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THE TORSION FREE PIERI FORMULA

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ABSTRACT. Central to the study of simple infinite dimensional $g\ell(n,\mathbb{C})$-modules having finite dimensional weight spaces are the torsion free modules. All degree 1 torsion free modules are known. Torsion free modules of arbitrary degree can be constructed by tensoring torsion free modules of degree 1 with finite dimensional simple modules. In this paper, the central characters of such a tensor product module are shown to be given by a Pieri-like formula, complete reducibility is established when these central characters are distinct and an example is presented illustrating the existence of a nonsimple indecomposable submodule when these characters are not distinct.

0. **Introduction.** Let G be a finite dimensional simple Lie algebra over the complex numbers $\mathbb{C}$ with Cartan subalgebra H. Let V be a simple H-diagonalizable G-module having finite dimensional weight spaces. The problem of classifying such modules V is progressing. The classical case when V is a highest weight module is well known. Since this includes all simple finite dimensional modules, we focus our attention on infinite dimensional V. Recently, [BBL2] gave an explicit construction of all multiplicity free modules V (*i.e.* having only 1-dimensional weight spaces). Earlier, [BL1] gave a less explicit construction of such modules under the weaker condition that they have at least one 1-dimensional weight space realizing them as quotients of the universal enveloping algebra. This work relied heavily on a paper by Fernando [F] which reduces the general classification problem to one of determining the simple torsion free modules of finite degree and shows that torsion free modules exist for G of type A and C only. When G is $A_2 = s\ell(3,\mathbb{C})$, the simple torsion free modules are completely classified and found to be submodules of the tensor product of a multiplicity free simple torsion free G-module and a simple finite dimensional one [BFL]. The definition and basic properties of torsion free modules are presented in Section 1c).

Let $s\ell(n,\mathbb{C})$ denote the $n \times n$ traceless complex matrices. If ω_1 denotes the first fundamental weight of $s\ell(n,\mathbb{C})$ then for each positive integer K, the simple $s\ell(n,\mathbb{C})$-module $L(K\omega_1)$ with highest weight $K\omega_1$ is multiplicity free and provides a finite dimensional analogue of the multiplicity free torsion free $s\ell(n,\mathbb{C})$-modules. Moreover, the tensor product of $L(K\omega_1)$ with any finite dimensional simple $s\ell(n,\mathbb{C})$ module $L(\lambda)$ is completely reducible and its simple constituents are described by the Pieri formula [FH].

Working under mild constraints which are justified by example, we establish a modified Pieri formula for the tensor product of a multiplicity free, torsion free $s\ell(n,\mathbb{C})$-

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module M with a simple finite dimensional module $L(\lambda)$. In fact, we transport information from the finite setting to the infinite one via the Pieri formula. The main result of this paper is stated in Theorem 3.4. This result provides an explicit specialization of Kostant's results [K] on the general decomposition of the tensor product of an infinite dimensional module, admitting a central character, with a finite dimensional module.

Each submodule of $M \otimes L(\lambda)$ is a torsion free module of finite degree and we conjecture that every simple torsion free module of finite degree can be realized in this manner. As indicated above, this conjecture is valid when $n = 3$.

1. **Preliminaries.** In this section, we set down some background results in a fashion which facilitates our use of them and prove some general results concerning torsion free modules.

(a) FINITE DIMENSIONAL MODULES. Throughout, let $G_n = \mathrm{gl}(n, \mathbb{C})$ be the Lie algebra of $n \times n$ matrices over $\mathbb{C}$ determined by the commutator product. Let $\{e_{ij} \mid i, j = 1, 2, \ldots, n\}$ be the set of standard matrix units of G_n. The subalgebra of G_n having basis $\{e_{ij} \mid i, j = 1, \ldots, r\}$ is denoted G_r. The Cartan subalgebra of diagonal matrices of G_r is denoted H_r and the subalgebra of traceless matrices in G_r is denoted $s\ell(r, \mathbb{C})$. Let ϵ_i be the linear transformation which maps any $n \times n$ diagonal matrix to its (i, i)-th component. The weight functions of $s\ell(n, \mathbb{C})$ are normally described in terms of the fundamental weights $\{\omega_i \mid i = 1, \ldots, n-1\}$, see for example [H]. However, we find it more convenient to describe weight functions in terms of the ϵ_i's. Since there are n independent ϵ's and only $n-1$ ω_i's this means that a normalization choice is to be made. This is equivalent to specifying the action of $I_{n\times n} \in G_n$. This point is illustrated below.

