundamental domain, between every two zeros of $j_m(z)$, there exists a zero of $j_n(z)$.*

Let us briefly describe the basic idea used in the works of [1, 10]. Suppose that $0 \leq x \leq 1/2$, and let $z = x + iy$ be over the unit circle between i and $e^{\pi i/3}$. Both works were based on studying the function

$$F_n(x) := j_n\left(x + i\sqrt{1-x^2}\right) e^{-2\pi n\sqrt{1-x^2}} = j_n(z)e^{-2\pi n y},$$

which is a real function over this interval. Clearly, $F_n(x) = 0$ if and only if $j_n(z) = 0$. Furthermore, the function $F_n(x)$ is closely approximated by $2\cos(2\pi n x)$. A nice bound on the remainder ([1, Key Lemma] and [10, Lemma 2.1])

$$T_n(x) := F_n(x) - 2\cos(2\pi n x),$$

then enabled the authors of both papers to prove the desired results on the location of zeros of F_n or j_n. This idea in a certain sense implies that if a separation property holds for the zeros of $2\cos(2\pi n x)$, then the same property will likely hold for the zeros of F_n. In this aspect, the following result serves as the starting point as well as a motivation for Theorem 1.2.

Proposition 1.3 *Let $n > m \geq 2$ be positive integers. Then on the interval $[0, \frac{1}{2}]$, between any two zeros of $2\cos(2\pi mx)$, there exists a zero of $2\cos(2\pi nx)$.*

We will prove this proposition after some preparation in Section 2.

We give an outline of the paper. This paper follows a similar structure to our previous paper [5], and many lemmas and propositions have similar proofs. In Section 2, we give some preliminary definitions and lemmas. In particular, we will prove Proposition 1.3, which serves as a prototype of the proof for Theorem 1.2 and the Stieltjes interlacing property in general. We then split the proof of Theorem 1.2 into multiple cases. In Section 3, we consider the easier case when $\frac{n}{m}$ is relatively large. When $\frac{n}{m}$ is relatively small, we follow the method in [10] and divide the interval $[0, \frac{1}{2}]$ into two subintervals; we consider the zeros on the subinterval $\left[0, \frac{1}{2} - \frac{\log m}{5m}\right]$ in Section 4 and zeros on $\left[\frac{1}{2} - \frac{\log m}{2m}, \frac{1}{2}\right]$ in Section 5. In Section 6, we combine all of the cases to complete the proof of Theorem 1.2 and discuss an application on the polynomials φ_n.

2 Notation and preliminary lemmas

In this section, we will set up some necessary notation and prove some lemmas for later applications.

Recall from Section 1 that for $z = x + iy$ on the unit circle between i and $e^{\pi i/3}$ and $x \in [0, \frac{1}{2}]$,

$$F_m(x) = j_m(z)e^{-2\pi my}$$

and

$$T_m(x) = F_m(x) - 2\cos(2\pi mx).$$

Definition 2.1 ([10, (12)]) For $0 \leq k \leq m-1$, let $x_{m,k}$ be the kth zero of $\cos(2\pi mx)$ on $[0, \frac{1}{2}]$, counting from $\frac{1}{2}$. More precisely,

$$x_{m,k} = \frac{1}{2} - \frac{k}{2m} - \frac{1}{4m} = \frac{1}{2} - \frac{2k+1}{4m}.$$

Definition 2.2 [10] For $0 \leq k \leq m-1$, let $u_{m,k}$ be the real part of the kth zero of $j_m(z)$ on the unit circle between i and $e^{\pi i/3}$, counting from $e^{\pi i/3}$. Equivalently, $u_{m,k}$ is the kth zero of $F_m(x)$ on $[0, \frac{1}{2}]$, counting from $\frac{1}{2}$. Thus, $u_{m,k} < u_{m,k-1}$ for all $1 \leq k \leq m-1$. Throughout the paper, we simply refer to $u_{m,k}$ as the zeros of $j_m(z)$.

Remark We typically use k for an index for the zeros of $j_m(z)$ and ℓ as an index for the zeros of $j_n(z)$.

A good upper bound for the distance between $x_{m,k}$ and $u_{m,k}$ plays a fundamental role in studying the location of zeros of j_m. Jermann developed the following upper bound in [10].

Lemma 2.3 ([10, Lemma 2.3]) *Let $m \geq 4$. Then*

$$|u_{m,k} - x_{m,k}| < \frac{1}{11m}.$$

Borrowing an idea from [8, Lemma 3.5], we will present a stronger upper bound for the distance $|u_{m,k} - x_{m,k}|$.

Note that [10, Corollary 2.2] gives the bound

$$|T_m(x)| \leq e^{-2\pi m(\frac{1}{2}-x)} + e^{-\frac{\sqrt{3}}{2}\pi m} m < 1.1.$$

The improved upper bound for $|u_{m,k} - x_{m,k}|$ is stated as follows.

Lemma 2.4 *Let $\delta_{m,k} = \frac{1}{2} - \frac{k}{2m} = x_{m,k} + \frac{1}{4m}$ and*

$$\gamma_{m,k} = \frac{1}{11.919m}\left(e^{-2\pi m(\frac{1}{2}-\delta_{m,k})} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right) = \frac{1}{11.919m}\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right).$$

Then for $m \geq 1$ and $k \geq 0$,

$$|u_{m,k} - x_{m,k}| < \gamma_{m,k} = \frac{1}{11.919m}\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right).$$

As a corollary, we have the following slight improvement of Lemma 2.3:

$$|u_{m,k} - x_{m,k}| \leq \gamma_{m,0} < \frac{1}{11.139m}.$$

Proof We have computationally verified both upper bounds are valid for $m < 4$. For $m \geq 4$, we follow the methods of [10, Lemma 2.3] and [8, Lemma 3.5]. First note that

$$\begin{aligned} F_m(x_{m,k} \pm \gamma_{m,k}) &= 2\cos(2\pi m(x_{m,k} \pm \gamma_{m,k})) + T_m(x_{m,k} \pm \gamma_{m,k}) \\ &= \pm(-1)^{m+k} 2\sin(2\pi m\gamma_{m,k}) + T_n(x_{m,k} \pm \gamma_{m,k}). \end{aligned}$$

On the interval $|x - x_{m,k}| \leq \frac{1}{4m}$, one has $|T_m(x)| \leq e^{-2\pi m(\frac{1}{2}-\delta_{m,k})} + e^{-\frac{\sqrt{3}}{2}\pi m} m$, according to [10, Corollary 2.2]. Also,

$$0 < 2\pi m|\gamma_{m,k}| < \frac{\pi}{6}(1.07),$$

because for $m \geq 1$, one has $e^0 + e^{-\frac{\sqrt{3}}{2}\pi m} m < 1.07$.

