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SELF-EMBEDDINGS OF MODELS OF ARITHMETIC; FIXED POINTS, SMALL SUBMODELS, AND EXTENDABILITY

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Abstract. In this paper we will show that for every cut I of any countable nonstandard model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$, each I-small Σ_1-elementary submodel of $\mathcal{M}$ is of the form of the set of fixed points of some proper initial self-embedding of $\mathcal{M}$ iff I is a strong cut of $\mathcal{M}$. Especially, this feature will provide us with some equivalent conditions with the strongness of the standard cut in a given countable model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$. In addition, we will find some criteria for extendability of initial self-embeddings of countable nonstandard models of $\mathrm{I}\Sigma_1$ to larger models.

§1. Introduction. In 1973, Harvey Friedman proved a striking result for countable nonstandard models of finite set theory, and consequently for countable models of Peano arithmetic (PA) stating that *every countable nonstandard model of* PA *carries a proper initial self-embedding*; here an *initial self-embedding* is a self-embedding whose image is an initial segment of the ground model [5]. Afterward, many versions of Friedman's style theorem appeared in the literature of model theory of arithmetic (e.g., see [3] or [16]). In [1], it is shown that some results on the set of fixed points of automorphism of countable recursively saturated models of PA can be generalized for initial self-embeddings of countable nonstandard models of $\mathrm{I}\Sigma_1$ (see Theorem 2.4). In this paper, inspired by results about automorphisms of models of PA, we will investigate some more properties of countable models of $\mathrm{I}\Sigma_1$ through initial self-embeddings.

In [4], Enayat generalized the notion of a *small* submodel from [14], to *I-small*[1] for a given cut I of a model of PA (see Definition 1), and proved that:

THEOREM 1.1 (Enayat). *Suppose $\mathcal{M} \models$ PA is countable, recursively saturated, and I is a strong cut of $\mathcal{M}$. Moreover, let $\mathcal{M}_0$ be an I-small elementary submodel of $\mathcal{M}$. Then there exists some automorphism j of $\mathcal{M}$ such that M_0 is equal to the set of fixed points of j.*

In Section 3 of this paper, after investigating some basic properties of I-small Σ_1-elementary submodels of a countable model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$ for some cut I of $\mathcal{M}$, we will refine the above theorem for initial self-embeddings; i.e., we will show that I is strong in $\mathcal{M}$ iff every I-small Σ_1-elementary submodel of $\mathcal{M}$ is equal to the set of

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[1] In his paper [4], Enayat called such submodels I-coded. The name I-small is borrowed from Kossak–Schmerl's book [13].

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fixed points of some proper initial self-embedding of $\mathcal{M}$. This result also generalizes one of the main theorems of [1] (see Corollary 4.3).

Section 4 of this paper is devoted to the investigation of equivalent conditions to strongness of the standard cut, denoted by $\mathbb{N}$, in a countable model of $\mathrm{I}\Sigma_1$, through the set of fixed points of initial self-embeddings. In [12], it is shown that:

THEOREM 1.2 (Kossak–Schmerl). *Suppose $\mathcal{M}$ is a countable recursively saturated model of* PA. *If $\mathbb{N}$ is not strong in $\mathcal{M}$, then for every automorphism j of $\mathcal{M}$ the set of fixed points of j is isomorphic to $\mathcal{M}$.*

In Corollary 4.2, we will show that for every countable nonstandard model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$, if $\mathbb{N}$ is not strong in $\mathcal{M}$, then the set of fixed points of any initial self-embedding j of $\mathcal{M}$ is either a model of $\neg\mathrm{B}\Sigma_1$, or is isomorphic to some proper initial segment of $\mathcal{M}$. Then, we conclude that $\mathbb{N}$ is strong in a countable recursively saturated model $\mathcal{M}$ of PA iff there exists some proper initial self-embedding j of $\mathcal{M}$ such that the set of fixed points of j is small in $\mathcal{M}$ and consequently it is not isomorphic to any proper initial segment of $\mathcal{M}$.

In Section 5, we will study the extendability of initial embeddings of models of $\mathrm{I}\Sigma_1$ to larger models. In particular, we will prove that any isomorphism between two Σ_1-elementary initial segment of a countable nonstandard model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$ is extendable to some initial self-embedding of $\mathcal{M}$ iff it preserves coded subsets (for the case of automorphisms of countable recursively saturated models of PA this condition is only a necessary condition for extendability to larger models [10]).

§2. Preliminaries. In this section we will review some definitions and results which are used through this paper. All unexplained notions can be found in [6, 7].

- Through this paper, we will work in the language of arithmetic $\mathcal{L}_A := \{+, ., <, 0, 1\}$. For a given class Γ of $\mathcal{L}$-formulas (where $\mathcal{L} \supseteq \mathcal{L}_A$), $\mathrm{I}\Gamma$ is the fragment of $\mathrm{PA}^* := \mathrm{PA}(\mathcal{L})$ with the induction scheme limited to formulas of Γ. The Γ-*Collection* scheme, denoted by $\mathrm{B}\Gamma$, consists of the formulas of the following form for every $\varphi \in \Gamma$:

$$\forall \bar{z}, u \,((\forall x < u \,\exists y \,\varphi(x, y, \bar{z})) \rightarrow \exists v \,(\forall x < u \,\exists y < v \,\varphi(x, y, \bar{z}))).$$

  Moreover, the *strong* Γ-*Collection* scheme, denoted by $\mathrm{B}^+\Gamma$, consists of the formulas of the following form for every $\varphi \in \Gamma$:

$$\forall \bar{z}, u \,\exists v \,\forall x < u \,(\exists y \,\varphi(x, y, \bar{z}) \rightarrow \exists y < v \,\varphi(x, y, \bar{z})).$$

  It is folklore that $\mathrm{I}\Sigma_{n+1} \vdash \mathrm{B}^+\Sigma_{n+1} \vdash \mathrm{B}\Sigma_{n+1}$ for all $n \in \omega$; moreover, for every $n \in \omega$, $\mathrm{I}\Sigma_n \nvdash \mathrm{B}\Sigma_{n+1}$ and $\mathrm{I}\Sigma_n \nvdash \neg\mathrm{B}\Sigma_{n+1}$ (see [6, Chapter I]).
- Within $\mathrm{I}\Delta_0 + \mathrm{Exp}$, the Δ_0-formula $x\mathrm{E}y$ denotes the *Ackermann's membership relation*, asserting that *"the x-th bit of the binary expansion of y is 1."* For every $\mathcal{M} \models \mathrm{I}\Delta_0 + \mathrm{Exp}$ and each $a \in M$, a_{E} denotes the set of E-members of a in $\mathcal{M}$. Moreover, the Δ_0-formulas $\mathrm{Card}(x) = y$, $\langle \bar{x} \rangle = y$, $\mathrm{Len}(x) = y$, $(x)_y = z$, and $x \upharpoonright_y = z$, respectively, express that *"there exists some bijection between y and the set coded by x," "the sequence number of $\bar{x}$ is y," "length of the sequence coded by x is y," "the yth element of the sequence number x is z,"* and *"the restriction*

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of the sequence number x to y is z." In addition, for every formula $\varphi(x)$, by the formula $y = \mu_x\ \varphi(x)$ we mean *"y is the least element such that $\varphi(y)$ holds."* Furthermore, for every $n \in \omega$ there exist $\mathcal{L}_A$-formulas Sat_{Σ_n} and Sat_{Π_n} which define the satisfaction predicate for Σ_n-formulas and Π_n-formulas, respectively, in an ambient model. For every natural number $n > 0$, it can be shown that Sat_{Σ_n} and Sat_{Π_n} are Σ_n and Π_n, respectively, in $\mathrm{I}\Sigma_1$. Moreover, $\mathrm{Sat}_{\Delta_0} \in \Delta_1^{\mathrm{I}\Sigma_1}$ [6, Chapter I, Theorem 1.75]. If $\mathcal{M}$ is a nonstandard model of $\mathrm{I}\Sigma_n$, the aforementioned feature along with Σ_n-Overspill in $\mathcal{M}$ imply that every coded Σ_n-type and every coded bounded Π_n-type is realized in $\mathcal{M}$.

- **Σ_n-Pigeonhole Principle.** For every $n > 0$, if $\mathcal{M} \models \mathrm{I}\Sigma_n$, $a \in M$, and φ is a Σ_1-formula which defines a function from $a+1$ into a in $\mathcal{M}$, then φ is not one-to-one [6].
- Given $\mathcal{L}_A$-structure $\mathcal{M}$ and subset X of M, for every $n > 0$, we define:
  - $\mathrm{K}^n(\mathcal{M}; X)$:=the set of all Σ_n-definable element of $\mathcal{M}$ with parameters from X.
  - $\mathrm{I}^n(\mathcal{M}; X) := \{x : x \leq a \text{ for some } a \in \mathrm{K}^n(\mathcal{M}; X)\}$.
  - $\mathrm{H}^n(\mathcal{M}; X) := \bigcup_{k\in\omega} \mathrm{H}^n_k(\mathcal{M}; X)$, where:

$$\mathrm{H}^n_0(\mathcal{M}; X) := \mathrm{I}^n(\mathcal{M}; X), \text{ and}$$
$$\mathrm{H}^n_{k+1}(\mathcal{M}; X) := \mathrm{I}^n(\mathcal{M}; \mathrm{H}^n_k(\mathcal{M}; X)).$$

  - $\mathrm{K}(\mathcal{M}; X) := \cup_{n\in\omega} \mathrm{K}^n(\mathcal{M}; X)$.

  (When $X = \emptyset$, we omit X from the notations.) Clearly, $\mathrm{I}^n(\mathcal{M}; X)$ and $\mathrm{H}^n(\mathcal{M}; X)$ are initial segments of $\mathcal{M}$. The following properties of these submodels of $\mathcal{M}$ are well-known (see, e.g., [6, Chapter IV, Theorem 1.33]):

THEOREM 2.1. *Suppose $n > 0$, and $\mathcal{M} \models \mathrm{I}\Sigma_n$ and $X \subseteq M$, then the following hold*:

(1) $\mathrm{K}^n(\mathcal{M}; X) \prec_{\Sigma_n} \mathcal{M}$, *and if* $\mathrm{K}^n(\mathcal{M})$ *is nonstandard, then* $\mathrm{K}^n(\mathcal{M}) \models \neg\mathrm{B}\Sigma_n$.
(2) $\mathrm{I}^n(\mathcal{M}; X) \prec_{\Sigma_{n-1}} \mathcal{M}$ *and* $\mathrm{I}^n(\mathcal{M}; X) \models \mathrm{B}\Sigma_n$.
(3) $\mathrm{H}^n(\mathcal{M}; X) \prec_{\Sigma_n} \mathcal{M}$ *and* $\mathrm{H}^n(\mathcal{M}; X) \models \mathrm{B}\Sigma_{n+1}$.

- A given structure $\mathcal{M}$ is called *recursively saturated* if it realizes every recursive type with finite number of parameters in M. In [2], Barwise and Shilipf showed that *any countable model $\mathcal{M}$of* PA *is recursively saturated iff it carries an inductive satisfaction class*; here an *inductive satisfaction class S* of $\mathcal{M}$ is a subset of M which contains $\langle\varphi, a\rangle$ such that (1) $\mathcal{M} \models \mathrm{Form}(\varphi)$, (2) $(\mathcal{M}; S) \models \mathrm{PA}^*$, and (3) $(\mathcal{M}; S)$ satisfies Tarski's inductive conditions for satisfaction (for a more precise definition see [7]). Smoryński in [15], by generalizing Barwise–Ressayre expandability result, proved that *for every countable recursively saturated model $\mathcal{M}$of* PA *there exists some inductive satisfaction class S such that $(\mathcal{M}; S)$ is also recursively saturated.*
- For every cut I of $\mathcal{M}$ the *I-Standard System* of $\mathcal{M}$, denoted by $\mathrm{SSy}_I(\mathcal{M})$, is the family of subsets of I of the form $I \cap a_{\mathrm{E}}$ for some $a \in M$. By $\mathrm{SSy}(\mathcal{M})$ we mean $\mathrm{SSy}_{\mathbb{N}}(\mathcal{M})$. It is well-known that for every model $\mathcal{M}$ of $\mathrm{I}\Sigma_n$ (for $n > 0$), $\mathrm{SSy}_I(\mathcal{M})$ is equal to the family of intersections of Σ_n-definable (with parameters) subsets of $\mathcal{M}$ with I (see [6, Chapter I]). Moreover, it is easy to

check that if $\mathcal{N}$ is an initial segment and a submodel of $\mathcal{M}$ containing I, then $\mathrm{SSy}_I(\mathcal{M}) = \mathrm{SSy}_I(\mathcal{N})$ (see [7]).

- A given model $\mathcal{M}$ of $\mathrm{I}\Delta_0$ is called *1-tall* if $\mathrm{K}^1(\mathcal{M}; a)$ is cofinal in $\mathcal{M}$ for no $a \in M$; and it is called *1-extendable* if it possesses some end extension $\mathcal{N} \models \mathrm{I}\Delta_0$ such that $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) = \mathrm{Th}_{\Sigma_1}(\mathcal{N})$. Dimitracopoulos and Paris [3] showed that:

THEOREM 2.2 (Dimitracopoulos–Paris). (1) *For any two countable and nonstandard models $\mathcal{M}$ and $\mathcal{N}$ of $\mathrm{I}\Delta_0 + \mathrm{Exp}$ such that $\mathcal{M}$ is 1-extendable and $\mathcal{N}$ is 1-tall, there exists a proper initial embedding from $\mathcal{M}$ into $\mathcal{N}$ iff $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$ and $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{N})$.*
(2) *Any 1-tall countable model $\mathcal{M}$ of $\mathrm{B}\Sigma_1 + \mathrm{Exp}$ in which $\mathbb{N}$ is not Π_1-definable (without parameters), is 1-extendable.*

- A cut I of a model $\mathcal{M}$ is called *strong* if for every coded function f of $\mathcal{M}$ whose domain contains I, there exists some $e > I$ such that $f(i) \in I$ iff $f(i) < e$ for all $i \in I$. Paris and Kirby, in [9], proved that *I is a strong cut of a model $\mathcal{M}$ of $\mathrm{I}\Delta_0 + \mathrm{Exp}$ iff $(I, \mathrm{SSy}_I(\mathcal{M})) \models \mathrm{ACA}_0$* (here ACA_0 is the subsystem of second order arithmetic with the comprehension scheme restricted to formulas with no second order quantifiers).
- For given $\mathcal{L}_A$-structures $\mathcal{M}$ and $\mathcal{N}$, an *(a proper) initial embedding j* is an embedding from $\mathcal{M}$ into $\mathcal{N}$ whose image is an (a proper) initial segment of $\mathcal{N}$.