Zelobenko, [Z], modified the Gel'fand-Zetlin presentation of simple finite dimensional G_n-modules V by redefining the action. In both presentations, a decreasing sequence of integers $m_{1n} \geq m_{2n} \geq \cdots \geq m_{nn}$ uniquely labels $V = V(m_{1n}, \ldots, m_{nn})$ having linear basis $B = \{\zeta(\mathrm{m}) \mid \mathrm{m} \in I(m_{1n}, \ldots, m_{nn})\}$ indexed by the set $I(m_{1n}, \ldots, m_{nn})$ of all triangular patterns of integers,

$$\mathrm{m} = \begin{bmatrix} m_{1n} & & m_{2n} & & \cdots & & m_{nn} \\ & m_{1,n-1} & & m_{2,n-1} & \cdots & m_{n-1,n-1} & \\ & & & & \cdots & & \\ & & & & m_{11} & & \end{bmatrix} \tag{1.1}$$

where the components m_{ij} satisfy the inequalities

$$m_{i,j+1} \geq m_{i,j} \geq m_{i+1,j+1} \quad \text{for } 1 \leq i \leq j \leq n-1. \tag{1.2}$$

For convenience, we set $\mathbf{m}_r = [m_{1r}, \ldots, m_{rr}]$, $R_r(\mathrm{m}) = \sum_{i=1}^{r} m_{ir}$ and $R_0(\mathrm{m}) = 0$. In order to describe the module action of G_n on V, it suffices to give the actions of the generators e_{kk}, $e_{k-1,k}$ and $e_{k,k-1}$. Following Zelobenko, we have

$$e_{kk}\zeta(\mathrm{m}) = \big(R_k(\mathrm{m}) - R_{k-1}(\mathrm{m})\big)\zeta(\mathrm{m}); \tag{1.3}$$

$$e_{k-1,k}\zeta(\mathrm{m}) = \sum_{j=1}^{k-1} -\frac{\Pi_{i=1}^{k}(l_{i,k} - l_{j,k-1})}{\Pi_{i=1, i\neq j}^{k-1}(l_{i,k-1} - l_{j,k-1})}\zeta(\mathrm{m} + \delta_{j,k-1}); \tag{1.4}$$

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$$e_{k,k-1}\zeta(\mathrm{m}) = \sum_{j=1}^{k-1} \frac{\prod_{i=1}^{k-2}(l_{i,k-2} - l_{j,k-1})}{\prod_{i=1,i\neq j}^{k-1}(l_{i,k-1} - l_{j,k-1})}\zeta(\mathrm{m} - \delta_{j,k-1}); \tag{1.5}$$

where $l_{i,j} = m_{i,j} - i$, and $\mathrm{m} \pm \delta_{j,k-1}$ denotes the pattern m with the $(j, k-1)$ component replaced by $m_{j,k-1} \pm 1$. By convention, we assume that $\zeta(\mathrm{m}) = 0$ for any pattern $\mathrm{m} \notin \mathcal{I}(m_{1n}, \ldots, m_{nn})$. Evidently, $\zeta(\mathrm{m})$ is a weight vector belonging to the functional

$$\lambda = \sum_{i=1}^{n} \big(R_i(\mathrm{m}) - R_{i-1}(\mathrm{m})\big)\epsilon_i.$$

We further note that subpatterns give us a natural branching of V with respect to subalgebras $\mathcal{G}_r$ for each $r = 1, \ldots, n$ and we define the symbol m_r by

$$\begin{bmatrix} m_{1r} & & m_{2r} & & \cdots & & m_{rr} \\ & m_{1,r-1} & & m_{2,r-1} & \cdots & m_{r-1,r-1} & \\ & & & & \cdots & & \\ & & & & m_{11} & & \end{bmatrix} \in \mathcal{I}(m_{1r}, \ldots, m_{rr}).$$