Using the fact that $\sin(\theta) > \frac{\sin(\frac{\pi}{6}(1.07))\theta}{(\frac{\pi}{6}(1.07))}$ when $0 < \theta < \pi(1.07)/6$, we get that

$$\begin{aligned} |2\sin(2\pi m\gamma_{m,k})| &> \frac{24\sin\left(\frac{\pi(1.07)}{6}\right)}{1.07} m\gamma_{m,k} > 11.919 m\gamma_{m,k} \\ &= e^{-2\pi m(\frac{1}{2}-\delta_{m,k})} + e^{-\frac{\sqrt{3}}{2}\pi m} m \geq |T_m(x_{m,k} \pm \gamma_{m,k})|. \end{aligned}$$

This implies that $F_m(x_{m,k} + \gamma_{m,k})F_m(x_{m,k} - \gamma_{m,k}) < 0$, and thus we have the desired bound. ∎

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

In the next result, we derive a condition on the values of m, n, k, and ℓ in order for the Stieltjes interlacing to happen.

Lemma 2.5 *Let $n > m \geq 2$, $0 \leq k \leq m-1$, and $0 \leq \ell \leq n-1$. Then $x_{m,k} > x_{n,\ell} > x_{m,k+1}$ if and only if*

$$\frac{2\ell+1}{2k+3} < \frac{n}{m} < \frac{2\ell+1}{2k+1}.$$

Proof By Definition 2.1,

$$\frac{1}{2} - \frac{2k+1}{4m} > \frac{1}{2} - \frac{2\ell+1}{4n} > \frac{1}{2} - \frac{2k+3}{4m}$$

happens if and only if

$$\frac{2\ell+1}{2k+3} < \frac{n}{m} < \frac{2\ell+1}{2k+1},$$

as desired. ∎

Corollary 2.6 *If $n > m$ and $x_{m,k} > x_{n,\ell} > x_{m,k+1}$, then $\ell > k$.*

Proof Suppose on the contrary that $\ell \leq k$. Then by Lemma 2.5,

$$\frac{n}{m} < \frac{2\ell+1}{2k+1} \leq 1,$$

contradicting the assumption that $n > m$. Thus, $\ell > k$, as desired. ∎

To show the Stieltjes interlacing property, special attention is needed for the behavior of the zeros of F_n (or $2\cos(2\pi nx)$) closest to the endpoints 0 or $\frac{1}{2}$.

Lemma 2.7 compares the zeros of $2\cos(2\pi nx)$ and $2\cos(2\pi mx)$ closest to $\frac{1}{2}$.

Lemma 2.7 *If $n > m \geq 2$, then $x_{n,0} > x_{m,0}$.*

Proof By Definition 2.1,

$$x_{n,0} - x_{m,0} = \left(\frac{1}{2} - \frac{1}{4n}\right) - \left(\frac{1}{2} - \frac{1}{4m}\right) = \frac{1}{4m} - \frac{1}{4n} > 0,$$

which completes the proof. ∎

The next lemma compares the zeros of F_n and F_m closest to $\frac{1}{2}$. It is a direct corollary of the interlacing property in [10, Theorem 3.1].

Lemma 2.8 ([10, Theorem 3.1]) *If $n > m \geq 2$, then $u_{n,0} > u_{m,0}$.*

Lemma 2.9 compares the zeros of $2\cos(2\pi nx)$ and $2\cos(2\pi mx)$ closest to 0.

Lemma 2.9 *If $n > m \geq 2$, then $x_{m,m-1} > x_{n,n-1}$.*

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

Proof By Definition 2.1, since $n > m$,

$$x_{m,m-1} - x_{n,n-1} = \frac{2(n-1)+1}{4n} - \frac{2(m-1)+1}{4m} = \frac{1}{4m} - \frac{1}{4n} > 0,$$

hence the claim. ∎

The next lemma compares the zeros of F_n and F_m closest to 0. It is again a corollary of the interlacing property in [10, Theorem 3.1].

Lemma 2.10 ([10, Theorem 3.1]) *If $n > m \geq 2$, then $u_{m,m-1} > u_{n,n-1}$.*

We now prove Proposition 1.3. This will illustrate how Lemmas 2.7–2.10 are applied to show Stieltjes interlacing. Similar to the proof of [5, Proposition 1.5], the main part of the proof is simple, but extra care must be taken to ensure there are no problems with the zeros closest to 0 and $\frac{1}{2}$. Similar work will be carried out in later sections as well.

Proof of Proposition 1.3 It suffices to prove the proposition for consecutive zeros of $2\cos(2\pi mx)$. By Definition 2.1, the distance between two consecutive zeros of $2\cos(2\pi mx)$ is $\frac{1}{2m}$, and similarly, the distance between two consecutive zeros of $2\cos(2\pi nx)$ is $\frac{1}{2n}$. Since $m < n$, $\frac{1}{2n} < \frac{1}{2m}$, which means that the distance between any two consecutive zeros of $2\cos(2\pi nx)$ is smaller than the distance between any two consecutive zeros of $2\cos(2\pi mx)$. This, combined with Lemmas 2.7 and 2.9, concludes the proof. ∎

For ease of notation, define

$$I_{m,k,n,\ell} = \left[\frac{2\ell + 1 - \frac{4}{11.919}\left(e^{-\ell\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right)}{2k + 1 + \frac{4}{11.919}\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right)}, \frac{2\ell + 1 + \frac{4}{11.919}\left(e^{-\ell\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right)}{2k + 1 - \frac{4}{11.919}\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right)} \right].$$

Lemma 2.11 ([5, Lemma 2.4]) *If $(x_{m,k} - x_{n,\ell})(u_{m,k} - u_{n,\ell}) \leq 0$, then*

$$\frac{n}{m} \in I_{m,k,n,\ell}.$$

Proof Following the proof of [5, Lemma 2.4], we note that if $(x_{m,k} - x_{n,\ell})(u_{m,k} - u_{n,\ell}) \leq 0$, then

$$|x_{m,k} - x_{n,\ell}| \leq |x_{m,k} - u_{m,k}| + |x_{n,\ell} - u_{n,\ell}|.$$

By Lemma 2.4, we conclude that

$$\left| \frac{2\ell+1}{4n} - \frac{2k+1}{4m} \right| < \frac{1}{11.919m}\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right) + \frac{1}{11.919n}\left(e^{-\ell\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right),$$

or equivalently,

$$\left|(2\ell+1) - (2k+1)\frac{n}{m}\right| < \frac{n}{m} \cdot \frac{4}{11.919}\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right) + \frac{4}{11.919}\left(e^{-\ell\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right).$$

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

If $\frac{n}{m} \leq \frac{2\ell+1}{2k+1}$, then

$$(2\ell+1)-(2k+1)\frac{n}{m} < \frac{n}{m}\cdot\frac{4}{11.919}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)+\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right),$$

so

$$\frac{2\ell+1-\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)}{2k+1+\frac{4}{11.919}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)} < \frac{n}{m} \leq \frac{2\ell+1}{2k+1} < \frac{2\ell+1+\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)}{2k+1-\frac{4}{11.919}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)},$$

as desired.