To every self-embedding j of $\mathcal{M}$, we associate two subsets of M:

$$\mathrm{I}_{\mathrm{fix}}(j) := \{m \in M : \forall x \leq m \; j(x) = x\}, \text{ and}$$
$$\mathrm{Fix}(j) := \{m \in M : j(m) = m\}.$$

In [1], it is shown that for every model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$, and any self-embedding j of $\mathcal{M}$, it holds that $K^1(\mathcal{M}) \prec_{\Sigma_1} \mathrm{Fix}(j) \prec_{\Sigma_1} \mathcal{M}$. Consequently, $\mathrm{Fix}(j) \models \mathrm{I}\Delta_0 + \mathrm{Exp}$. The following results on the set of fixed points of initial self-embeddings were also proved in [1]:

THEOREM 2.3 (B-Enayat). *Let $\mathcal{M}$ and $\mathcal{N}$ be countable nonstandard models of $\mathrm{I}\Sigma_1$, $c \in M$ and $d, b \in N$, and let I be a proper cut shared by $\mathcal{M}$ and $\mathcal{N}$ which is closed under exponentiation. Then the following are equivalent:*
(1) *There exists some proper initial embedding j from $\mathcal{M}$ into $\mathcal{N}$ such that $I \subseteq \mathrm{I}_{\mathrm{fix}}(j)$, $j(M) < b$, and $j(c) = d$.*
(2) *$\mathrm{SSy}_I(\mathcal{M}) = \mathrm{SSy}_I(\mathcal{N})$, and for every Δ_0-formula $\delta(z, x, y)$ and every $i \in I$ it holds that*

$$\mathcal{M} \models \exists z\, \delta(z, c, i) \;\Rightarrow\; \mathcal{N} \models \exists z < b\, \delta(z, d, i).$$

REMARK 1. With the above assumptions, suppose $a \in M \cap N$ such that for all Δ_0-formula δ and for every $i \in I$ it holds that

$$\mathcal{M} \models \exists z\, \delta(z, c, (a)_i) \;\Rightarrow\; \mathcal{N} \models \exists z < b\, \delta(z, d, (a)_i).$$

Then, by an appropriate modification in the proof of Theorem 2.3, we can manage to construct the above proper initial embedding j with the additional feature that $j((a)_i) = (a)_i$ for every $i \in I$.

THEOREM 2.4 (B-Enayat). *Suppose $\mathcal{M} \models \mathrm{I}\Sigma_1$ is countable and nonstandard and I is a cut of $\mathcal{M}$. Then the following hold*:

(1) *I is closed under exponentiation iff there exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{I}_{\mathrm{fix}}(j) = I$.*
(2) *I is strong in $\mathcal{M}$ and $I \prec_{\Sigma_1} \mathcal{M}$, iff there exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) = I$.*
(3) *$\mathbb{N}$ is strong in $\mathcal{M}$ iff there exists some proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) = \mathrm{K}^1(\mathcal{M})$.*

The following lemma from [1] will be useful in Section 4 of this paper:

LEMMA 2.5. *Suppose $\mathcal{M} \models \mathrm{I}\Delta_0 + \mathrm{Exp}$ in which $\mathbb{N}$ is not a strong cut, then for any self-embedding j of $\mathcal{M}$, the following hold*:

(1) *The nonstandard fixed points of j are downward cofinal in the nonstandard part of $\mathcal{M}$.*
(2) *For every element $a \in M$, and $m \in \mathrm{Fix}(j)$ there exists an element $b \in \mathrm{Fix}(j)$ such that*

$$\mathrm{Th}_{\Sigma_1}(\mathcal{M}; a, m) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}; b, m).$$

Convention. Suppose $\mathcal{M} \models \mathrm{I}\Sigma_1$ and $\langle \delta_r : r \in M \rangle$ is a canonical enumeration of all Δ_0-formulas within $\mathcal{M}$ as in [6, Chapter I]. To be more precise, $\langle \delta_r : r \in M \rangle$ is an enumeration of all Δ_0-formulas of $\mathcal{M}$ (containing nonstandard formulas) such that for every $r \in M$ it holds that

- δ_r is a standard Δ_0-formula for all standard $r \in M$, and
- for every Δ_0-formula δ there exists some standard $r \in M$ such that $\mathcal{M}$ believes that r is the Gödel number of δ.

(For more details see [6] or [7].)
Now, for every $r \in M$, we define the following notations:

- $f_r(\diamondsuit) = \blacklozenge$ denotes the following partial Σ_1-function in $\mathcal{M}$:

$$\exists z((z)_0 = \blacklozenge \ \wedge \ z = \mu_y \mathrm{Sat}_{\Delta_0}(\delta_r(\diamondsuit, (y)_0, (y)_1))).$$

- The notation $[f_r(\bar{x}) \downarrow]$ denotes the Σ_1-formula $\exists z, y \ \mathrm{Sat}_{\Delta_0}(\delta_r(\bar{x}, y, z))$, and $[f_r(\bar{x}) \downarrow]^{<w}$ stands for the formula $\exists z, y < w \ \mathrm{Sat}_{\Delta_0}(\delta_r(\bar{x}, y, z))$.
- Let $\mathcal{F}(\mathcal{M})$ to be the collection of all $\emptyset$-definable partial Σ_1-functions in $\mathcal{M}$.

First note that, it is shown in [1] that

$$\mathrm{K}^1(\mathcal{M}; a) = \{f(a) : \ f \in \mathcal{F}(\mathcal{M}) \text{ and } \mathcal{M} \models [f(a) \downarrow]\}.$$

Clearly, $f_r \in \mathcal{F}(\mathcal{M})$ for all standard $r \in M$. Moreover, by a little bit of effort we can show that for every $f \in \mathcal{F}(\mathcal{M})$ there exists some standard $r \in M$ such that $f = f_r$ (for details see [1, Lemma 3.1.2]).

As a result, if $\mathcal{M}$ and $\mathcal{N}$ are two models of $\mathrm{I}\Delta_0$ such that $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) = \mathrm{Th}_{\Sigma_1}(\mathcal{N})$, then $\mathcal{F}(\mathcal{M}) = \mathcal{F}(\mathcal{N}) = \mathcal{F} := \{f_n : n \in \mathbb{N}\}$.

§3. *I*-small Σ_1-elementary submodels. In [14], Lascar introduced a class of submodels of models of arithmetic, namely *small* submodels, which resemble those submodels of a model of set theory whose cardinality is less than the cardinality of the ground model. Then, Enayat inspired by a result of Schmerl (stated without proof as Theorem 5.7 in [8]), generalized this notion in [4]. In this section we will prove some results about these submodels.

DEFINITION 1. For a given proper cut I of a model $\mathcal{M}$ of $\mathrm{I}\Delta_0 + \mathrm{Exp}$, subset X of M is called *I-small* in $\mathcal{M}$ if there exists some $a \in M$ such that $X = \{(a)_i : i \in I\}$, and $(a)_i \neq (a)_j$ for all distinct $i, j \in I$. When $I = \mathbb{N}$, we simply use *small* for $\mathbb{N}$-small.

It is easy to see that for every model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$, each proper cut I of $\mathcal{M}$ is I-small. Moreover, for every $a \in M$, $\mathrm{K}^1(\mathcal{M}; a)$ is small in $\mathcal{M}$. In [12], it is shown that every recursively saturated model $\mathcal{M}$ of PA possesses some small submodel which is not finitely generated. This result can be generalized for I-small submodels, when I is a strong cut of $\mathcal{M}$ (see Theorem 3.2). Furthermore, By using compactness arguments, for every model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$, we can find some elementary extension of $\mathcal{M}$ in which it is small. And finally, in [11] it is shown that every nonstandard small submodel is a *mixed* submodel (i.e., neither cofinal, nor initial segment). In a similar manner, for every cut I of a model $\mathcal{M}$ of $\mathrm{I}\Sigma_1$, and each I-small submodel $\mathcal{M}_0$ of $\mathcal{M}$, if $I \subsetneq M_0$ then M_0 is mixed in $\mathcal{M}$. (Since if $M_0 := \{(a)_i : i \in I\}$, and $A := \{i \in I : \mathcal{M} \models \neg i \mathrm{E}(a)_i\}$, then $A \in \mathrm{SSy}_I(\mathcal{M}) \setminus \mathrm{SSy}_I(\mathcal{M}_0)$. So $\mathcal{M}_0$ cannot be an initial segment of $\mathcal{M}$.)

In the following lemma we will show that in the definition of I-small, if I is a strong cut or it is equal to $\mathbb{N}$, then the condition $(a)_i \neq (a)_j$ for all distinct $i, j \in I$, can be eliminated:

LEMMA 3.1. *Suppose $\mathcal{M} \models \mathrm{I}\Sigma_1$ is nonstandard, $I \subsetneq_e \mathcal{M}$, $\mathcal{M}_0$ is a submodel of $\mathcal{M}$ such that $M_0 = \{(a)_i : i \in I\}$ for some $a \in M$. Then the following hold*:

(1) *If $I = \mathbb{N}$, then $\mathcal{M}_0$ is small.*
(2) *If I is strong in $\mathcal{M}$, then $\mathcal{M}_0$ is I-small.*

PROOF. First, we will inductively define the following Δ_0-function (with parameters) in $\mathcal{M}$:

$$g(0) := (a)_0,$$
and
$$g(x+1) := y \text{ iff}$$
$$\exists r < \mathrm{Len}(a) \left(r = \mu_z \left(\exists u < z \left(\begin{array}{c} y = (a)_r \wedge \\ g(x) = (a)_u \wedge \\ \forall w < z\, ((a)_w \neq (a)_z \wedge \exists v \leq u((a)_w = (a)_v)) \end{array} \right) \right) \right).$$

Note that by the way we defined g, its domain is an initial segment of $\mathcal{M}$, and $\mathrm{Dom}(g) \leq \mathrm{Len}(a)$. Moreover, since I and M_0 are not Δ_0-definable in $\mathcal{M}$, then $I \subsetneq \mathrm{Dom}(g)$. So by Σ_1-induction in $\mathcal{M}$, we can find some $d \in M$ such that $(d)_i = g(i)$ for every $i \in I$. Clearly, $(d)_i \neq (d)_j$ for every distinct $i, j \in I$, and $M_0 \subseteq \{(d)_i : i \in I\}$. Now, in each case of the statement of the theorem we will prove that $\{(d)_i : i \in I\} \subseteq M_0$:

(1) Suppose $I = \mathbb{N}$. If $\{(d)_n : n \in \mathbb{N}\} \nsubseteq M_0$, then there exists the least number $n \in \mathbb{N}$ such that $(d)_n \notin M_0$. So by the definition of g, there exist some $m \in \mathbb{N}$ and some $r \in M \setminus \mathbb{N}$ such that $(d)_{n-1} = (a)_m$ and $(d)_n = (a)_r$. Therefore, by the definition of g, it holds that $M_0 = \{(a)_0, \dots, (a)_m\}$, which is a contradiction.

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(2) In the general case with the extra assumption that I is strong in $\mathcal{M}$, consider the following partial Δ_0-function in $\mathcal{M}$:

$$h(x) := \mu_r((d)_x = (a)_r).$$

Since I is strong and $I \subseteq \mathrm{dom}(h)$ (because g is well-defined on I), there exists some $e \in M$ such that $h(i) \in I$ iff $h(i) < e$, for all $i \in I$. Moreover, by the definition of d, g and h, for every $i \in I$ it holds that $(d)_i = (a)_{h(i)}$. So it suffices to prove that $h(i) < e$ for every $i \in I$. Suppose not; so there exists some $i_0 \in I$ which is the least element of M such that $h(i_0) > e$. Now, by the way we defined g and h, it holds that

$$\mathcal{M} \models \forall i < h(i_0)\ ((a)_i \neq (a)_{h(i_0)} \wedge \exists j \leq h(i_0 - 1)((a)_i = (a)_j)).$$

Therefore, $M_0 = \{x \in M : \ \mathcal{M} \models \exists i \leq h(i_0 - 1) \ \ x = (a)_i\}$. So M_0 is Δ_0-definable in $\mathcal{M}$, which is a contradiction. ⊣

In the following theorem, we will show that when I is strong, the basic properties which hold for small submodels, also hold for I-small ones.