The $s\ell(n,\mathbb{C})$-module V obtained by restricting the action above to elements of $s\ell(n,\mathbb{C})$ remains simple. However, the labeling by decreasing sequences is no longer a one-to-one correspondence. In particular, $V(m_{1n}, \ldots, m_{nn})$ and $V(m_{1n} - c, \ldots, m_{nn} - c)$ are isomorphic $s\ell(n,\mathbb{C})$-modules since the only elements in $\mathcal{G}_n$ which distinguish these modules are the nonzero multiples of the identity $I_{n\times n}$. Since we are primarily interested in $s\ell(n,\mathbb{C})$-modules, we may pick c so that $m_{nn} - c$ is zero. This allows us to label our simple finite dimensional modules by partitions

$$\pi = \{\pi_1 \geq \pi_2 \geq \cdots \geq \pi_{n-1} \geq \pi_n = 0\}.$$

with π_i positive integers whenever convenient. Our basis is then labeled by $\mathcal{I}(\pi)$ where the conditions on our triangular patterns in $\mathcal{I}(\pi)$ satisfy conditions described in (1.2) with $m_{in} = \pi_i$. Our notation for this module is $V(\pi)$.

Of considerable interest to us are the multiplicity free simple modules $V(\pi)$ corresponding to $\pi = \{K \geq 0 \geq \cdots \geq 0\}$. In addition to the pattern realization, there is a very elementary realization of $V(K, 0, \ldots, 0)$ as being isomorphic to the module

$$\mathrm{span}_{\mathbb{C}}\Big\{x^q = x_1^{q_1}x_2^{q_2}\cdots x_n^{q_n} \mid q = (q_1, \ldots, q_n), q_i \in \mathbb{Z}_{\geq 0} \text{ with } K = \sum_{i=1}^{n} q_i\Big\}$$

under the action

$$e_{ij}x^q = q_j x^{q+\delta_i-\delta_j}$$

where $q + \delta_i - \delta_j$ denotes the vector obtained from q by adding 1 to its i-th coordinate and subtracting 1 from its j-th coordinate so that formally e_{ij} is multiplication by x_i and partial differentiation with respect to x_j. We notice that x^q is a weight vector belonging to weight $\sum_{i=1}^n q_i\epsilon_i$, since $e_{ii}x^q = q_i x^q$.

Vector exponential notation such as x^q is used throughout this paper to denote the product of the x_i's raised to the corresponding coordinate powers, *i.e.* $x^q = x_1^{q_1}x_2^{q_2}\cdots x_n^{q_n}$. Also,

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throughout this paper, we assume that $\pi = \{\pi_1 \geq \pi_2 \geq \cdots \geq \pi_{n-1} \geq \pi_n = 0\}$ is a partition of the non-negative integer N and consider it in $\mathbb{Z}^n_{\geq 0}$. For $p = (p_1, p_2, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1}_{\geq 0}$, we write

$$p \prec \pi \iff \pi_1 \geq p_1 \geq \pi_2 \geq \cdots \geq p_{n-1} \geq \pi_n = 0.$$

To facilitate later generalizations, fix $\ell \in \mathbb{Z}^n_{\geq 0}$ with $\sum_{i=1}^n \ell_i = K > n\pi_1$ and set

$$M(\ell; N) = \mathrm{span}_{\mathbb{C}}\Big\{x^{\ell+h} \mid \ell_i + h_i \in \mathbb{Z}_{\geq 0} \text{ with } \sum_{i=1}^n h_i = -N\Big\}. \tag{1.6}$$

Clearly $M(\ell; N) \simeq V(K-N, 0, \ldots, 0)$. We now consider the tensor product module

$$T(\ell; \pi) = M(\ell; N) \otimes V(\pi) \simeq V(K-N, 0, \ldots, 0) \otimes V(\pi). \tag{1.7}$$

In the setting of the Littlewood Richardson algorithm and Young tableaux (see for example [BBL1], pp. 79–81 for a brief description), there are a total of K boxes representing $T(\ell, \pi)$. Applying this algorithm, we obtain a direct sum of simple modules:

$$T(\ell; \pi) \simeq \bigoplus \sum_{p \prec \pi} V\Big(K - \sum_{i=1}^{n-1} p_i, p_1, \ldots, p_{n-1}\Big). \tag{1.8}$$

This restricted case of the Littlewood Richardson algorithm is known as the Pieri Formula (see for example [FH]).