Otherwise, if $\frac{n}{m} > \frac{2\ell+1}{2k+1}$, then

$$(2k+1)\frac{n}{m}-(2\ell+1) < \frac{n}{m}\cdot\frac{4}{11.919}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)+\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right),$$

so

$$\frac{2\ell+1-\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)}{2k+1+\frac{4}{11.919}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)} < \frac{2\ell+1}{2k+1} < \frac{n}{m} < \frac{2\ell+1+\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)}{2k+1-\frac{4}{11.919}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)},$$

as desired. ∎

The next lemma will give a sufficient condition for the zeros of F_n to interlace.

Lemma 2.12 ([5, Lemma 2.11]) *Suppose $n > m \geq 2$. If $\frac{n}{m}$ lies in the gap between $I_{m,k+1,n,\ell}$ and $I_{m,k,n,\ell}$, or equivalently,*

$$\frac{2\ell+1+\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)}{2k+3-\frac{4}{11.919}\left(e^{-(k+1)\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)} < \frac{n}{m} < \frac{2\ell+1-\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)}{2k+1+\frac{4}{11.919}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)},$$

then $u_{m,k} > u_{n,\ell} > u_{m,k+1}$.

Proof The proof follows similarly to [5, Lemma 2.11]. Since

$$\frac{2\ell+1}{2k+3} < \frac{2\ell+1+\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)}{2k+3-\frac{4}{11.919}\left(e^{-(k+1)\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)}$$

and

$$\frac{2\ell+1-\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)}{2k+1+\frac{4}{11.919}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)} < \frac{2\ell+1}{2k+1},$$

we know that $x_{m,k} > x_{n,\ell} > x_{m,k+1}$ by Lemma 2.5. Moreover, since $\frac{n}{m} \notin I_{m,k,n,\ell}$ and $\frac{n}{m} \notin I_{m,k+1,n,\ell}$, by Lemma 2.11, we conclude that $u_{m,k} > u_{n,\ell} > u_{m,k+1}$, as desired.

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

We now check to make sure that the interval is nonempty to ensure that the lemma is nontrivial. Using the bound $e^0 + e^{-\frac{\sqrt{3}}{2}\pi m} m < 1.07$ for $m \geq 1$, note that

$$
\begin{aligned}
&\left(2\ell+1-\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)\right)\left(2k+3-\frac{4}{11.919}\left(e^{-(k+1)\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)\right)\\
&\qquad-\left(2\ell+1+\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)\right)\left(2k+1+\frac{4}{11.919}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)\right)\\
&>\left(2\ell+1-\frac{4}{11.919}1.07\right)\left(2k+3-\frac{4}{11.919}1.07\right)\\
&\qquad-\left(2\ell+1+\frac{4}{11.919}1.07\right)\left(2k+1+\frac{4}{11.919}1.07\right)\\
&>(2\ell+0.64)(2k+2.64)-(2\ell+1.36)(2k+1.36)\\
&=2.56\ell-1.44k-0.16\\
&>0,
\end{aligned}
$$

since $\ell > k$ by Corollary 2.6. ∎

3 $\frac{n}{m}$ is large

Throughout the rest of this paper, improving on the estimate in [10, Corollary 2.2], we use the bounds

$$e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m < 1.01 \qquad \text{and} \qquad e^{-(k+1)\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m < 0.052$$

for $k \geq 0$ and $m \geq 2$.

This section treats the case when n is large relative to m.

Proposition 3.1 *Let $2 \leq m < n$ be positive integers such that $\frac{n}{m} > 1.434$. Then between every two zeros of $F_m(x)$, there exists a zero of $F_n(x)$.*

Proof By Definition 2.1 and Lemma 2.4, for each $0 \leq k \leq m-1$ and $0 \leq \ell \leq n-1$,

$$
\begin{aligned}
u_{m,k}-u_{m,k+1} &\geq x_{m,k}-x_{m,k+1}-\left|x_{m,k}-u_{m,k}\right|-\left|x_{m,k+1}-u_{m,k+1}\right|\\
&>\frac{1}{2m}-\frac{1}{11.919m}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)-\frac{1}{11.919m}\left(e^{-(k+1)\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)\\
&>\frac{1}{2m}-\frac{1}{11.919m}(1.062)\\
&>\frac{1}{2n}+\frac{1}{11.919n}(1.062)\\
&>\frac{1}{2n}+\frac{1}{11.919n}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)+\frac{1}{11.919n}\left(e^{-(\ell+1)\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)\\
&>x_{n,\ell}-x_{n,\ell+1}+\left|x_{n,\ell}-u_{n,\ell}\right|+\left|x_{n,\ell+1}-u_{n,\ell+1}\right|\\
&\geq u_{n,\ell}-u_{n,\ell+1},
\end{aligned}
$$

where the fourth inequality follows since $\frac{n}{m} > 1.434$.

Combined with Lemmas 2.8 and 2.10, the proof is complete. ∎

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

4 Zeros close to 0

In this section, we investigate the behavior of zeros of j_n and j_m close to the endpoint 0 when $\frac{n}{m}$ is relatively small.

First, we correct a minor mistake in [10]. Jermann claims in [10, (26)] that if $x_{m,k} \in [0, \frac{1}{2} - \frac{\log m}{5m}]$, then $k \geq \frac{\log m}{3}$. However, this is not true for $10 < m < 1809$. In fact, if $x_{m,k} \in [0, \frac{1}{2} - \frac{\log m}{5m}]$ and $10 < m < 1809$, then one can only get $k \geq \frac{2\log m}{5} - \frac{1}{2}$. Thus, we need to modify [10, Lemma 3.3] and its proof slightly; we also give a weaker version of [10, Lemma 3.3] that is more applicable to us.