Theorem 3.2. *Let $\mathcal{M} \models \mathrm{I}\Sigma_1$ be nonstandard, and let I be a strong cut of $\mathcal{M}$. Then*:

(1) *For every $a \in M$, $\mathrm{K}^1(\mathcal{M}; I \cup \{a\})$ is I-small.*
(2) *If $\mathcal{M}_0$ is an I-small submodel of $\mathcal{M}$, then $I \subseteq M_0$.*
(3) *If $\mathcal{M} \models \mathrm{PA}$ is countable and recursively saturated, then there exists some I-small elementary submodel of $\mathcal{M}$ which is not of the form of $\mathrm{K}(\mathcal{M}; I \cup \{a\})$ for any $a \in M$.*

Proof. (1) First fix some arbitrary $s > I$. So by using strong Σ_1-Collection in $\mathcal{M}$ for the formula $\mathrm{Sat}_{\Delta_0}(\delta_r(i, a, z))$, we will find some $b \in M$ such that

$$\mathcal{M} \models \forall \langle r, i\rangle < s\ ([f_r(i, a)\downarrow] \rightarrow [f_r(i, a)\downarrow]^{<b}).$$

Then, by using Σ_1-induction we observe that $\mathcal{M} \models \exists y \ \ \forall \langle r, i\rangle < s\ \varphi(y, r, i, a, b)$, in which $\varphi(y, r, i, a, b)$ is the following Δ_0-formula:

$$\big(([f_r(i, a)\downarrow]^{<b} \rightarrow (y)_{\langle r,i\rangle} = f_r(i, a)) \ \wedge \ (\neg[f_r(i, a)\downarrow]^{<b} \rightarrow (y)_{\langle r,i\rangle} = 0)\big).$$

As a result, if $d \in M$ is such that $\mathcal{M} \models \forall \langle r, i\rangle < s\ \varphi(d, r, i, a, b)$, then

$$\mathrm{K}^1(\mathcal{M}; I \cup \{a\}) = \{(d)_i : \ i \in I\}.$$

So by Lemma 3.1, $\mathrm{K}^1(\mathcal{M}; I \cup \{a\})$ is I-small in $\mathcal{M}$.

(2) The exact argument used in [4, Theorem 4.5.1] works here: let $M_0 = \{(a)_i : \ i \in I\}$ for some $a \in M$ such that $(a)_i \neq (a)_j$ for all distinct $i, j \in I$. Then put

$$Z := \{\langle y, z\rangle \in M : \ \mathcal{M} \models (a)_y = z\}.$$

Since Z is Δ_0-definable in $\mathcal{M}$, then $X := I \cap Z \in \mathrm{SSy}_I(\mathcal{M})$. As a result, because I is strong in $\mathcal{M}$, $(I; X) \models \mathrm{PA}^*$. Now, suppose $I \not\subseteq M_0$. So $(I; X) \models \exists x \ \ (\forall y \ \ \langle y, x\rangle \notin X)$. Let $(I, X) \models \mathbf{x}_0 := \mu_x(\forall y \ \ \langle y, x\rangle \notin X)$. Therefore, $\mathbf{x}_0 \notin M_0$. So since $\mathbf{x}_0 \neq 0$, and by the definition of $\mathbf{x}_0$, we conclude that $\mathbf{x}_0 - 1 \in M_0$, which contradicts the fact that $\mathcal{M}_0$ is a submodel of $\mathcal{M}$.

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(3) We will generalize the method used in [12, Proposition 2.10]: let S be a nonstandard partial inductive satisfaction class for $\mathcal{M}$ such that $(\mathcal{M}; S)$ is recursively saturated. Put $\mathcal{M}^* := (\mathcal{M}; S)$, and $\mathcal{N} := \mathrm{K}(\mathcal{M}^*; I \cup \{s\})$ for some $s > I$. First, note that N is I-small in $\mathcal{M}$: since $\mathcal{M}^*$ is a countable recursively saturated model of PA^*, so it also possesses an inductive satisfaction class. Moreover, I is also strong in $\mathcal{M}^*$. Therefore, by repeating the argument used in the proof of part (1) of this theorem, and Lemma 3.1(2), we can show that N is I-small in $\mathcal{M}$.
Moreover, on one hand, it is easy to see that $S \cap N$ is a nonstandard satisfaction class for the $\mathcal{L}_A$-structure $\mathcal{N}$. So $\mathcal{N}$ is also a recursively saturated model of PA. On the other hand, I is a proper initial segment of $\mathcal{N}$ (because $s > I$). Therefore, $\mathcal{N}$ is of the form of $\mathrm{K}(\mathcal{M}; I \cup \{a\})$ for no $a \in M$. ⊣

REMARK 2. In part (2) of the above theorem, as the anonymous referee suggested, we do not need the whole strength of PA^*; it suffices that $(I; X) \models \mathrm{I}\Sigma_1$. As a result, if I is a *semiregular* cut of $\mathcal{M}$, then part (2) of the above theorem holds (for the definition of a semiregular cut and their properties, see [13]).

The following lemma will be useful in the proof of the main theorem of this section:

LEMMA 3.3. *Suppose $\mathcal{M} \models \mathrm{I}\Sigma_1$, I is a strong cut of $\mathcal{M}$, and $a \in M \setminus I$ such that $(a)_i \neq (a)_j$ for all distinct $i, j \in I$. Moreover, let $M_0 = \{(a)_i : i \in I\}$ be a Σ_1-elementary submodel of $\mathcal{M}$, $X \subseteq M_0$ be coded in $\mathcal{M}$, and $i_0 \in I$ such that $i < i_0$ for all $(a)_i \in X$. Then X is coded in $\mathcal{M}_0$.*

PROOF. Suppose $\alpha \in M$ codes X in $\mathcal{M}$. So $\mathcal{M} \models \overbrace{\alpha = \sum_{i<i_1} 2^{(\sigma)_i}}^{\delta(\alpha,\sigma,i_1)}$, in which $i_1 = \mathrm{Card}(X) \leq i_0$ and $\sigma := \langle x : x \mathrm{E} \alpha \rangle$ (so $\mathrm{Len}(\sigma) = i_1$). Since $\delta(x, y, z)$ is a Δ_0-formula and $\mathcal{M}_0 \prec_{\Sigma_1} \mathcal{M}$, it suffices to prove that $\sigma \in M_0$. For this purpose let $Y := \{i < i_0 : \mathcal{M} \models (a)_i \mathrm{E} \alpha\}$. Then there exists some $\gamma \in I$ which codes Y.
Now, we define

$$h(z) := \begin{cases} \mu_u(\langle (a)_x : x \mathrm{E} z \rangle = (a)_u \wedge u < \mathrm{Len}(a)), & \text{if } \mathcal{M} \models \exists u < \mathrm{Len}(a)\ \langle (a)_x : x \mathrm{E} z \rangle = (a)_u, \\ 0, & \text{otherwise.} \end{cases}$$

Since I is strong in $\mathcal{M}$, there exists some e such that $h(i) > e$ iff $h(i) > I$, for all $i \in I$. We claim that $\mathcal{M} \models \forall x\ \varphi(x, a, \gamma, e)$, where $\varphi(x, a, \gamma, e)$ is the following Δ_0-formula:

$$\forall y < \mathrm{Len}(x)\ \exists z \mathrm{E} \gamma\ ((x)_y = (a)_z) \rightarrow \exists w < \min\{e, \mathrm{Len}(a)\}\ (x = (a)_w).$$

Therefore, $\mathcal{M} \models \varphi(\sigma, a, \gamma, e)$, which implies that $\sigma = (a)_c$ for some $c < \min\{e, \mathrm{Len}(a)\}$. So $\sigma = (a)_{h(\gamma)}$ and $h(\gamma) < e$, which implies that $\sigma \in M_0$.

In order to prove the above claim, we will use Δ_0-induction inside $\mathcal{M}$: let $\mathbf{x} \in M$ such that $\mathcal{M} \models \varphi(w, a, \gamma, e)$ for every $w < \mathbf{x}$, and $\mathcal{M} \models \forall y < \mathrm{Len}(\mathbf{x})\ \exists z \mathrm{E} \gamma\ ((\mathbf{x})_y = (a)_z)$. So by induction hypothesis $\mathcal{M} \models \mathbf{x} \restriction_{\mathrm{Len}(\mathbf{x})-1} = (a)_{\mathbf{z}}$ for some $\mathbf{z} < \min\{e, \mathrm{Len}(a)\}$. Then, we put $Z := \{i < \gamma : \mathcal{M} \models \exists y < \mathrm{Len}(\mathbf{x}) - 1\ (\mathbf{x})_y = (a)_i\}$, and let $\mathbf{z}_0 \in I$ code Z. As a result, $h(\mathbf{z}_0) \leq \mathbf{z} < \min\{e, \mathrm{Len}(a)\}$, which implies that $\mathbf{x} \restriction_{\mathrm{Len}(\mathbf{x})-1} = (a)_{h(\mathbf{z}_0)} \in M_0$. So since $\mathcal{M}_0 \prec_{\Sigma_1} \mathcal{M}$, then $\mathbf{x}$ is in M_0. Therefore, $\mathbf{x} = (a)_i$ for some $i \in I < \min\{e, \mathrm{Len}(a)\}$. ⊣

Now we are ready to prove the main theorem and corollary of this section. The method we use for proving Theorem 3.4 is a combination of the back-and-forth method used in [1, Theorem 6.1] and [8, Theorem 5.6].

THEOREM 3.4. *Assume $\mathcal{N} \models \mathrm{I}\Sigma_1$ is countable and nonstandard, I is a strong cut of $\mathcal{N}$, and $\mathcal{N}_0$ is an I-small Σ_1-elementary submodel of $\mathcal{N}$ such that $I \neq N_0$. If $\mathcal{M} := \mathrm{H}^1(\mathcal{N}; N_0)$, then there exists some proper initial self-embedding j of $\mathcal{M}$ such that $N_0 = \mathrm{Fix}(j)$.*

In order to prove Theorem 3.4, we will first prove the following lemmas:

LEMMA 3.4.1. *Suppose $\mathcal{N}$, N_0, I, and $\mathcal{M}$ are as in Theorem 3.4. Then I is strong in $\mathcal{M}$ and there exists some $a \in M$ such that $N_0 = \{(a)_i : i \in I\}$ and $(a)_i \neq (a)_i$ for distinct $i, j \in I$.*

PROOF. By Theorem 2.1, $\mathcal{M}$ is a Σ_1-elementary initial segment of $\mathcal{N}$ such that $\mathcal{M} \models \mathrm{I}\Sigma_1$. So it is easy to see that I is also strong in $\mathcal{M}$. Moreover, since $N_0 \neq I$, by using Σ_1-Overspill in $\mathcal{M}$ we can find the desired $a \in M$. ⊣

LEMMA 3.4.2. *Suppose $\mathcal{N}$, N_0, I, $\mathcal{M}$, and $a \in M$ are as in Theorem 3.4 and Lemma* 3.4.1. *Moreover, let $b \in M$, $\bar{u} := u_1, \dots, u_n$, and $\bar{v} := v_1, \dots, v_n < b$ be finite tuples in M. Then the following holds*:

(i) *For every $m \in M$ there exists some $\alpha \in M$ such that $I \cap \alpha_{\mathrm{E}}$ equals to the following set*:

$$C := \left\{\langle r, i\rangle \in I : \mathcal{M} \models [f_r(\bar{u}, m, (a)_i)\downarrow] \text{ and } f_r(\bar{u}, m, (a)_i) \notin \mathrm{K}^1(\mathcal{M}; N_0 \cup \{\bar{u}\})\right\}.$$

(ii) *For every $m' \in M$ there exists some $\alpha' \in M$ such that $I \cap \alpha'_{\mathrm{E}}$ equals to the following set*:

$$C' := \{\langle r, i\rangle \in I : \mathcal{M} \models [f_r(\bar{v}, m', (a)_i)\downarrow]^{<b} \text{ and } f_r(\bar{v}, m', (a)_i) \notin \mathrm{K}^1(\mathcal{M}; N_0 \cup \{\bar{v}\})\}.$$

PROOF. We will prove part (i), and part (ii) will be proved similarly. Let

$$R := \left\{\langle\langle r, i\rangle, k, t\rangle \in I : \mathcal{M} \models \begin{pmatrix} ([f_r(\bar{u}, m, (a)_i)\downarrow] \wedge [f_t(\bar{u}, (a)_k)\downarrow]) \rightarrow \\ f_r(\bar{u}, m, (a)_i) = f_t(\bar{u}, (a)_k) \end{pmatrix}\right\}.$$

On one hand, since R is Π_1-definable in $\mathcal{M}$, then $R \in \mathrm{SSy}_I(\mathcal{M})$. On the other hand, by Lemma 3.2(2), it holds that

$$I \setminus C = \overbrace{\{\langle r, i\rangle \in I : (I, R) \models \exists k, t\ \langle\langle r, i\rangle, k, t\rangle \in R\}}^{B}.$$

Since I is strong in $\mathcal{M}$, which implies that $(I, \mathrm{SSy}_I(\mathcal{M})) \models \mathrm{ACA}_0$, and because B is arithmetical in R and $R \in \mathrm{SSy}_I(\mathcal{M})$, we may deduce that $B \in \mathrm{SSy}_I(\mathcal{M})$, and consequently $C \in \mathrm{SSy}_I(\mathcal{M})$. ⊣

LEMMA 3.4.3. *Suppose $\mathcal{N}$, I, N_0, $\mathcal{M}$, and $a \in M$ are as in Theorem 3.4 and Lemma* 3.4.1. *Moreover, let $b \in M$, $\bar{u} := u_1, \dots, u_n$, and $\bar{v} := v_1, \dots, v_n$ be finite tuples in M such that*

$$\mathcal{M} \models (\bar{v} < b \ \wedge\ \mathrm{P}(\bar{u}, \bar{v}) \ \wedge \mathrm{Q}(\bar{u}, \bar{v})), \ \textit{where}$$

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$\mathrm{P}(\bar{u},\bar{v}) \equiv [f(\bar{u},(a)_i)\downarrow] \to [f(\bar{v},(a)_i)\downarrow]^{<b}$, *for all* $f \in \mathcal{F}$ *and* $i \in I$;
and
$\mathrm{Q}(\bar{u},\bar{v}) \equiv \begin{pmatrix}[f(\bar{u},(a)_i)\downarrow] \wedge [f(\bar{v},(a)_i)\downarrow]^{<b} \wedge \\ f(\bar{u},(a)_i) \notin N_0\end{pmatrix} \Rightarrow f(\bar{u},(a)_i) \neq f(\bar{v},(a)_i)$, *for all* $f \in \mathcal{F}$ *and all* $i \in I$.

Then the following holds:

(i) *If* $(a)_{\mathrm{i}} \in N_0$ *and* $m \in M$ *such that* $m \leq t(\bar{u},(a)_{\mathrm{i}})$ *for some* $t \in \mathcal{F}$ *and* $\alpha \in M$ *as in Lemma* 3.4.2(i), *then for every natural number* $k > 0$ *and any* $(k+1)$*-many elements* $f, f_{n_1}, \dots, f_{n_k}$ *of* $\mathcal{F}$ *and each* $z \in N_0$ *there exist some* $s > I$ *and some* $m' < b$ *such that* $\mathcal{M} \models \Psi(f, f_{n_1}, \dots, f_{n_k}, \bar{u}, m, \bar{v}, m', b, a, s, \alpha, z, (a)_{\mathrm{i}})$, *where* Ψ *is the following* Π_1*-formula*:

$$m' \leq t(\bar{v},(a)_{\mathrm{i}}) \;\wedge \begin{pmatrix}\forall i < s([f(\bar{u},m,(a)_i,z)\downarrow] \to [f(\bar{v},m',(a)_i,z)\downarrow]^{<b}) \wedge \\ \forall i < s \bigwedge_{t\leq k}\begin{pmatrix}([f_{n_t}(\bar{v},m',(a)_i)\downarrow]^{<b} \wedge \langle n_t, i\rangle \mathrm{E}\alpha) \to \\ f_{n_t}(\bar{u},m,(a)_i) \neq f_{n_t}(\bar{v},m',(a)_i)\end{pmatrix}\end{pmatrix}.$$

(ii) *If* $m' < \max\{\bar{v}\}$ *and* $\alpha' \in M$ *as in Lemma* 3.4.2(ii), *then for every natural number* $k > 0$ *and any* $(k+1)$*-many elements* $f, f_{n_1}, \dots, f_{n_k}$ *of* $\mathcal{F}$ *and each* $z \in N_0$ *there exist some* $s > I$ *and some* $m < \max\{\bar{u}\}$ *such that* $\mathcal{M} \models \Psi'(f, f_{n_1}, \dots, f_{n_k}, \bar{u}, m, \bar{v}, m', b, a, s, \alpha, z)$, *where* Ψ' *is the following* Π_1*-formula*:

$$\begin{pmatrix}\forall i < s(\neg[f(\bar{v},m',(a)_i,z)\downarrow]^{<b} \to \neg[f(\bar{u},m,(a)_i,z)\downarrow]) \wedge \\ \forall i < s \bigwedge_{t\leq k}\begin{pmatrix}(\langle n_t, i\rangle \mathrm{E}\alpha' \wedge [f_{n_t}(\bar{u},m,(a)_i)\downarrow]) \to \\ f_{n_t}(\bar{u},m,(a)_i) \neq f_{n_t}(\bar{v},m',(a)_i)\end{pmatrix}\end{pmatrix}.$$

PROOF. (i) First note that since $\mathrm{P}(\bar{u},\bar{v})$ holds in $\mathcal{M}$, Theorem 2.3 and Remark 1 imply that

(1): There exists some initial self-embedding j_0 of $\mathcal{M}$ such that $j_0(M) < b$, $j_0(\bar{u}) = \bar{v}$, and $N_0 \subseteq \mathrm{Fix}(j_0)$.

Now, suppose that part (i) of this lemma does not hold; i.e., there is some $k > 0$ for which there exist $(k+1)$-many elements $\{f, f_{n_1}, \dots, f_{n_k}\}$ of $\mathcal{F}$, and some $\mathbf{z} \in N_0$ such that for all $s > I$ it holds that

$$\mathcal{M} \models \forall y < b\; \neg\Psi(f, f_{n_1}, \dots, f_{n_k}, \bar{u}, m, \bar{v}, y, b, a, s, \alpha, \mathbf{z}, (a)_{\mathrm{i}}).$$

Therefore, by Σ_1-Underspill in $\mathcal{M}$, there exists some $s \in I$ such that

$$\mathcal{M} \models \forall y < b\; \neg\Psi(f, f_{n_1}, \dots, f_{n_k}, \bar{u}, m, \bar{v}, y, b, a, s, \alpha, \mathbf{z}, (a)_{\mathrm{i}}).$$

(2): Let $k_0 \in \mathbb{N}$ be the least natural number, for which there exists a set $\{f, f_{n_1}, \dots, f_{n_{k_0}}\}$ of elements of $\mathcal{F}$, some $\mathbf{z}_0 \in N_0$, and some $s_0 \in I$ such that

$$\mathcal{M} \models \forall y < b\; \neg\Psi(f, f_{n_1}, \dots, f_{n_{k_0}}, \bar{u}, m, \bar{v}, y, b, a, s_0, \alpha, \mathbf{z}_0, (a)_{\mathrm{i}}).$$

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Put

$$X := \{x \in M : \mathcal{M} \models \exists i < s_0(x = (a)_i \wedge [f(\bar{u}, m, (a)_i, \mathbf{z}_0)\downarrow])\}$$
and
$$X' := \{\langle n, x\rangle \in M : \mathcal{M} \models \exists i < s_0(x = (a)_i \wedge \bigvee_{t=1}^{k_0} n = n_t \wedge \langle n, i\rangle \mathrm{E}\alpha)\}.$$

By Lemma 3.3, there exist $(a)_\xi \in N_0$ and $(a)_\zeta \in N_0$ which code X and X', respectively. So we can restate statement (2) in the following form:

(3): Let $k_0 \in \mathbb{N}$ be the least natural number, for which there exists a set $\{f, f_{n_1}, \ldots, f_{n_{k_0}}\}$ of elements of $\mathcal{F}$, some $\mathbf{z}_0, (a)_\zeta, (a)_\xi \in N_0$ such that

$$\mathcal{M} \models \forall y \leq t(\bar{v}, (a)_{\mathrm{i}}) \left(\begin{array}{c} \forall \varepsilon < (a)_\xi (\varepsilon \mathrm{E}(a)_\xi \to [f(\bar{v}, y, \varepsilon, \mathbf{z}_0)\downarrow]^{<b} \to \\ \exists \varepsilon < \mathrm{E}(a)_\zeta \bigvee_{t=1}^{k_0} \left(\begin{array}{c} \langle n_t, \varepsilon\rangle \mathrm{E}(a)_\zeta \ \wedge [f_{n_t}(\bar{v}, y, \varepsilon)\downarrow]^{<b} \wedge \\ f_{n_t}(\bar{u}, m, \varepsilon) = f_{n_t}(\bar{v}, y, \varepsilon) \end{array} \right) \end{array} \right).$$

Our plan is to consider two cases $k_0 = 1$ and $k_0 > 1$, and in each case obtain a contradiction. But before dividing the cases we will define some Σ_1-functions which will help us in achieving the desired contradictions. First, by considering the code of the sequence $\langle f_{n_t}(\bar{u}, m, \varepsilon) : \langle n_t, \varepsilon\rangle \mathrm{E}(a)_\zeta\rangle$ in $\mathcal{M}$, we may quantify out $f_{n_t}(\bar{u}, m, \varepsilon)$s from the formula in statement (3); in other words, there exists some $\mathbf{x} \in M$ such that $(\mathbf{x})_{\langle n_t, \varepsilon\rangle} = f_{n_t}(\bar{u}, m, \varepsilon)$ for every $\langle n_t, \varepsilon\rangle \mathrm{E}(a)_\zeta$. So we will deduce that

(4): $\mathcal{M} \models \exists x \forall y \leq t(\bar{v}, (a)_{\mathrm{i}})\ \ \theta(y, b, \bar{v}, x, (a)_\xi, (a)_\zeta, \mathbf{z}_0)$, where $\theta(y, b, \bar{v}, x, (a)_\xi, (a)_\zeta, \mathbf{z}_0)$ is the following Δ_0-formula:

$$\left(\begin{array}{c} \forall \varepsilon < (a)_\xi (\varepsilon \mathrm{E}(a)_\xi \to [f(\bar{v}, y, \varepsilon, \mathbf{z}_0)\downarrow]^{<b}) \to \\ \exists \langle n_t, \varepsilon\rangle \mathrm{E}(a)_\zeta \left([f_{n_t}(\bar{v}, y, \varepsilon)\downarrow]^{<b} \wedge (x)_{\langle n_t, \varepsilon\rangle} = f_{n_t}(\bar{v}, y, \varepsilon) \right) \end{array} \right).$$

Then, we will define Σ_1-definable partial functions $b(\diamondsuit, y, (a)_\xi, (a)_\zeta, \mathbf{z}_0)$ and $g(\diamondsuit, (a)_\xi, (a)_\zeta, \mathbf{z}_0, (a)_{\mathrm{i}})$, as follows (we omit the parameters $(a)_\xi$, $(a)_\zeta$, $(a)_{\mathrm{i}}$, and $\mathbf{z}_0$ in the presentations of these functions for the sake of simplicity):

– $b(\diamondsuit, y) := \min \left\{ w : \left(\begin{array}{c} \forall \varepsilon \mathrm{E}(a)_\xi \ ([f(\diamondsuit, y, \varepsilon, \mathbf{z}_0)\downarrow]^{<w}) \wedge \\ \forall \langle n_t, \varepsilon\rangle \mathrm{E}(a)_\zeta \ ([f_{n_t}(\diamondsuit, y, \varepsilon)\downarrow]^{<w}) \end{array} \right) \right\}.$

– $g(\diamondsuit) := x$ iff $\exists z \left(z = \mu_w \ \forall y \leq t(\diamondsuit, (a)_{\mathrm{i}}) \left(\begin{array}{c} (z)_0 = x \wedge \\ [b(\diamondsuit, y)\downarrow]^{<(w)_1} \to \\ \theta(y, b(\diamondsuit, y), \diamondsuit, (w)_0, (a)_\xi, (a)_\zeta, \mathbf{z}_0) \end{array} \right) \right)$;
and $g_t(\diamondsuit, \varepsilon) := (g(\diamondsuit))_{\langle n_t, \varepsilon\rangle}$, for every $\langle n_t, \varepsilon\rangle \mathrm{E}(a)_\zeta$.

From the definition of $g_t(\bar{v}, \varepsilon)$s and statement (4) we may infer that

(5): $\mathcal{M} \models \forall y \leq t(\bar{v}, (a)_{\mathrm{i}}) \left(\begin{array}{c} ([b(\bar{v}, y)\downarrow]^{<b} \ \wedge \ \forall \varepsilon < (a)_\xi (\varepsilon \mathrm{E}(a)_\xi \to [f(\bar{v}, y, \varepsilon, \mathbf{z}_0)\downarrow]^{<b(\bar{v}, y)})) \to \\ \exists \langle n_t, \varepsilon\rangle \mathrm{E}(a)_\zeta \left(\begin{array}{c} [f_{n_t}(\bar{v}, y, \varepsilon)\downarrow]^{<b(\bar{v}, y)} \wedge [g_t(\bar{v}, \varepsilon)\downarrow]^{<b(\bar{v}, y)} \wedge \\ g_t(\bar{v}, \varepsilon) = f_{n_t}(\bar{v}, y, \varepsilon) \end{array} \right) \end{array} \right).$

It is not difficult to express the formula in the statement (5) in the form of $\forall z < b\, \delta(\bar{v}, (a)_\xi, (a)_\zeta, \mathbf{z}_0)$ for some Δ_0-formula δ. Therefore, by the property $\mathrm{P}(\bar{u},\bar{v})$, the definition of function s, and statement (5) we deduce that

$$(6):\ \mathcal{M} \models \forall y \leq t(\bar{u},(a)_{\mathrm{i}}) \begin{pmatrix} ([b(\bar{u},y)\downarrow] \wedge \forall \varepsilon < (a)_\xi(\varepsilon \mathrm{E}(a)_\xi \to [f(\bar{u},y,\varepsilon,\mathbf{z}_0)\downarrow]^{<b(\bar{u},y)})) \to \\ \exists \langle n_t,\varepsilon\rangle \mathrm{E}(a)_\zeta \begin{pmatrix} [f_{n_t}(\bar{u},y,\varepsilon)\downarrow]^{<b(\bar{u},y)} \wedge [g_t(\bar{u},\varepsilon)\downarrow]^{<b(\bar{u},y)} \wedge \\ g_t(\bar{u},\varepsilon) = f_{n_t}(\bar{u},y,\varepsilon) \end{pmatrix} \end{pmatrix}.$$

Finally, we will simultaneously define two more Σ_1-definable functions in $\mathcal{M}$:

$$\langle o(\diamondsuit,y),h(\diamondsuit,y)\rangle$$
$$:= \min \left\{ \langle n_t,\varepsilon\rangle \mathrm{E}(a)_\zeta : \begin{pmatrix} [b(\diamondsuit,y)\downarrow]\wedge \\ [f_{n_t}(\diamondsuit,y,\varepsilon)\downarrow]^{<b(\diamondsuit,y)} \wedge [g_t(\diamondsuit,\varepsilon)\downarrow]^{<b(\diamondsuit,y)}\wedge \\ g_t(\diamondsuit,\varepsilon) = f_{n_t}(\diamondsuit,y,\varepsilon) \end{pmatrix} \right\}.$$

(Note that, similar to the way we defined function g, we can express the above definition by a Σ_1-formula.) Then, by statement (5) it holds that

$$(7):\ \mathcal{M} \models \forall y \leq t(\bar{v},(a)_{\mathrm{i}}) \begin{pmatrix} ([b(\bar{v},y)\downarrow]^{<b} \wedge \forall \varepsilon < (a)_\xi(\varepsilon \mathrm{E}(a)_\xi \to [f(\bar{v},y,\varepsilon,\mathbf{z}_0)\downarrow]^{<b})) \to \\ [\langle o(\bar{v},y),h(\bar{v},y)\rangle\downarrow] \end{pmatrix}.$$