Lemma 4.1 ([10, Lemma 3.3]) *For $m \geq 191$ and $k \geq \frac{2\log m}{5} - \frac{1}{2}$,*

$$|x_{m,k} - u_{m,k}| < \frac{1}{10m(m+1)}.$$

Moreover, for $m \geq 30$ and $k \geq \frac{2\log m}{5} - \frac{1}{2}$,

$$|x_{m,k} - u_{m,k}| < \frac{1}{6m(m+1)}.$$

Proof We will follow the proof of [10, Lemma 3.3] closely here with necessary modifications to take care of the aforementioned mistake.

Assume on the contrary that

$$\frac{1}{4m} > |x_{m,k} - u_{m,k}| \geq \frac{1}{cm(m+1)}.$$

Here, we write c in place of 6 or 10. Then by the Taylor approximation for $\sin(x)$,

$$\begin{aligned} |2\cos(2\pi m u_{m,k})| = |2\sin(2\pi m(x_{m,k} - u_{m,k}))| &\geq 2\sin\left(\frac{2\pi m}{cm(m+1)}\right) \\ &\geq \frac{4\pi}{c(m+1)} - \frac{8\pi^3}{3c^3(m+1)^3}. \end{aligned}$$

On the other hand, for $k \geq \frac{2\log m}{5} - \frac{1}{2}$, estimates by taking the derivative and [10, Corollary 2.2] gives that

$$\begin{aligned} |2\cos(2\pi m u_{m,k})| &\leq e^{-\pi k} + e^{-\pi m \frac{\sqrt{3}}{2}} m \\ &\leq e^{\frac{\pi}{2}} m^{-\frac{2\pi}{5}} + e^{-\pi m \frac{\sqrt{3}}{2}} m < \frac{4\pi}{c(m+1)} - \frac{8\pi^3}{3c^3(m+1)^3} \end{aligned}$$

for $m \geq 30$ when $c = 6$ and $m \geq 191$ when $c = 10$, which gives a contradiction. ∎

The correctness of [10, Theorem 3.1] thus relies on making sure that j_n and j_{n+1} interlace for $11 \leq n \leq 190$. We have computationally verified that they do.

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

We may now prove the main result of this section.

Proposition 4.2 *Let $30 \le m < n$ be positive integers such that $\frac{n}{m} \le 1.434$. Let r be the smallest value for k such that $x_{m,k}$ lies in the interval $[0, \frac{1}{2} - \frac{\log m}{5m}]$. Then for $r \le k \le m-2$, between $u_{m,k}$ and $u_{m,k+1}$, there exists a zero of $F_n(x)$.*

Proof This proof follows similarly to [5, Proposition 4.2]. By [10, Theorem 3.1], we are done if $n = m+1$. Thus, we assume that $n \ge m+2$.

We know that if $x_{m,k} \in [0, \frac{1}{2} - \frac{\log m}{5m}]$, then by Definition 2.1, $k \ge \frac{2\log m}{5} - \frac{1}{2}$. Similarly, if $x_{n,\ell} \in [0, \frac{1}{2} - \frac{\log n}{5n}]$, then $\ell \ge \frac{2\log n}{5} - \frac{1}{2}$. Since $m < n$, $\frac{1}{2} - \frac{\log m}{5m} < \frac{1}{2} - \frac{\log n}{5n}$, so if $x_{n,\ell} \in [0, \frac{1}{2} - \frac{\log m}{5m}]$, then $\ell \ge \frac{2\log n}{5} - \frac{1}{2}$.

Also,

$$\frac{1}{3n(n+1)} + \frac{1}{3m(m+1)} < \frac{2}{3m(m+1)} < \frac{1}{1.434m^2} \le \frac{1}{mn} \le \frac{(n-m)}{2mn} = \frac{1}{2m} - \frac{1}{2n}. \tag{4.1}$$

Thus, by Lemma 4.1, for k and ℓ such that $x_{m,k}, x_{m,k+1}, x_{n,\ell}$, and $x_{n,\ell+1}$ lie in the interval $[0, \frac{1}{2} - \frac{\log m}{5m}]$,

$$\begin{aligned} u_{m,k} - u_{m,k+1} &\ge x_{m,k} - x_{m,k+1} - |x_{m,k} - u_{m,k}| - |x_{m,k+1} - u_{m,k+1}| \\ &> \frac{1}{2m} - \frac{1}{3m(m+1)} \\ &> \frac{1}{2n} + \frac{1}{3n(n+1)} \\ &\ge x_{n,\ell} - x_{n,\ell+1} + |x_{n,\ell} - u_{n,\ell}| + |x_{n,\ell+1} - u_{n,\ell+1}| \\ &\ge u_{n,\ell} - u_{n,\ell+1}, \end{aligned} \tag{4.2}$$

where the the third inequality is due to (4.1).

We note that k and ℓ such that $x_{m,k}, x_{m,k+1}, x_{n,\ell}$, and $x_{n,\ell+1}$ lie in the interval $[0, \frac{1}{2} - \frac{\log m}{5m}]$ exists because $10 < m < n$ and Proposition 1.3.

By the distance inequality (4.2), to show the Stieltjes interlacing on the interval $[0, \frac{1}{2} - \frac{\log m}{5m}]$, we only need to check the zeros closest to the endpoints. The two zeros closest to 0 are covered by Lemma 2.10, so it remains to verify that there is a zero of F_n between $u_{m,r}$ and $u_{m,r+1}$. Let $u_{n,\ell+1}$ be the closest zero of F_n smaller than $u_{m,r+1}$. By Lemma 4.1 and (4.1),

$$\begin{aligned} x_{n,\ell} &= x_{n,\ell+1} + \frac{1}{2n} \\ &\le u_{n,\ell+1} + |x_{n,\ell+1} - u_{n,\ell+1}| + \frac{1}{2n} \\ &< u_{m,r+1} + |x_{n,\ell+1} - u_{n,\ell+1}| + \frac{1}{2n} \\ &\le x_{m,r+1} + |x_{n,\ell+1} - u_{n,\ell+1}| + |x_{m,r+1} - u_{m,r+1}| + \frac{1}{2n} \end{aligned}$$

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

$$
\begin{aligned}
&= x_{m,r} + \frac{1}{2n} - \frac{1}{2m} + |x_{n,\ell+1} - u_{n,\ell+1}| + |x_{m,r+1} - u_{m,r+1}| \\
&< x_{m,r} + \frac{1}{2n} - \frac{1}{2m} + \frac{1}{6n(n+1)} + \frac{1}{6m(m+1)} \\
&< x_{m,r}.
\end{aligned}
$$

Then (4.2) applies, so

$$u_{n,\ell} < u_{n,\ell+1} + u_{m,r} - u_{m,r+1} < u_{m,r}.$$

By the choice of $u_{n,\ell+1}$, we conclude that $u_{n,\ell}$ lies between $u_{m,r}$ and $u_{m,r+1}$, as desired. ■

5 Zeros close to $\frac{1}{2}$

In this section, we study the case when $\frac{n}{m}$ is relatively small and the zeros involved are close to the endpoint $\frac{1}{2}$. This case is difficult because the distance inequality $u_{n,\ell} - u_{n,\ell+1} < u_{m,k} - u_{m,k+1}$ that was used in the previous cases is hard to establish. To overcome this difficulty we will follow the treatment in Section 5 of [5] closely. Our goal is to prove the following result.