Similarly, from statement (6) we may deduce that

$$(8):\ \mathcal{M} \models \forall y \leq t(\bar{u},(a)_{\mathrm{i}}) \begin{pmatrix} ([b(\bar{u},y)\downarrow] \wedge \forall \varepsilon < (a)_\xi(\varepsilon \mathrm{E}(a)_\xi \to [f(\bar{u},y,\varepsilon,\mathbf{z}_0)\downarrow]^{<b(\bar{u},y)})) \to \\ [\langle o(\bar{u},y),h(\bar{u},y)\rangle\downarrow] \end{pmatrix}.$$

Now, we are ready to examine the mentioned underlined cases for k_0:

– If $k_0 > 1$: By using Lemma 3.3, let $(a)_\rho \in N_0$ be the code of the following subset of N_0:

$$A := \left\{ \langle o(\bar{v},y),h(\bar{v},y)\rangle : \mathcal{M} \models \begin{pmatrix} y \leq t(\bar{v},(a)_{\mathrm{i}}) \wedge [\langle o(\bar{v},y),h(\bar{v},y)\rangle\downarrow]^{<b} \wedge \\ \forall \varepsilon < (a)_\xi(\varepsilon \mathrm{E}(a)_\xi \to [f(\bar{v},y,\varepsilon,\mathbf{z}_0)\downarrow]^{<b} \wedge \\ \exists \varepsilon < (a)_\zeta \begin{pmatrix} \langle n_1,\varepsilon\rangle \mathrm{E}(a)_\zeta \wedge [f_{n_1}(\bar{v},y,\varepsilon)\downarrow]^{<b} \wedge \\ f_{n_1}(\bar{v},y,\varepsilon) = f_{n_1}(\bar{u},m,\varepsilon) \end{pmatrix} \end{pmatrix} \right\}.$$

So, by statements (3), (7), and the definition of $(a)_\rho$, we conclude that

$$(9):\ \mathcal{M} \models \forall y \leq t(\bar{v},(a)_{\mathrm{i}}) \begin{pmatrix} \forall \varepsilon < (a)_\xi \begin{pmatrix} \varepsilon \mathrm{E}(a)_\xi \to [f(\bar{v},y,\varepsilon,\mathbf{z}_0)\downarrow]^{<b} \wedge \\ [\langle o(\bar{v},y),h(\bar{v},y)\rangle\downarrow]^{<b} \wedge \\ \neg\langle o(\bar{v},y),h(\bar{v},y)\rangle \mathrm{E}(a)_\rho \end{pmatrix} \to \\ \exists \varepsilon < \mathrm{E}(a)_\zeta \bigvee_{t=2}^{k_0} \begin{pmatrix} \langle n_t,\varepsilon\rangle \mathrm{E}(a)_\zeta \wedge [f_{n_t}(\bar{v},y,\varepsilon)\downarrow]^{<b} \wedge \\ f_{n_t}(\bar{u},m,\varepsilon) = f_{n_t}(\bar{v},y,\varepsilon) \end{pmatrix} \end{pmatrix}.$$

Let $f' \in \mathcal{F}$ such that

$$f'(\diamondsuit,y,\varepsilon,\langle \mathbf{z}_0,(a)_\rho,(a)_\zeta,(a)_\xi\rangle) = x$$
$$\text{iff}$$
$$x = f(\diamondsuit,y,\varepsilon,\mathbf{z}_0) \wedge [\langle o(\diamondsuit,y),h(\diamondsuit,y)\rangle\downarrow] \wedge \neg\langle o(\diamondsuit,y),h(\diamondsuit,y)\rangle \mathrm{E}(a)_\rho.$$

So by considering f' instead of f in statement (3) and $\langle \mathbf{z}_0,(a)_\rho,(a)_\zeta,(a)_\xi\rangle$ instead of $\mathbf{z}_0$, statement (9) leads to contradiction with the minimality of k_0.

– If $k_0 = 1$: In this case our plan for obtaining a contradiction is as follows: on one hand, by the definitions of $h(\bar{u},y)$ and $(a)_\zeta$, $\mathcal{M}$ thinks that the cardinality of $\{h(\bar{u},y) : \mathcal{M} \models (y < t(\bar{u},(a)_{\mathrm{i}}) \wedge [h(\bar{u},y)\downarrow])\}$ is at most s_0. On the other hand, for every $i \in I$ we will inductively

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define some Σ_1-functions, namely $w(\bar{u}, i)$s, such that $\mathcal{M} \models [w(\bar{u}, i)\downarrow]^{<t(\bar{u},(a)_{\mathrm{i}})}$ and $\mathcal{M}$ believes that there is a bijection between members of $\{h(\bar{u}, w(\bar{u}, i)) : \ i \in I$ and $\mathcal{M} \models [h(\bar{u}, w(\bar{u}, i))\downarrow]\}$ and the elements of I. So the contradiction is achieved. To be more precise, we define

$$w(\diamondsuit, 0) := \min \left\{ y \leq t(\diamondsuit, (a)_{\mathrm{i}}) : \begin{pmatrix} [b(\diamondsuit, y)\downarrow] \ \wedge \ [h(\diamondsuit, y)\downarrow]^{<(a)_\zeta} \wedge \\ \forall \varepsilon < (a)_\xi (\varepsilon \mathrm{E}(a)_\xi \rightarrow [f(\diamondsuit, y, \varepsilon, \mathbf{z}_0)\downarrow]^{<b(\diamondsuit, y)}) \end{pmatrix} \right\},$$

and
$w(\diamondsuit, i+1) := \min\{y \leq t(\diamondsuit, (a)_{\mathrm{i}}) : \varphi(\diamondsuit, i, y, (a)_\zeta, (a)_\xi, \mathbf{z}_0)\}$, where $\varphi(\diamondsuit, i, y, (a)_\zeta, (a)_\xi, \mathbf{z}_0)$ is the following formula:

$$\begin{pmatrix} [b(\diamondsuit, y)\downarrow] \ \wedge [h(\diamondsuit, y)\downarrow]^{<(a)_\zeta} \wedge \\ \forall \varepsilon < (a)_\xi (\varepsilon \mathrm{E}(a)_\xi \rightarrow [f(\diamondsuit, y, \varepsilon, \mathbf{z}_0)\downarrow]^{<b(\diamondsuit, y)}) \wedge \\ \forall x \leq i \begin{pmatrix} \begin{pmatrix} [h(\diamondsuit, w(\diamondsuit, x))\downarrow]^{<(a)_\zeta} \wedge \\ [f_{n_1}(\diamondsuit, y, h(\diamondsuit, w(\diamondsuit, x)))\downarrow]^{<b(\diamondsuit, y)} \wedge \\ [f_{n_1}(\diamondsuit, w(\diamondsuit, x), h(\diamondsuit, w(\diamondsuit, x)))\downarrow]^{<b(\diamondsuit, w(\diamondsuit, x))} \end{pmatrix} \rightarrow \\ f_{n_1}(\diamondsuit, y, h(\diamondsuit, w(\diamondsuit, x))) \neq f_{n_1}(\diamondsuit, w(\diamondsuit, x), h(\diamondsuit, w(\diamondsuit, x))) \end{pmatrix} \end{pmatrix}.$$

First, we will show that $\mathcal{M} \models [w(\bar{u}, i)\downarrow]$ for all $i \in I$. Otherwise, there exists the least $0 < i_0 \in I$ such that

(10): $\mathcal{M} \models \forall y \leq t(\bar{u}, (a)_{\mathrm{i}}) \ \neg\varphi(\bar{u}, i_0, y, (a)_\zeta, (a)_\xi, \mathbf{z}_0)$.

Note that by the definition of $(a)_\xi$ and $(a)_\zeta$ it holds that

(11): $\mathcal{M} \models ([b(\bar{u}, m)\downarrow] \ \wedge \ \forall \varepsilon < (a)_\xi (\varepsilon \mathrm{E}(a)_\xi \rightarrow [f(\bar{u}, m, \varepsilon, \mathbf{z}_0)\downarrow]^{<b(\bar{u}, m)})$.

So by statements (8), (10), and (11), there exists some $i_1 < i_0$ such that

$$(12): \mathcal{M} \models \begin{pmatrix} [h(\bar{u}, w(\bar{u}, i_1))\downarrow] \wedge [f_{n_1}(\bar{u}, m, h(\bar{u}, w(\bar{u}, i_1)))\downarrow] \wedge \\ [f_{n_1}(\bar{u}, w(\bar{u}, i_1), h(\bar{u}, w(\bar{u}, i_1)))\downarrow] \wedge \\ f_{n_1}(\bar{u}, m, h(\bar{u}, w(\bar{u}, i_1))) = f_{n_1}(\bar{u}, w(\bar{u}, i_1), h(\bar{u}, w(\bar{u}, i_1))) \end{pmatrix}.$$

Clearly, $f_{n_1}(\bar{u}, w(\bar{u}, i_1), h(\bar{u}, w(\bar{u}, i_1))) \in \mathrm{K}^1(\mathcal{M}; N_0 \cup \{\bar{u}\})$. So by statement (12), $f_{n_1}(\bar{u}, m, h(\bar{u}, w(\bar{u}, i_1))) \in \mathrm{K}^1(\mathcal{M}; M_0 \cup \{\bar{u}\})$. So $\mathcal{M} \models \neg\langle n_1, h(\bar{u}, w(\bar{u}, i_1))\rangle \mathrm{E}(a)_\zeta$ (by the definition of $(a)_\zeta$), which is in contradiction with the definition of the function h.

Then, by the definition of $w(\bar{u}, i)$s and statement (8), the function $i \mapsto h(\bar{u}, w(\bar{u}, i))$ from $\{i : \ i \leq s_0 + 1\}$ into $((a)_\zeta)_{\mathrm{E}}$ is well-defined and coded in $\mathcal{M}$. So, since the cardinality of $(a)_\zeta$ is less than $s_0 + 1$, by Σ_1-Pigeonhole Principle in $\mathcal{M}$, there exists some distinct $i_0 < i_1 \leq s_0 + 1$ such that

(13): $\mathcal{M} \models h(\bar{u}, w(\bar{u}, i_0)) = h(\bar{u}, w(\bar{u}, i_1))$.

Therefore, by statement (13) and the definition of h we conclude that

$$(14): \mathcal{M} \models \begin{pmatrix} [g_1(\bar{u}, h(\bar{u}, w(\bar{u}, i_0)))\downarrow] \wedge [g_1(\bar{u}, h(\bar{u}, w(\bar{u}, i_1)))\downarrow] \wedge \\ g_1(\bar{u}, h(\bar{u}, w(\bar{u}, i_0))) = g_1(\bar{u}, h(\bar{u}, w(\bar{u}, i_1))) \end{pmatrix}.$$

Moreover, by the definition of $h(\bar{u}, w(\bar{u}, i))$, for $i = i_0, i_1$ it holds that

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(15): $\mathcal{M} \models \begin{pmatrix} [f_{n_1}(\bar{u}, w(\bar{u}, i), h(\bar{u}, w(\bar{u}, i))) \downarrow] \wedge [g_1(\bar{u}, h(\bar{u}, w(\bar{u}))) \downarrow] \wedge \\ f_{n_1}(\bar{u}, w(\bar{u}, i), h(\bar{u}, w(\bar{u}, i))) = g_1(\bar{u}, h(\bar{u}, w(\bar{u}))) \end{pmatrix}$.

So statements (13–15) imply that

(16): $\mathcal{M} \models f_{n_1}(\bar{u}, w(\bar{u}, i_1), h(\bar{u}, w(\bar{u}, i_0))) = f_{n_1}(\bar{u}, w(\bar{u}, i_0), h(\bar{u}, w(\bar{u}, i_0)))$.
But statement (16) is in contradiction with the definition of $w(\bar{u}, i_1)$.

(ii) Part (ii) of this lemma will be proved exactly like part (i), except we need an extra statement between statements (3) and (4): by using Σ_1-Collection, we deduce that

(3'): $\mathcal{M} \models \exists w \forall x < u_0 \begin{pmatrix} \forall \varepsilon < (a)_\lambda \, ([f(\bar{u}, x, \varepsilon, \mathbf{z}_0) \downarrow]^{<w} \to \varepsilon \mathrm{E}(a)_\lambda) \to \\ \exists \varepsilon < (a)_\eta \bigvee_{t \leq k_1} \begin{pmatrix} \langle n_t, \varepsilon \rangle \mathrm{E}(a)_\eta \wedge [f_{n_t}(\bar{u}, x, \varepsilon) \downarrow]^{<w} \wedge \\ f_{n_t}(\bar{u}, x, \varepsilon) = f_{n_t}(\bar{v}, m', \varepsilon) \end{pmatrix} \end{pmatrix}$;

in which $(a)_\lambda \in N_0$ and $(a)_\eta \in N_0$ code the following Y and Y', respectively:

$$Y := \{x \in M : \mathcal{M} \models \exists i < s_0(x = (a)_i \wedge [f(\bar{v}, m', (a)_i, \mathbf{z}_0) \downarrow]^{<b})\}, \text{ and}$$

$$Y' := \{\langle n, x \rangle \in M : \mathcal{M} \models \exists i < s_0(x = (a)_i \wedge \bigvee_{t=0}^{k_0} n = n_t \wedge \langle n, i \rangle \mathrm{E}\alpha')\}.$$

⊣

Now we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. By Lemma 3.4.1, there exists some $a \in M$ such that $N_0 = \{(a)_i : i \in I\}$ and $(a)_i \neq (a)_i$ for distinct $i, j \in I$. Now, in order to construct j, first by using strong Σ_1-Collection in $\mathcal{M}$, we will find some $b \in M$ such that

$$\mathcal{M} \models [f((a)_i) \downarrow] \to [f((a)_i) \downarrow]^{<b}, \text{ for all } f \in \mathcal{F} \text{ and all } i \in I.$$

Then, by using back-and-forth method we will inductively build finite functions $\bar{u} \mapsto \bar{v}$ such that $\bar{u}, \bar{v} \in M$, and $\mathcal{M} \models (\bar{v} < b \ \wedge \ \mathrm{P}(\bar{u}, \bar{v}) \ \wedge \mathrm{Q}(\bar{u}, \bar{v}))$, where $\mathrm{P}(\bar{u}, \bar{v})$ and $\mathrm{Q}(\bar{u}, \bar{v})$ are properties as stated in Lemma 3.4.3. Through the "forth" stages of back-and-forth we shall make the domain of j to be equal to M, and "back" stages are for making the range of j to be an initial segment of $\mathcal{M}$. For the first step of induction, we will choose $0 \mapsto 0$. Then, suppose $\bar{u} \mapsto \bar{v}$ is built such that $\mathcal{M} \models (\bar{v} < b \ \wedge \ \mathrm{P}(\bar{u}, \bar{v}) \ \wedge \mathrm{Q}(\bar{u}, \bar{v}))$.