Proposition 5.1 *Let $10 < m < n$ be positive integers such that $\frac{n}{m} \leq 1.434$. Then on the interval $\left[\frac{1}{2} - \frac{\log m}{2m}, \frac{1}{2}\right]$, between any two zeros of F_m, there exists a zero of F_n.*

Proposition 5.1 is implied by Proposition 1.3 and the following two lemmas whose proofs will occupy the remaining section.

Lemma 5.2 *Let $10 < m < n$ be positive integers such that $\frac{n}{m} \leq 1.434$. Suppose that $\frac{1}{2} \geq x_{m,k} > x_{n,\ell} > x_{m,k+1} \geq \frac{1}{2} - \frac{\log m}{2m}$ and $u_{n,\ell} \geq u_{m,k}$. Then $u_{m,k} > u_{n,\ell+1} > u_{m,k+1}$.*

Lemma 5.3 *Let $10 < m < n$ be positive integers such that $\frac{n}{m} \leq 1.434$. Suppose that $\frac{1}{2} \geq x_{m,k} > x_{n,\ell} > x_{m,k+1} \geq \frac{1}{2} - \frac{\log m}{2m}$ and $u_{n,\ell} \leq u_{m,k+1}$. Then $u_{m,k} > u_{n,\ell-1} > u_{m,k+1}$.*

5.1 Proof of Lemma 5.2

Proof of Lemma 5.2 By the assumptions, we know that $\frac{n}{m} \in I_{m,k,n,\ell}$ by Lemma 2.11. We wish to show that

$$
\frac{2\ell + 3 + \frac{4}{11.919}\left(e^{-(\ell+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right)}{2k + 3 - \frac{4}{11.919}\left(e^{-(k+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right)} < \frac{2\ell + 1 - \frac{4}{11.919}\left(e^{-\ell\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right)}{2k + 1 + \frac{4}{11.919}\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right)} \tag{5.1}
$$

and

$$
\frac{2\ell + 1 + \frac{4}{11.919}\left(e^{-\ell\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right)}{2k + 1 - \frac{4}{11.919}\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right)} < \frac{2\ell + 3 - \frac{4}{11.919}\left(e^{-(\ell+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right)}{2k + 1 + \frac{4}{11.919}\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right)}, \tag{5.2}
$$

so $\frac{n}{m} \in I_{m,k,n,\ell}$ implies that $u_{m,k} > u_{n,\ell+1} > u_{m,k+1}$ by Proposition 2.12.

Again by our assumptions, we know that $\ell > k$ by Corollary 2.6, $10 < m < n$, and by Lemma 2.5, $\frac{2\ell+1}{2k+3} < \frac{n}{m} \le 1.434$. Moreover, since we are on the interval $\left[\frac{1}{2} - \frac{\log m}{2m}, \frac{1}{2}\right]$, as mentioned in [10, (27)], $k \le \log m$. Thus, (5.1) holds by Lemma 5.4, and (5.2) holds by Lemma 5.5. ■

The proofs of Lemmas 5.4 and 5.5, which were needed above, are now shown.

Lemma 5.4 *If $\ell > k$ are non-negative integers and $10 < m < n$ are positive integers such that $k \le \log m$, then (5.1) holds.*

Proof Note that the function $f(m) := e^{-\frac{\sqrt{3}}{2}\pi m} m$ is decreasing in m for $m \ge 1$. Thus, since $\ell > k$, $m < n$, and $k \le \log m$ (or equivalently, $e^k \le m$),

$$\begin{aligned}
&\left(2\ell + 1 - \frac{4}{11.919}\left(e^{-\ell\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right)\right)\left(2k + 3 - \frac{4}{11.919}\left(e^{-(k+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right)\right) \\
&\quad - \left(2\ell + 3 + \frac{4}{11.919}\left(e^{-(\ell+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right)\right)\left(2k + 1 + \frac{4}{11.919}\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right)\right) \\
&= 4(\ell - k) - \frac{4}{11.919}(2k+3)\left(e^{-\ell\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right) \\
&\quad - \frac{4}{11.919}(2\ell+1)\left(e^{-(k+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right) \\
&\quad - \frac{4}{11.919}(2k+1)\left(e^{-(\ell+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right) - \frac{4}{11.919}(2\ell+3)\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right) \\
&\quad + \left(\frac{4}{11.919}\right)^2\left(e^{-\ell\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right)\left(e^{-(k+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right) \\
&\quad - \left(\frac{4}{11.919}\right)^2\left(e^{-(\ell+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi n} n\right)\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right) \\
&> 4(\ell - k) - \frac{16}{11.919}(\ell - k) - \frac{4}{11.919}(2k+3)\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right) \\
&\quad - \frac{4}{11.919}(2k+1)\left(e^{-(k+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right) \\
&\quad - \frac{4}{11.919}(2k+1)\left(e^{-(k+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right) \\
&\quad - \frac{4}{11.919}(2k+3)\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m} m\right) - \left(\frac{4}{11.919}\right)^2 (1.01)^2 \\
&> 2.54 - \frac{8}{11.919}(2k+3)\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi e^k} e^k\right) \\
&\quad - \frac{8}{11.919}(2k+1)\left(e^{-(k+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi e^k} e^k\right).
\end{aligned}$$

Moreover,

$$g(k) := \frac{8}{11.919}(2k+3)\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi e^k} e^k\right) + \frac{8}{11.919}(2k+1)\left(e^{-(k+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi e^k} e^k\right)$$

is decreasing for $k \ge 0$, and $g(0) < 2.54$. Thus, for $k \ge 0$, (5.1) holds. ■

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

Lemma 5.5 *Let k, ℓ be non-negative integers such that $\frac{2\ell+1}{2k+3} < 1.434$. Moreover, if $k \leq 1$, suppose that $\ell > k$ and $u_{n,\ell} \geq u_{m,k}$. If m and n are positive integers greater than 10 such that $\frac{n}{m} \leq 1.434$, then (5.2) holds.*

Proof First, suppose $k \geq 2$. Then

$$\begin{aligned}
&\left(2\ell+3-\frac{4}{11.919}\left(e^{-(\ell+1)\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)\right)\left(2k+1-\frac{4}{11.919}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)\right)\\
&\quad-\left(2\ell+1+\frac{4}{11.919}\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)\right)\left(2k+1+\frac{4}{11.919}\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)\right)\\
&=2(2k+1)-\frac{4}{11.919}(2k+1)\left(e^{-(\ell+1)\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n+e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)\\
&\quad-\frac{4}{11.919}(2\ell+3)\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)-\frac{4}{11.919}(2\ell+1)\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)\\
&\quad+\left(\frac{4}{11.919}\right)^2\left(e^{-(\ell+1)\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)\\
&\quad-\left(\frac{4}{11.919}\right)^2\left(e^{-\ell\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)\left(e^{-k\pi}+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)\\
&>2(2k+1)-\frac{4(1.062)}{11.919}(2k+1)-\frac{16}{11.919}(\ell+1)(1.01)-\left(\frac{4}{11.919}\right)^2(1.01)^2\\
&=(2k+3)\\
&\left(\left(2-\frac{4(1.062)}{11.919}\right)-\frac{8(1.01)}{11.919}\frac{2\ell+1}{2k+3}-\frac{\left(\frac{4}{11.919}\right)^2(1.01)^2+2\left(2-\frac{4(1.062)}{11.919}\right)+\frac{8(1.01)}{11.919}}{2k+3}\right)\\
&>(2k+3)\\
&\left(\left(2-\frac{4(1.062)}{11.919}\right)-\frac{8(1.01)}{11.919}1.434-\frac{\left(\frac{4}{11.919}\right)^2(1.01)^2+2\left(2-\frac{4(1.062)}{11.919}\right)+\frac{8(1.01)}{11.919}}{7}\right)\\
&>0,
\end{aligned}$$

as desired. We are now left with the cases for $k = 0, 1$. Since $\ell > k$ and $\frac{2\ell+1}{2k+3} < 1.434$, we are able to explicitly write the three cases that are possible.

Case 1: $k = 0$.

We must have $\ell = 1$. Then since $\frac{n}{m} \leq 1.434$, by Definition 2.1,

$$\begin{aligned}
x_{m,0}-x_{n,1}&=\frac{3}{4n}-\frac{1}{4m}\\
&>\frac{1.01}{11.919m}+\frac{0.052}{11.919n}\\
&>\frac{1}{11.919m}\left(1+e^{-\frac{\sqrt{3}}{2}\pi m}m\right)+\frac{1}{11.919n}\left(e^{-\pi}+e^{-\frac{\sqrt{3}}{2}\pi n}n\right)\\
&>|x_{m,0}-u_{m,0}|+|x_{n,1}-u_{n,1}|,
\end{aligned}$$

so $u_{m,0} > u_{n,1}$, contradicting the assumption that $u_{n,1} \geq u_{m,0}$. Thus, this case is not possible.

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

Case 2: $k = 1, \ell = 2$.

Since $\frac{n}{m} \leq 1.434$, by Definition 2.1,

$$\begin{aligned} x_{m,1} - x_{n,2} &= \frac{5}{4n} - \frac{3}{4m} \\ &> \frac{0.052}{11.919m} + \frac{0.002}{11.919n} \\ &> \frac{1}{11.919m}\left(e^{-\pi} + e^{-\frac{\sqrt{3}}{2}\pi m}m\right) + \frac{1}{11.919n}\left(e^{-2\pi} + e^{-\frac{\sqrt{3}}{2}\pi n}n\right) \\ &> |x_{m,1} - u_{m,1}| + |x_{n,2} - u_{n,2}|, \end{aligned}$$

so $u_{m,1} > u_{n,2}$, contradicting the assumption that $u_{n,2} \geq u_{m,1}$. Thus, this case is not possible.

Case 3: $k = 1, \ell = 3$.

Since $\frac{n}{m} \leq 1.434$, by Definition 2.1,

$$\begin{aligned} x_{m,1} - x_{n,3} &= \frac{7}{4n} - \frac{3}{4m} \\ &> \frac{0.052}{11.919m} + \frac{0.001}{11.919n} \\ &> \frac{1}{11.919m}\left(e^{-\pi} + e^{-\frac{\sqrt{3}}{2}\pi m}m\right) + \frac{1}{11.919n}\left(e^{-3\pi} + e^{-\frac{\sqrt{3}}{2}\pi n}n\right) \\ &> |x_{m,1} - u_{m,1}| + |x_{n,3} - u_{n,3}|, \end{aligned}$$

so $u_{m,1} > u_{n,3}$, contradicting the assumption that $u_{n,3} \geq u_{m,1}$. Thus, this case is not possible.

We have covered all possible cases, and thus conclude the proof of Lemma 5.5.

We now prove the second lemma necessary for Proposition 5.1.

5.2 Proof of Lemma 5.3

Proof of Lemma 5.3 By the assumptions, we have that $\frac{n}{m} \in I_{m,k+1,n,\ell}$ by Lemma 2.11. Moreover, recall that $\ell > k \geq 0$ by Corollary 2.6. We wish to show that

$$\frac{2\ell - 1 + \frac{4}{11.919}\left(e^{-(\ell-1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi n}n\right)}{2k + 3 - \frac{4}{11.919}\left(e^{-(k+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi m}\right)} < \frac{2\ell + 1 - \frac{4}{11.919}\left(e^{-\ell\pi} + e^{-\frac{\sqrt{3}}{2}\pi n}\right)}{2k + 3 + \frac{4}{11.919}\left(e^{-(k+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi m}m\right)} \tag{5.3}$$

and

$$\frac{2\ell + 1 + \frac{4}{11.919}\left(e^{-\ell\pi} + e^{-\frac{\sqrt{3}}{2}\pi n}n\right)}{2k + 3 - \frac{4}{11.919}\left(e^{-(k+1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi m}m\right)} < \frac{2\ell - 1 - \frac{4}{11.919}\left(e^{-(\ell-1)\pi} + e^{-\frac{\sqrt{3}}{2}\pi n}n\right)}{2k + 1 + \frac{4}{11.919}\left(e^{-k\pi} + e^{-\frac{\sqrt{3}}{2}\pi m}m\right)}, \tag{5.4}$$

so $\frac{n}{m} \in I_{m,k+1,n,\ell}$ implies that $u_{m,k} > u_{n,\ell-1} > u_{m,k+1}$ by Lemma 2.12.