"Forth" stages: Let $m \in M \setminus \{\bar{u}\}$ be arbitrary. By the definition of $\mathcal{M}$, without loss of generality, we can assume that $m \leq t(\bar{u}, (a)_\mathrm{i})$ for some $t \in \mathcal{F}$ and i $\in I$. Moreover, let $\alpha \in M$ be as in Lemma 3.4.2. In order to find some image for m, for every $s \in M$ we define the following bounded Π_1-type:

$$p_s(y) := \{y \leq t(\bar{v}, (a)_\mathrm{i})\} \cup p_{s1}(y) \cup p_{s2}(y), \text{where}$$
$$p_{s1}(y) := \{\forall i < s([f(\bar{u}, m, (a)_i) \downarrow] \to [f(\bar{v}, y, (a)_i) \downarrow]^{<b}) : f \in \mathcal{F}\};$$
and
$$p_{s2}(y) := \left\{ \forall i < s \begin{pmatrix} ([f_n(\bar{v}, y, (a)_i) \downarrow]^{<b} \wedge \langle n, i \rangle \mathrm{E}\alpha) \to \\ f_n(\bar{u}, m, (a)_i) \neq f_n(\bar{v}, y, (a)_i) \end{pmatrix} : n \in \mathbb{N} \right\}.$$

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We claim that there exists some $s > I$ such that the type $p_s(y)$ is finitely satisfiable in $\mathcal{M}$. Then since p_s is a Π_1, bounded and recursive type in $\mathcal{M}$, there exists some m' which realizes $p_s(y)$ in $\mathcal{M}$. Therefore, $\mathrm{P}((\bar{u}, m), (\bar{v}, m'))$ and $\mathrm{Q}((\bar{u}, m), (\bar{v}, m'))$ hold in $\mathcal{M}$, and this finishes the "forth" stage.

In order to prove the above claim, let $d > I$ be an arbitrary and fixed element of M. Moreover, suppose $i, s \in M$, and $\Theta(s, i, \bar{u}, m, \bar{v}, b, a, \alpha, \beta, (a)_{\mathrm{i}})$ is the following Δ_0-formula:

$$\forall r < i\ \exists y \leq t(\bar{v}, (a)_{\mathrm{i}}) \left(\begin{array}{c} \forall w < s(\langle r, w\rangle \mathrm{E}\beta \rightarrow [f_r(\bar{v}, y, (a)_w)\downarrow]^{<b}) \wedge \\ \forall w < s \forall r' < i \begin{pmatrix} ([f_{r'}(\bar{v}, y, (a)_w)\downarrow]^{<b} \wedge \langle r', w\rangle \mathrm{E}\alpha) \rightarrow \\ f_{r'}(\bar{u}, m, (a)_w) \neq f_{r'}(\bar{v}, y, (a)_w) \end{pmatrix} \end{array} \right),$$

where β is the code of the following Σ_1-definable set in $\mathcal{M}$:

$$L := \{\langle r, w\rangle < d : \mathcal{M} \models [f_r(\bar{u}, m, (a)_w)\downarrow]\}.$$

Note that the parameter i in Θ is for bounding indexes of functions which appear in Θ. Moreover, the parameter s is for bounding elements of N_0 appearing in Θ; i.e., if Θ contains some element of the form $(a)_w$, then $w < s$. Now, for every $i \in M$, we let $\mathrm{G}(i)$ be an upper bound for parameters s as above. To be more precise, we define

$$\mathrm{G}(i) := \max\{x < d : \mathcal{M} \models \Theta(x, i, \bar{u}, m, \bar{v}, b, a, \alpha, \beta, (a)_{\mathrm{i}})\}.$$

Clearly G is Δ_0-definable function in $\mathcal{M}$, and $I \subseteq \mathrm{Dom}(\mathrm{G})$ (we assume $\max(\emptyset) = 0$). Therefore, since I is strong, there exists some $e > I$ such that for all $i \in I$, $\mathrm{G}(i) > I$ iff $\mathrm{G}(i) > e$. We will show that $p_e(y)$ is finitely satisfiable in $\mathcal{M}$:

First, note that, $p_{e1}(y)$ is closed under conjunctions. (This holds similar to the way statement (1) in the proof of Lemma 3.4.3 holds.) So let $f_n, f_{n_1}, \dots, f_{n_k}$ be some finite number of elements of $\mathcal{F}$, and let $n^* = \max\{n, n_0, \dots, n_k\}$. Then, use Lemma 3.4.3(i), (n^*+2)-many times; i.e., for every $t = 0, \dots, n^*+1$ consider f_t instead of f in the assertion of Lemma 3.4.3(i), $0 \in M_0$ instead of z, and $f_1, \dots, f_{n^*}$. So by Lemma 3.4.3(i), for every $t = 0, \dots, n^*+1$ there exists some $s_t > I$ such that

(1): $\mathcal{M} \models \exists y < b\ \Psi(f_t, f_1, \dots, f_{n^*}, \bar{u}, m, \bar{v}, y, b, a, s_t, \alpha, 0, (a)_{\mathrm{i}})$.
Then, let $s^* := \min\{s_t : t < n^*+1\}$. Therefore, by statement (1) and the definitions of Θ and Ψ it holds that

(2): $\mathcal{M} \models \Theta(s^*, n^*, \bar{u}, m, \bar{v}, b, a, \alpha, \beta, (a)_{\mathrm{i}})$.
So from statement (2) and the definition of G, we infer that $\mathrm{G}(n^*) > I$ and it holds that

(3): $\mathcal{M} \models \Theta(\mathrm{G}(n^*), n^*, \bar{u}, m, \bar{v}, b, a, \alpha, \beta, (a)_{\mathrm{i}})$.
Consequently $\mathrm{G}(n^*) > e$, and again by the definition of Θ and statement (3), we deduce that

(4): $\mathcal{M} \models \Theta(e, n^*, \bar{u}, m, \bar{v}, b, a, \alpha, \beta, (a)_{\mathrm{i}})$.
Statement (4) implies that

(4): $$\mathcal{M} \models \exists y \leq t(\bar{v}, (a)_{\mathrm{i}}) \left(\begin{array}{c} \forall i < e([f_n(\bar{u}, m, (a)_i)\downarrow] \rightarrow [f_n(\bar{v}, y, (a)_i)\downarrow]^{<b}) \wedge \\ \bigwedge_{t=1}^{k} \forall i < e \begin{pmatrix} ([f_{n_t}(\bar{v}, y, (a)_i)\downarrow]^{<b} \wedge \langle n_t, i\rangle \mathrm{E}\alpha) \rightarrow \\ f_{n_t}(\bar{u}, m, (a)_i) \neq f_{n_t}(\bar{v}, y, (a)_i) \end{pmatrix} \end{array} \right).$$

So statement (4) finishes the proof of the claim.

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"Back" stages: Let $m' \in M \setminus \{\bar{v}\}$ such that $m' < v_0 := \max\{\bar{v}\}$, and $u_0 := \max\{\bar{u}\}$. In order to find some element of M whose image is m', we modify the proof of the "forth" stage in the following way:

- Replace $p_s(y)$ by

$$q_s(x) := \{x < u_0\} \cup q_{s1}(x) \cup q_{s2}(x), \text{where},$$

$$q_{s1}(x) := \{\forall i < s(\neg[f(\bar{v}, m', (a)_i)\downarrow]^{<b} \to \neg[f(\bar{u}, x, (a)_i)\downarrow]) : f \in \mathcal{F}\}; \text{and}$$

$$q_{s2}(x) := \left\{\forall i < s \begin{pmatrix} \begin{pmatrix} \langle n, i\rangle \mathrm{E}\alpha' \wedge \\ [f_n(\bar{u}, x, (a)_i)\downarrow] \end{pmatrix} \to \\ f_n(\bar{u}, x, (a)_i) \neq f_n(\bar{v}, m', (a)_i) \end{pmatrix} : n \in \mathbb{N}\right\}.$$

- Replace $\Theta(s, i, \bar{u}, m, \bar{v}, b, a, \alpha, \beta)$ with $\Theta'(s, i, \bar{u}, \bar{v}, m', b, a, \alpha', \beta')$:

$$\forall r < i\ \exists x < u_0 \begin{pmatrix} \forall w < s(\langle x, r, w\rangle \mathrm{E}\beta' \to [f_r(\bar{v}, m', (a)_w)\downarrow]^{<b}) \wedge \\ \forall w < s \forall r' < i \begin{pmatrix} (\langle r', w\rangle \mathrm{E}\alpha' \wedge \langle x, r', w\rangle \mathrm{E}\beta') \to \\ f_{r'}(\bar{u}, m, (a)_w) \neq f_{r'}(\bar{v}, y, (a)_w)) \end{pmatrix} \end{pmatrix},$$

where β' is the code of the following Σ_1-definable set in $\mathcal{M}$:

$$L' := \{\langle x, r, w\rangle : \mathcal{M} \models (x < u_0 \wedge w < d \wedge r < d \wedge [f_r(\bar{u}, x, (a)_w)\downarrow])\}.$$

The rest of the argument goes smoothly by modifying the "forth" stage according to the above changes, and this completes the proof. ⊣

COROLLARY 3.5. *Assume $\mathcal{M} \models \mathrm{I}\Sigma_1$ is countable and nonstandard, I is a proper cut of $\mathcal{M}$, and $\mathcal{M}_0$ is an I-small Σ_1-elementary submodel of $\mathcal{M}$. Then the following are equivalent*:

(1) *I is strong in $\mathcal{M}$.*
(2) *There exists some proper initial self-embedding j of $\mathcal{M}$ such that $M_0 = \mathrm{Fix}(j)$.*

PROOF. Suppose $M_0 = \{(a)_i : i \in I\}$, for some $a \in M$ such that $(a)_i \neq (a)_j$ for all distinct $i, j \in I$.

(1) ⇒ (2): If $M_0 = I$, then by Theorem 2.4(2), we are done. So suppose $I \subsetneq M_0$. First, by using Theorem 3.4 let h be some proper initial self-embedding of $\mathrm{H}^1(\mathcal{M}; M_0)$ such that $\mathrm{Fix}(h) = M_0$. Moreover, fix some $b \in \mathrm{H}^1(\mathcal{M}; M_0) \setminus M_0$ such that $h(\mathrm{H}^1(\mathcal{M}; M_0)) < b$. Now, by using strong Σ_1-Collection in $\mathrm{H}^1(\mathcal{M}; M_0)$, and since $\mathrm{H}^1(\mathcal{M}; M_0) \prec_{\Sigma_1} \mathcal{M}$, we can find some $d \in \mathrm{H}^1(\mathcal{M}; M_0)$ such that

$$\mathcal{M} \models [f((a)_i, b)\downarrow] \to [f((a)_i, b)\downarrow]^{<d}, \text{ for all } f \in \mathcal{F} \text{ and all } i \in I.$$

Therefore, by Theorems 2.1 and 2.3 and Remark 1, there exists some proper initial embedding $k : \mathcal{M} \hookrightarrow \mathrm{H}^1(\mathcal{M}; M_0)$ such that $M_0 \subseteq \mathrm{Fix}(k)$, $k(M) < d$, and $b \in k(M)$ (note that since $\mathrm{H}^1(\mathcal{M}; M_0)$ is an initial segment of $\mathcal{M}$, then $\mathrm{SSy}_I(\mathcal{M}) = \mathrm{SSy}_I(\mathrm{H}^1(\mathcal{M}; M_0))$). Finally, we put $j := k^{-1}hk$. It is easy to check that j is a well-defined proper initial self-embedding of $\mathcal{M}$ such that $\mathrm{Fix}(j) = M_0$.

(2) ⇒ (1): We combine the methods used in the proofs of Theorems 5.1 and 6.1 of [1]. Suppose I is not strong; i.e., there exists some coded function f in $\mathcal{M}$ such that $I \subseteq \mathrm{Dom}(f)$, and the set $D := \{f(i) : i \in I \wedge I < f(i)\}$ is downward cofinal in $M \setminus I$.

Let $b \in M \setminus M_0$ and $g := j(f)$. For every $k \in M$, we put

$$A_k := \{\langle r, y\rangle < k : \mathcal{M} \models \mathrm{Sat}_{\Delta_0}(\delta_r((a)_y, b))\}.$$

Since A_k is bounded and Δ_1-definable, it is coded by some s_k in $\mathcal{M}$. Moreover, the function $k \mapsto s_k$ is Δ_1-definable in $\mathcal{M}$. Now, we define

$$h(k) := \mu_x\ (\forall\langle r, y\rangle < k\ (\langle r, y\rangle \mathrm{E} s_k \to \mathrm{Sat}_{\Delta_0}(\delta_r((a)_y, x)))).$$

So note that:

(I) For every $k > I$, we have $\mathrm{Th}_{\Delta_0}(\mathcal{M}; b, \{(a)_i\}_{i\in I}) \subseteq \mathrm{Th}_{\Delta_0}(\mathcal{M}; h(k), \{(a)_i\}_{i\in I})$.