Note that by letting $k = k' - 1$ and $\ell = \ell' + 1$, (5.3) is the same as (5.2), which was proven in Lemma 5.5, but with k' and ℓ' instead of k and ℓ. We see that $\frac{2\ell'+1}{2k'+3} \leq \frac{2\ell+1}{2k+3} < \frac{n}{m} \leq 1.434$ by Lemma 2.5. Moreover, if $k \geq 1$, then $k' \geq 2$, so (5.3) holds by Lemma 5.5. Thus, it remains to consider the case when $k = 0$.

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

Similarly, by letting $\ell = \ell' + 1$ and leaving k unchanged, (5.4) is the same as (5.1), which was proven in Lemma 5.4, but with ℓ' instead of ℓ. Since we are on the interval $\left[\frac{1}{2} - \frac{\log m}{2m}, \frac{1}{2}\right]$, as mentioned in [10, (27)], $k \leq \log m$. Then if $\ell > k+1$, (5.4) holds by Lemma 5.4. Thus, it remains to consider the case where $\ell = k+1$.

Thus, Lemma 5.3 holds in all cases except when $k = 0$ and $\ell \geq 2$, or when $\ell = k+1$. We now show that each of these cases cannot happen.

Case 1: $k = 0$ and $\ell \geq 2$.

Recall that $x_{m,0} > x_{n,\ell} > x_{m,1}$ by our assumptions. By Lemma 2.5, we have that

$$\frac{n}{m} > \frac{2\ell+1}{2k+3} \geq \frac{5}{3} > 1.434,$$

contradicting the assumption that $\frac{n}{m} \leq 1.434$. Thus, this case is not possible.

Case 2: $\ell = k+1$. Recall that $x_{m,k} > x_{n,k+1} > x_{m,k+1}$ by our assumptions. Since we are on the interval $\left[\frac{1}{2} - \frac{\log m}{2m}, \frac{1}{2}\right]$, we know that $k+1 \leq \log m < \log n$ by [10, (27)]. Let $\hat{x}_{n,k}$ be as defined in [10, Section 3]. Then by [10, Lemma 3.4],

$$\hat{x}_{n,k+1} - x_{n,k+1} = \frac{w_{k+1}}{n} \qquad \text{and} \qquad \hat{x}_{m,k+1} - x_{m,k+1} = \frac{w_{k+1}}{m},$$

where w_k is independent of n (or m) and $|w_k| \leq \frac{1}{12}$ for all $k \leq \log m$. Following the method of [10, Lemma 3.5] and applying Definition 2.1,

$$\begin{aligned}
\hat{x}_{n,k+1} - \hat{x}_{m,k+1} &= x_{n,k+1} - x_{m,k+1} - w_{k+1}\left(\frac{1}{m} - \frac{1}{n}\right) \\
&\geq \frac{2k+3}{4}\left(\frac{1}{m} - \frac{1}{n}\right) - \frac{1}{12}\left(\frac{1}{m} - \frac{1}{n}\right) \\
&= \frac{3k+4}{6}\left(\frac{n-m}{mn}\right) > 0.
\end{aligned}$$

Moreover, by [10, Lemma 3.6],

$$|\hat{x}_{n,k+1} - u_{n,k+1}| < \frac{1}{20n(n+1)} \qquad \text{and} \qquad |\hat{x}_{m,k+1} - u_{m,k+1}| < \frac{1}{20m(m+1)}.$$

Since $\frac{n}{m} \leq 1.434$, we get

$$\begin{aligned}
|\hat{x}_{n,k+1} - u_{n,k+1}| + |\hat{x}_{m,k+1} - u_{m,k+1}| &< \frac{1}{20n(n+1)} + \frac{1}{20m(m+1)} \\
&< \frac{1}{10m(m+1)} \\
&< \frac{4}{6 \cdot 1.434 m^2} \\
&\leq \frac{3k+4}{6}\left(\frac{n-m}{nm}\right) \\
&< \hat{x}_{n,k+1} - \hat{x}_{m,k+1},
\end{aligned}$$

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

so $u_{n,k+1} > u_{m,k+1}$. This contradicts our initial assumption that $u_{n,k+1} \leq u_{m,k+1}$, so we conclude that this case is not possible.

We have exhausted all cases, so the proof of Lemma 5.3 is complete. ∎

6 Proof of Theorem 1.2 and discussion

Proof of Theorem 1.2 It suffices to show the theorem for consecutive zeros of F_m. By Proposition 3.1, the theorem is proved for $2 \leq m < n$ such that $\frac{n}{m} > 1.434$. For $m \geq 30$, as the length of the intersection interval

$$\left[\frac{1}{2} - \frac{\log m}{2m}, \frac{1}{2} - \frac{\log m}{5m}\right] = \left[0, \frac{1}{2} - \frac{\log m}{5m}\right] \cap \left[\frac{1}{2} - \frac{\log m}{2m}, \frac{1}{2}\right]$$

is

$$\frac{1}{2} - \frac{\log m}{5m} - \frac{1}{2} + \frac{\log m}{2m} = \frac{3 \log m}{10} \frac{1}{m} \geq \frac{3 \log 30}{10} \frac{1}{m} > \frac{1}{2m} + \frac{2}{11m},$$

by Lemma 2.3 there is at least one zero (not two zeros as incorrectly and unnecessarily claimed in the proof of [10, Theorem 3.1]) of F_m in this interval. Thus, by Propositions 4.2 and 5.1, we conclude that for $30 \leq m < n$ such that $\frac{n}{m} \leq 1.434$, the theorem is proven. It remains to check the case when $2 \leq m \leq 29$ and $\frac{n}{m} \leq 1.434$, which has been verified computationally and by using [10, Theorem 3.1].

As a corollary we obtain the following indivisibility among the polynomials φ_n in Section 1.

Corollary 6.1 *If $2 \leq m < n \leq 2m$, then $\varphi_m \nmid \varphi_n$.*

Proof Suppose on the contrary that $\varphi_m \mid \varphi_n$.

By Theorem 1.2, there are at least $m - 1$ zeros of φ_n that are not shared by φ_m. There are m more zeros of φ_n from φ_m, and two extra zeros of φ_n closest to the two endpoints by Lemmas 2.8 and 2.10. Thus, the total number of zeros of φ_n is at least $m + (m - 1) + 2 = 2m + 1$, which is a contradiction. ∎

We end the paper with the following well-known conjecture due to Ono [12, Problem 4.30]. It has been verified by us using SAGE for $n \leq 500$, see also [9] for a partial result towards the conjecture. Clearly this conjecture implies Corollary 6.1.

Conjecture 6.2 *Each polynomial φ_n for $n \geq 1$ is irreducible over $\mathbb{Q}$.*

APPENDIX A: Data

In this appendix, we supply the relevant data for $n \leq 10$. In Table A.1, the real parts of zeros of $j_n(z)$ in the fundamental domain are calculated using Mathematica. For comparison, the zeros of $\cos(2\pi n x)$ in the interval $(0, 0.5)$ are given in Table A.2. Note that these data are ordered decreasingly according to the convention in this paper. In Figure A.1, we plot the real parts of zeros of $j_n(z)$ to help visualize the statement of Theorem 1.2.

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

Table A.1. Real parts of zeros of $j_n(z)$.

$n=1$	0.2311									
$n=2$	0.3713	0.1252								
$n=3$	0.4145	0.2501	0.0833							
$n=4$	0.4360	0.3125	0.1875	0.0625						
$n=5$	0.4488	0.3500	0.2450	0.1500	0.0450					
$n=6$	0.4574	0.3750	0.2917	0.2083	0.1250	0.0417				
$n=7$	0.4635	0.3927	0.3214	0.2500	0.1786	0.1071	0.0357			
$n=8$	0.4680	0.4063	0.3437	0.2813	0.2187	0.1563	0.0938	0.0313		
$n=9$	0.4716	0.4167	0.3611	0.3056	0.2450	0.1944	0.1389	0.0833	0.0278	
$n=10$	0.4744	0.4250	0.3750	0.3250	0.2745	0.2250	0.1750	0.1250	0.0750	0.0250

Table A.2. Zeros of $\cos(2\pi nx)$ in $(0, \frac{1}{2})$.

$n=1$	1/4									
$n=2$	3/8	1/8								
$n=3$	5/12	1/4	1/12							
$n=4$	7/16	5/16	3/16	1/16						
$n=5$	9/20	7/20	1/4	3/20	1/20					
$n=6$	11/24	3/8	7/24	5/24	1/8	1/24				
$n=7$	13/28	11/28	9/28	1/4	5/28	3/28	1/28			
$n=8$	15/32	13/32	11/32	9/32	7/32	5/32	3/32	1/32		
$n=9$	17/36	5/12	13/36	11/36	1/4	7/36	5/36	1/12	1/36	
$n=10$	19/40	17/40	3/8	13/40	11/40	9/40	7/40	1/8	3/40	1/40

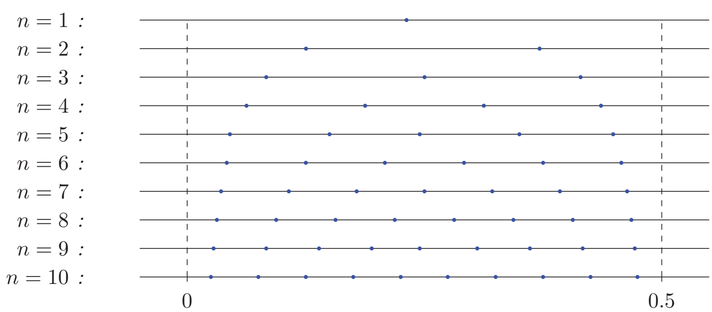

Figure A.1: Real parts of zeros of $j_n(z)$.

Acknowledgment This research was conducted during the 2021 REU program at Clemson University, supported by NSF-DMS Grant 1547399. We want to thank the referee for advice and for pointing out the references related to Conjecture 6.2.

References

[1] T. Asai, M. Kaneko, and H. Ninomiya, *Zeros of certain modular functions and an application.* Comment. Math. Univ. St. Paul. **46** (1997), no. 1, 93–101.

https://doi.org/10.4153/S0008439522000054 Published online by Cambridge University Press

[2] A. F. Beardon and K. A. Driver, *The zeros of linear combinations of orthogonal polynomials*. J. Approx. Theory **137** (2005), no. 2, 179–186.

[3] J. H. Bruinier, W. Kohnen, and K. Ono, *The arithmetic of the values of modular functions and the divisors of modular forms*. Compos. Math. **140** (2004), no. 3, 552–566.

[4] K. Driver, A. Jooste, and K. Jordaan, *Stieltjes interlacing of zeros of Jacobi polynomials from different sequences*. Electron. Trans. Numer. Anal. **38** (2011), 317–326.

[5] W. Frendreiss, J. Gao, A. Lei, A. Woodall, H. Xue, and D. Zhu, *A Stieltjes separation property of zeros of Eisenstein series*, to appear in Kyushu J. Math. (2022).

[6] E.-U. Gekeler, *Some observations on the arithmetic of Eisenstein series of the modular group* sl $(2, \mathbb{Z})$. Arch. Math. 77 (2001), 5–21.

[7] P. C. Gibson, *Common zeros of two polynomials in an orthogonal sequence*. J. Approx. Theory **105** (2000), no. 1, 129–132.

[8] T. Griffin, N. Kenshur, A. Price, B. Vandenberg-Daves, H. Xue, and D. Zhu, *Interlacing of zeros of Eisenstein series*. Kyushu J. Math. **75** (2021), no. 2, 249–272.

[9] P. Guerzhoy, *Irreducibility of some Faber polynomials*. Ramanujan J. **16** (2008), no. 1, 53–57.

[10] J. Jermann, *Interlacing property of the zeros of $j_n\ (\tau)$*. Proc. Am. Math. Soc. **140** (2012), 3385–3396.

[11] H. Nozaki, *A separation property of the zeros of Eisenstein series for* $\mathsf{SL}(2, \mathbb{Z})$. Lond. Math. Soc. **40** (2008), 26–36.

[12] Ken Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q -series* , CBMS Regional Conference Series in Mathematics, 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004.

[13] F. K. C. Rankin and H. P. F. Swinnerton-Dyer, *On the zeros of Eisenstein series*. Lond. Math. Soc. **2** (1970), 169–170.

[14] G. Szegő, *Orthogonal polynomials*, 4th ed., American Mathematical Society, Providence, RI, 1975.

Department of Mathematics, Texas A&M University, College Station, TX, 77840, USA
e-mail: wfrendreiss@tamu.edu

Department of Mathematics, Harvard University, Cambridge, MA, 02138, USA
e-mail: jgao@college.harvard.edu

Department of Mathematics, UC Berkeley, Berkeley, CA, 94720, USA
e-mail: 1austinlei@berkeley.edu

Mathematics Department, Brigham Young University, Provo, UT, 84602, USA
e-mail: amy.woodall713@gmail.com

School of Mathematical and Statistical Sciences, Clemson University, Clemson, SC, 29634, USA
e-mail: huixue@clemson.edu zdaozho@clemson.edu