(II) For every $i \in I$, $h(i)$ is well-defined and inside $M_0 = \mathrm{Fix}(j)$; the reason behind this statement is that for every $i \in I$ we consider the following set:

$$B_i := \{\langle r, \varepsilon\rangle : \mathcal{M} \models \exists y < i((a)_y = \varepsilon \wedge \langle r, y\rangle \mathrm{E} s_i)\}.$$

Then, by Lemma 3.3, B_i is coded by some $\alpha_i \in M_0 = \mathrm{Fix}(j)$. So it holds that

$$\mathcal{M} \models h(i) = \mu_x(\forall\langle r, \varepsilon\rangle \mathrm{E}\alpha_i\ (\mathrm{Sat}_{\Delta_0}(\delta_r(\varepsilon, x))).$$

As a result, since $\mathcal{M}_0 \prec_{\Sigma_1} \mathcal{M}$, statement (II) holds.

Now, let $h' := j(h)$. So for all $i \in I$, and all $u < i$ such that $f(u) < i$, statement (II) implies that

$$h'(g(u)) = j(h)(j(f)(u)) = j(h)(j(f)(j(u))) = j(h(f(u))) = h(f(u)).$$

Therefore, for all $i \in I$, $\mathcal{M} \models \varphi(i, f, g, h, h')$, where $\varphi(i, f, g, h, h')$ is the following Δ_1-formula:

$$\forall u < i\ (f(u) < i \to h(f(u)) = h'(g(u))).$$

So by Σ_1-Overspill in $\mathcal{M}$, there exists some $s > I$ such that

$$(\spadesuit):\ \ \forall u < s\ (f(u) < s \to h(f(u)) = h'(g(u))).$$

Since D is downward cofinal in $M \setminus I$, there is some $i_0 \in I$ such that $I < f(i_0) < s$. Let $c := h(f(i_0))$. On one hand, by (I), $\mathrm{Th}_{\Delta_0}(\mathcal{M}; b, \{(a)_i\}_{i\in I}) \subseteq \mathrm{Th}_{\Delta_0}(\mathcal{M}; c, \{(a)_i\}_{i\in I})$. As a result, because $b \notin M_0$, we have $c \notin M_0 = \mathrm{Fix}(j)$. On the other hand ($\spadesuit$) implies that

$$j(c) = j(h(f(i_0))) = j(h)(j(f)((j(i_0))) = h'(g(i_0)) = h(f(i_0)) = c.$$

As a result, I has to be strong in $\mathcal{M}$. ⊣

§4. Strongness of the standard cut and fixed points. In this section, we will show some properties of $\mathrm{Fix}(j)$, when $\mathbb{N}$ is not strong in $\mathcal{M}$. Then we will derive some criteria for strongness of $\mathbb{N}$ in a countable nonstandard model of $\mathrm{I}\Sigma_1$ in terms of sets of fixed points of its initial self-embeddings.

LEMMA 4.1. *Suppose $\mathcal{M}$ is a nonstandard model of $\mathrm{I}\Sigma_1$ in which $\mathbb{N}$ is not strong. Then for any self-embedding j of $\mathcal{M}$ the following hold*:

(1) $\mathrm{Fix}(j)$ *is* 1*-tall.*
(2) *If* $\mathrm{Fix}(j)$ *is a countable model of* $\mathrm{B}\Sigma_1$, *then it is* 1*-extendable.*

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PROOF. (1) Let $a \in \mathrm{Fix}(j)$ be arbitrary and fixed. Since $\mathrm{Fix}(j) \prec_{\Sigma_1} \mathcal{M}$, it suffices to prove that $\mathrm{K}^1(\mathcal{M}; a)$ is not cofinal in $\mathrm{Fix}(j)$. Since $\mathcal{M} \models \mathrm{B}^+\Sigma_1$, there exists some $t_0 \in M$ such that $\mathrm{K}^1(\mathcal{M}; a) < t_0$. Moreover, by Lemma 2.5(2) there exists some $t_{00} \in \mathrm{Fix}(j)$ such that $\mathrm{Th}_{\Sigma_1}(\mathcal{M}; t_0, a) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}; t_{00}, a)$. Therefore, $\mathrm{K}^1(\mathcal{M}; a) < t_{00}$.

(2) By Theorem 2.2(2), and part (1) of this lemma, it suffices to prove that $\mathbb{N}$ is not Π_1-definable in $\mathrm{Fix}(j)$. Suppose not; i.e., $\mathbb{N}$ is definable in $\mathrm{Fix}(j)$ by some Π_1-formula $\pi(x)$. By Lemma 2.5(1), $\mathrm{Fix}(j) \cap M \setminus \mathbb{N}$ is downward cofinal in $M \setminus \mathbb{N}$. So by Σ_1-Underspill in $\mathcal{M}$, there exists some $n \in \mathbb{N}$ such that $\mathcal{M} \models \neg\pi(n)$, and consequently since $\mathrm{Fix}(j) \prec_{\Sigma_1} \mathcal{M}$, $\mathrm{Fix}(j) \models \neg\pi(n)$, which is a contradiction. ⊣

The following corollary generalizes Theorem 1.2:

COROLLARY 4.2. *Let $\mathcal{M} \models \mathrm{I}\Sigma_1$ be countable and nonstandard in which $\mathbb{N}$ is not strong, and j is an initial self-embedding of $\mathcal{M}$ such that $\mathrm{Fix}(j) \models \mathrm{B}\Sigma_1$. Then $\mathrm{Fix}(j)$ is isomorphic to a proper cut of $\mathcal{M}$.*

PROOF. By Theorem 2.2(1) and the previous lemma, it is enough to prove that $\mathrm{SSy}(\mathrm{Fix}(j)) = \mathrm{SSy}(\mathcal{M})$. So let $X = \mathbb{N} \cap a_{\mathrm{E}}$ for some $a \in M$. Since $\mathbb{N}$ is not strong in $\mathcal{M}$, by Lemma 2.5(2) there exists some $b \in \mathrm{Fix}(j)$ such that $\mathrm{Th}_{\Sigma_1}(\mathcal{M}; a) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{M}; b)$. Therefore, $X = \mathbb{N} \cap b_{\mathrm{E}}$, and this finishes the proof. ⊣

We conclude this section with a generalization of a similar result about automorphisms of countable recursively saturated models of PA in [12]. Moreover, the following corollary defines Theorem 2.4(3).

COROLLARY 4.3. *Let $\mathcal{M} \models \mathrm{I}\Sigma_1$ be countable and nonstandard. Then the following are equivalent:*

(1) *$\mathbb{N}$ is strong in $\mathcal{M}$.*
(2) *There exists some proper initial self-embedding j of $\mathcal{M}$ such that* $\mathrm{Fix}(j) = \mathrm{K}^1(\mathcal{M})$.
(3) *There exists some proper initial self-embedding j of $\mathcal{M}$, and some small $\mathcal{M}_0 \prec_{\Sigma_1} \mathcal{M}$, such that* $\mathrm{Fix}(j) = M_0$.
(4) *For every small $\mathcal{M}_0 \prec_{\Sigma_1} \mathcal{M}$ there exists some proper initial self-embedding j of $\mathcal{M}$ such that* $\mathrm{Fix}(j) = M_0$.
(5) *There exists some proper initial self-embedding j of $\mathcal{M}$ such that* $\mathrm{Fix}(j) \subseteq \mathrm{I}^1(\mathcal{M})$.

If $\mathcal{M} \models$ PA and it is recursively saturated, then the above statements are equivalent to the following:

(6) *There exists some proper initial self-embedding j of $\mathcal{M}$ such that* $\mathrm{Fix}(j) \models \mathrm{B}\Sigma_1$ *and it is isomorphic to no proper initial segments of $\mathcal{M}$.*

PROOF. The equivalences of statements (1)–(5) is a straightforward implication of Corollary 3.5 and Lemma 4.1(1). Moreover, (6) $\Rightarrow$ (1) holds by Corollary 4.2. In order to prove (4) $\Rightarrow$ (6), similar to the proof of Theorem 3.2(3), we will find some small recursively saturated elementary submodel $\mathcal{M}_0$ of $\mathcal{M}$. So statement (4) will provide us with a proper initial self-embedding j of $\mathcal{M}$ such that $\mathrm{Fix}(j) = M_0$. Clearly $\mathrm{Fix}(j) \models \mathrm{B}\Sigma_1$. Moreover, as we mentioned in the beginning of Section 3,

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$\mathrm{SSy}(\mathcal{M}_0) \neq \mathrm{SSy}(\mathcal{M})$. As a result, $\mathrm{Fix}(j)$ is isomorphic to no proper initial segment of $\mathcal{M}$. ⊣

§5. Extendability. In this section, we will study the extendability of initial embeddings. Most of the theorems of this section are generalizations of results about automorphisms of countable recursively saturated models of PA obtained in [10, 11].

DEFINITION 2. Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of $\mathrm{I}\Sigma_1$, $\mathcal{M}_0$ and $\mathcal{N}_0$ are bounded submodels (or proper cuts) of $\mathcal{M}$ and $\mathcal{N}$, respectively. We call an initial embedding $j : \mathcal{M}_0 \hookrightarrow \mathcal{N}_0$ an *initial* $(\mathcal{M}, \mathcal{N})$*-embedding* if for every $A \subseteq M_0$ it holds that

$$A \in \mathrm{SSy}_I(\mathcal{M}) \text{ iff } j(A) \in \mathrm{SSy}_J(\mathcal{N}),$$
$$\text{where } I := \mathrm{I}^1(\mathcal{M}; M_0), \text{ and } J := \mathrm{I}^1(\mathcal{N}; j(M_0)).$$

If $\mathcal{M} = \mathcal{N}$, we call such j an *initial* $\mathcal{M}$*-embedding*.

First, in the next lemma we will show that the condition in the above definition, i.e., preserving coded subsets, is a necessary condition for extendability of an initial embedding.

LEMMA 5.1. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of $\mathrm{I}\Sigma_1$, $\mathcal{M}_0 \subseteq \mathcal{M}$ and $\mathcal{N}_0 \subseteq \mathcal{N}$ are bounded submodels (or proper cuts), and $j : \mathcal{M}_0 \hookrightarrow \mathcal{N}_0$ is an initial embedding. If j is extendable to some initial embedding $\hat{j} : \mathcal{M} \hookrightarrow \mathcal{N}$, then j is an initial $(\mathcal{M}, \mathcal{N})$-embedding.*

PROOF. Put $I := \mathrm{I}^1(\mathcal{M}; M_0)$, $J := \mathrm{I}^1(\mathcal{N}; j(M_0))$, and let $A \subseteq M_0$ be arbitrary. If $A = I \cap (\alpha_{\mathrm{E}})^{\mathcal{M}}$ for some α in $\mathcal{M}$, then clearly $j(A) = J \cap ((\hat{j}(\alpha))_{\mathrm{E}})^{\mathcal{N}}$. Conversely, suppose $j(A) \in \mathrm{SSy}_J(\mathcal{N})$. Since M_0 is bounded in $\mathcal{M}$, we have $J \subsetneq_e \hat{j}(M)$. As a result, $j(A) \in \mathrm{SSy}_J(\hat{j}(M))$, which implies that $A \in \mathrm{SSy}_I(\mathcal{M})$. ⊣

Converse of the above lemma holds, when $\mathcal{M}_0$ and $j(M_0)$ are Σ_1-elementary initial segments of $\mathcal{M}$ and $\mathcal{N}$.

THEOREM 5.2. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable and nonstandard models of $\mathrm{I}\Sigma_1$, and I and J are Σ_1-elementary initial segments of $\mathcal{M}$ and $\mathcal{N}$, respectively. Then for any isomorphism $j : I \to J$ which is an initial $(\mathcal{M}, \mathcal{N})$-embedding and each $b > J$, there exists some proper initial embedding $\hat{j} : \mathcal{M} \hookrightarrow \mathcal{N}$ such that $\hat{j} \restriction_I = j$ and $\hat{j}(M) < b$.*

SKETCH OF PROOF. The proof is conducted by a back-and-forth argument similar to the one used in the proof of [1, Theorem 3.3]; we will build finite partial functions $\bar{u} \mapsto \bar{v}$ such that the following induction hypothesis holds:

$$\text{If } \mathcal{M} \models [f(\bar{u}, i)\downarrow], \text{ then } \mathcal{N} \models [f(\bar{v}, j(i))\downarrow]^{<b},$$
$$\text{for every } f \in \mathcal{F} \text{ and } i \in I.$$

For the "forth" steps, if $\bar{u} \mapsto \bar{v}$ is built, for given $m \in M$ we define

$$H := \{\langle r, i\rangle \in I : \mathcal{M} \models [f_r(\bar{u}, m, i)\downarrow]\}.$$

Then, let $h \in M$ such that $H = I \cap h_{\mathrm{E}}$. Since j is an initial $(\mathcal{M}, \mathcal{N})$-embedding, there exists some $h' \in N$ such that $j(H) = J \cap h'_{\mathrm{E}}$. Therefore, by induction hypothesis for every $s \in I$ it holds that

(1): $\mathcal{N} \models \exists x, w < b\, \forall \langle r, i\rangle < j(s)\, (\langle r, i\rangle \mathrm{E} h' \to [f_r(\bar{v}, x, i))\downarrow]^{<w})$.

Since j is onto, statement (1) implies that for every $t \in J$ it hold that

(2): $\mathcal{N} \models \exists x, w < b\, \forall \langle r, i\rangle < t\, (\langle r, i\rangle \mathrm{E} h' \to [f_r(\bar{v}, x, i))\downarrow]^{<w})$.

Therefore, by using Σ_1-Overspill in $\mathcal{N}$, we will find some image for m, for which induction hypothesis holds. The "back" stages can be done similarly. ⊣

The proof of the above theorem can also be modified for I-small submodels.

THEOREM 5.3. *Suppose $\mathcal{M} \models \mathrm{I}\Sigma_1$ is countable and nonstandard, I is a strong cut of $\mathcal{M}$, $\mathcal{M}_0$ is an I-small Σ_1-elementary submodel of $\mathcal{M}$ such that $M_0 := \{(a)_i : i \in I\}$, and j is an initial embedding of $\mathcal{M}_0$ such that $j(I) \subseteq_e \mathcal{M}$. Then the following are equivalent*:

1. *$j \restriction_I$ is an initial $\mathcal{M}$-embedding, and there exists some $b \in M$ such that $\mathcal{M} \models j((a)_i) = (b)_{j(i)}$ for all $i \in I$.*
2. *j extends to some proper initial self-embedding of $\mathcal{M}$.*

SKETCH OF PROOF. (2) ⇒ (1) holds by Lemma 5.1. In order to prove (1) ⇒ (2), we will use a similar argument to the proof of [1, Theorem 3.3] to obtain an extension $\hat{j}$ of j. For this purpose, first we will fix some $d \in M$ which is an upper bound for M_0. Then, we will build finite partial functions $\bar{u} \mapsto \bar{v}$ such that the following induction hypothesis holds:

$$\mathcal{M} \models [f(\bar{u}, (a)_i)\downarrow] \to [f(\bar{v}, (b)_{j(i)})\downarrow]^{<d},$$
$$\text{for every } f \in \mathcal{F} \text{ and } i \in I.$$

Here, we outline the proof for the "back" steps and the proof of "forth" steps is left to the reader. Suppose $\bar{u} \mapsto \bar{v}$ is built, and $m < \max\{\bar{v}\}$ is given. We define

$$L := \{\langle r, i\rangle \in j(I) :\ \mathcal{M} \models \neg[f_r(\bar{v}, m, (b)_i)]^{<d}\}.$$

Then, let $l \in M$ such that $L = j(I) \cap l_{\mathrm{E}}$. Since $j \restriction_I$ is an initial $\mathcal{M}$-embedding, then there exists some $l' \in M$ such that $j^{-1}(L) = I \cap l'_{\mathrm{E}}$. Moreover, by using Lemma 3.3, for every $s \in I$ there exists some $(a)_{i_s} \in M_0$ which codes of the following subset of M_0:

$$A := \{\langle r, (a)_i\rangle :\ \mathcal{M} \models (\langle r, i\rangle < s\ \wedge \langle r, i\rangle \mathrm{E} l')\}.$$

By Π_1-Overspill, it suffices to prove that for every $s \in I$ it holds that

$$(\star):\ \ \mathcal{M} \models \exists x < \max\{\bar{u}\}\, \forall \langle r, i\rangle < s\, (\langle r, i\rangle \mathrm{E} l' \to \neg[f_r(\bar{u}, x, (a)_i)\downarrow]).$$

Suppose not; i.e., there exists some $s \in I$ which for statement $(\star)$ does not hold. So we have,

(i): $\mathcal{M} \models \forall x < \max\{\bar{u}\}\, \exists \langle r, i\rangle < s\, (\langle r, i\rangle \mathrm{E} l'\ \wedge [f_r(\bar{u}, x, (a)_i)\downarrow])$.

As a result, by using Σ_1-Collection in $\mathcal{M}$, from statement (i), induction hypothesis, and the way we chose $(a)_{i_s}$, we may conclude that

(ii): $\mathcal{M} \models \forall x < \max\{\bar{v}\}\, \exists \langle r, \varepsilon\rangle \mathrm{E}(b)_{j(i_s)}\, ([f_r(\bar{v}, x, \varepsilon)\downarrow]^{<d})$.

So by statement (ii), there exists some $\langle r, i\rangle < s$ such that

(iii): $\mathcal{M} \models (\langle r, i\rangle \mathrm{E} l'\ \wedge\ [f_{j(r)}(\bar{v}, m, (b)_{j(i)})\downarrow]^{<d})$.

But statement (iii) is in direct contradiction with the way we chose l'. ⊣

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In the last theorem, we investigate whether we can control the set of fixed points, while extending an isomorphism to an initial self-embeddings with larger domain:

THEOREM 5.4. *Suppose* $\mathcal{M} \models \mathrm{I}\Sigma_1$ *is countable and nonstandard,* I *is a strong* Σ_1*-elementary initial segment of* $\mathcal{M}$*, and* $j : I \to I$ *is an isomorphism and an initial* $\mathcal{M}$*-embedding. Then there exists some proper initial self-embedding* $\hat{j}$ *of* $\mathcal{M}$ *such that* $\hat{j} \restriction_I = j$*, and* $\mathrm{Fix}(\hat{j}) = \mathrm{Fix}(j)$.

SKETCH OF PROOF. First, we will fix some arbitrary $a > I$. Since I is strong in $\mathcal{M}$, there exists some $b > I$ such that:

$(\star)$: if $\mathcal{M} \models [f(a,i)\downarrow]$ and $f(a,i) > I$ then $f(a,i) > b$, for all $f \in \mathcal{F}$ and $i \in I$.

So by Theorem 5.2, there exists some proper initial self-embedding $\bar{j}$ of $\mathcal{M}$ such that $\bar{j} \restriction_I = j$ and $\bar{j}(M) < b$. If $\mathrm{Fix}(\bar{j}) = \mathrm{Fix}(j)$, then we are done. Otherwise, by using a similar argument to the proof of Theorem 3.4 (as we will briefly outline the modifications which should be made to the proof below), we will construct some proper initial self-embedding h of $\mathcal{N} := \mathrm{H}^1(\mathcal{M}; a)$ such that $h \restriction_I = j$, $\mathrm{Fix}(h) = \mathrm{Fix}(j)$, and $h(N) < b$. If $\mathcal{M} = \mathcal{N}$, then we are done. Otherwise, by using Theorem 2.3 we shall find some proper initial embedding $k : \mathcal{M} \hookrightarrow \mathcal{N}$ such that $I \subseteq \mathrm{I}_{\mathrm{fix}}(k)$ and $b \in k(M)$. Finally, we put $\hat{j} := k^{-1}hk$.

In order to construct the aforementioned h, we will inductively build finite functions $\bar{u} \mapsto \bar{v}$ such that

$$\mathrm{P}(\bar{u},\bar{v}) \equiv [f(\bar{u},i)\downarrow] \to [f(\bar{v},j(i))\downarrow]^{<b}, \text{ for all } f \in \mathcal{F} \text{ and } i \in I; \text{ and}$$

$$\mathrm{Q}(\bar{u},\bar{v}) \equiv \begin{pmatrix} [f(\bar{u},i)\downarrow] \wedge [f(\bar{v},j(i))\downarrow]^{<b} \wedge \\ f(\bar{u},i) \notin I \end{pmatrix} \Rightarrow f(\bar{u},i) \neq f(\bar{v},j(i)), \text{ for all } f \in \mathcal{F}$$

and all $i \in I$.

- For the first step of induction, we will take $a \mapsto \bar{j}(a)$; clearly $\mathrm{P}(a,\bar{j}(a))$ holds in $\mathcal{M}$. Moreover, by statement $(\star)$ and since $\mathrm{Fix}(\bar{j}) < b$, the property $\mathrm{Q}(a,\bar{j}(a))$ also holds in $\mathcal{M}$.
- Then suppose $\bar{u} \mapsto \bar{v}$ is built. We will just mention the changes that should be made in the "forth" steps of Theorem 3.4, and "back" steps should be modified similarly:
  - Suppose $m \in N \setminus \{\bar{u}\}$ is given. By the definition of $\mathcal{N}$, without loss of generality, we may assume that $m \leq t(\bar{u},a)$ for some $t \in \mathcal{F}$. Put
    $$C := \{\langle r,i\rangle \in I : \mathcal{N} \models [f_r(\bar{u},m,i)\downarrow] \wedge f_r(\bar{u},m,i) \notin \mathrm{K}^1(\mathcal{N}; I \cup \{\bar{u}\})\}.$$
    Let $\alpha,\alpha' \in N$ such that $C = I \cap \alpha_{\mathrm{E}}$ and $j(C) = I \cap \alpha'_{\mathrm{E}}$ (note that since $\mathcal{N}$ is a Σ_1-elementary initial segment of $\mathcal{M}$ containing I, j is an initial $\mathcal{N}$-embedding):
  - Let $L := \{\langle r,i\rangle \in I : \mathcal{N} \models [f_r(\bar{u},m,i)\downarrow]\}$, $L = I \cap \beta_{\mathrm{E}}$, and $j(L) = I \cap \beta'_{\mathrm{E}}$ for $\beta,\beta' \in N$.
  - For every $s \in \bar{j}(N)$ such that $\bar{j}(s') = s$ for some $s' \in N$, let
    $$p_s(y) := \{y < t(\bar{v},\bar{j}(a))\} \cup p_{s1}(y) \cup p_{s2}(y), \text{ where}$$
    $$p_{s1}(y) := \{\forall i < s(\langle n,i\rangle \mathrm{E}\beta' \to [f_n(\bar{v},y,i)\downarrow]^{<b}) : n \in \mathbb{N}\}, \text{ and}$$

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$$p_{s2}(y) := \left\{ \forall w < s' \, \forall i < s \left(\begin{pmatrix} \langle n, w\rangle \mathrm{E}\alpha \ \wedge \ \langle n, i\rangle \mathrm{E}\alpha' \wedge \\ [f_n(\bar{v}, y, i) \downarrow]^{<b} \end{pmatrix} \rightarrow \atop f_n(\bar{u}, m, w) \neq f_n(\bar{v}, y, i) \right) : \ n \in \mathbb{N} \right\}.$$

– In order to find some $s > I$ such that $s \in \bar{j}(N)$ and $p_s(y)$ is finitely satisfiable, we will adapt the rest of the proof of Theorem 3.4 accordingly; for instance, we will mention two of these adaptations:

(1) Let $d' \in N$ such that $d' > I$ and $d := \bar{j}(d')$. Moreover, for every $i, s, s' \in N$, let $\Theta(s, s', i, \bar{u}, m, \bar{v}, b, \bar{j}(a), \alpha, \alpha', \beta')$ be the following Δ_0-formula:

$$\forall r < i \, \exists y \leq t(\bar{v}, \bar{j}(a))$$
$$\times \left(\begin{matrix} \forall w < s(\langle r, w\rangle \mathrm{E}\beta' \rightarrow [f_r(\bar{v}, y, w) \downarrow]^{<b}) \wedge \\ \forall w < s \forall w' < s' \forall r' < i \left(\begin{pmatrix} \langle r', w'\rangle \mathrm{E}\alpha' \wedge \\ \langle r', w\rangle \mathrm{E}\alpha \wedge \\ [f_{r'}(\bar{v}, y, (a)_w) \downarrow]^{<b} \end{pmatrix} \rightarrow \atop f_{r'}(\bar{u}, m, w') \neq f_{r'}(\bar{v}, y, w) \right) \end{matrix} \right).$$

Then, for every $i \in M$, we define

$$\mathrm{G}(i) := \max\{w < d' : \ \mathcal{M} \models \exists x \leq d \ \Theta(x, w, i, \bar{u}, m, \bar{v}, b, \bar{j}(a), \alpha, \alpha', \beta')\}.$$

Since I is strong, there exists some $e' > I$ such that $e' \leq d'$, and for all $i \in I$, $\mathrm{G}(i) > I$ iff $\mathrm{G}(i) > e'$. Then, for every $i \in M$ put

$$l(i) := \max \left\{ x < \bar{j}(e') : \ \mathcal{M} \models \begin{pmatrix} [\mathrm{G}(i) \downarrow]^{<d'} \ \wedge \ \mathrm{G}(i) > e' \wedge \\ \Theta(x, \mathrm{G}(i), i, \bar{u}, m, \bar{v}, b, \bar{j}(a), \alpha, \alpha', \beta') \end{pmatrix} \right\}.$$

Again, since I is strong, there exists some $e > I$ such that $e \leq d$, and for all $i \in I$, $l(i) > I$ iff $l(i) > e$. Then $p_e(y)$ is a finitely satisfiable type.

(2) Instead of the function $\langle o(\diamondsuit, y), h(\diamondsuit, y)\rangle$ we need to define the following function:

$$\langle o(\diamondsuit, y), h(\diamondsuit, y), h'(\diamondsuit, y)\rangle$$
$$:= \min \left\{ \langle n_t, i, w\rangle \mathrm{E}\alpha_{s_0} : \begin{pmatrix} [b(\diamondsuit, y) \downarrow] \wedge \\ [f_{n_t}(\diamondsuit, y, i) \downarrow]^{<b(\diamondsuit, y)} \wedge \\ [g_t(\diamondsuit, w) \downarrow]^{<b(\diamondsuit, y)} \wedge \\ g_t(\diamondsuit, w) = f_{n_t}(\diamondsuit, y, i) \end{pmatrix} \right\},$$

where $\alpha_{s_0} \in I$ is the code of the following subset of I:

$$\{\langle n, i, w\rangle : \ \mathcal{M} \models i < s_0 \wedge w < j^{-1}(s_0) \wedge \langle n, w\rangle \mathrm{E}\alpha \wedge \langle n, i\rangle \mathrm{E}\alpha'\}.$$

The rest of the adaptations should be made similar to statements (1) and (2) in order to construct h. ⊣

REMARK 3. If we let j be the trivial automorphism of I, then Theorem 5.4 implies Theorem 2.4(2).

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