The action of $I_{n\times n}$ on $M(\ell; N)$ is multiplication by $K - N$ and on $V(\pi) \simeq V(K-N, 0, \ldots, 0)$ multiplication by N. Therefore $V(K - \sum_{i=1}^{n-1} p_i, p_1, \ldots, p_{n-1})$ has highest weight $\lambda = (K - \sum_{i=1}^{n-1} p_i)\epsilon_1 + p_1\epsilon_2 + p_2\epsilon_3 + \cdots + p_{n-1}\epsilon_n$. For each $p \prec \pi$, set $X_p^{(\ell)}$ equal to the character of the simple highest weight module $L\big((K - \sum_{i=1}^{n-1} p_i)\epsilon_1 + p_1\epsilon_2 + p_2\epsilon_3 + \cdots + p_{n-1}\epsilon_n\big)$. It should be noted that when $(K - \sum_{i=1}^{n-1} p_i)\epsilon_1 + p_1\epsilon_2 + p_2\epsilon_3 + \cdots + p_{n-1}\epsilon_n$ is restricted to $s\ell(n, \mathbb{C})$, it is equal to the weight given in the basis (1.1) by $(K - p_1 - \sum_{i=1}^{n-1} p_i)\omega_1 + (p_1 - p_2)\omega_2 + \cdots + (p_{n-2} - p_{n-1})\omega_{n-1}$ which is the usual expression for this weight.

This means that the set of highest weights and the central characters of the decomposition of $T(\ell; \pi)$ are

$$\Big\{(K - \sum_{i=1}^{n-1} p_i)\epsilon_1 + p_1\epsilon_2 + p_2\epsilon_3 + \cdots + p_{n-1}\epsilon_n \mid p \prec \pi\} \text{ and } \mathrm{Ch}(\ell; N) = \{X_p^{(\ell)} \mid p \prec \pi\Big\},$$

respectively.

PROPOSITION 1.9. *Fix any n-tuple $\ell \in \mathbb{Z}^n_{\geq 0}$ with $\ell_i \geq \pi_1$ for $i = 1, \ldots, n$ and $\sum_{i=1}^n \ell_i = K$. Let $\lambda^{(\ell)}$ denote the weight function $\sum_{i=1}^n \ell_i\epsilon_i$, $p \prec \pi$ with $P = \sum_{i=1}^{n-1} p_i$ and $q_i = \ell_1 + \cdots + \ell_i$. Then*

(i) the weight space $T(\ell; \pi)_{\lambda^{(\ell)}}$ of $T(\ell; \pi)$ belonging to $\lambda^{(\ell)} = \sum_{i=1}^n \ell_i\epsilon_i$ has basis

$$B_{\lambda^{(\ell)}} = \Big\{\prod_{r=1}^n x_r^{\ell_r - R_r(\mathrm{m}) + R_{r-1}(\mathrm{m})} \otimes \zeta(\mathrm{m}) \mid \mathrm{m} \in I(\pi)\Big\}$$

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and so $\dim_{\mathbb{C}} T(\ell;\pi)_{\lambda^{(\ell)}} = \dim_{\mathbb{C}} V(\pi)$,

(ii) $V(K - \sum_{i=1}^{n-1} p_i, p_1, \ldots, p_{n-1})_{\lambda^{(\ell)}}$ *has basis* $\{\zeta([\ell;\mathrm{m}]) \mid \mathrm{m} \in I(\pi), \mathrm{m}_{n-1} = p\}$ *where* $[\ell;\mathrm{m}]$ *is described using the bottom* $n-1$ *rows of* m *as given by:*

$$\begin{bmatrix} q_n - P & & p_1 & & \cdots & & p_{n-1} \\ & q_{n-1} - R_{n-2}(\mathrm{m}) & & m_{1,n-2} & \cdots & m_{n-2,n-2} & \\ & & & & \cdots & & \\ & & & q_2 - R_1(\mathrm{m}) & & m_{1,1} & \\ & & & & q_1 & & \end{bmatrix}.$$

PROOF. For any pattern $\mathrm{m} = (m_{ij}) \in I(\pi)$

$$0 \le m_{r,r} \le R_r(\mathrm{m}) - R_{r-1}(\mathrm{m}) \le m_{1r} \le \pi_1$$

for $r = 1, \ldots, n$. Since $\ell_r \ge \pi_1$, it follows that $\ell_r - R_r(\mathrm{m}) + R_{r-1}(\mathrm{m}) \ge 0$. Also, $\sum_{i=1}^{n}\big(-R_i(\mathrm{m}) + R_{i-1}(\mathrm{m})\big) = -R_n(\mathrm{m}) = -N$ and hence

$$\prod_{r=1}^{n} x_r^{\ell_r - R_r(\mathrm{m}) + R_{r-1}(\mathrm{m})} \in M(\ell;N).$$

Moreover, for each $r = 1, \ldots, n$ we have

$$e_{rr}\Big(\prod_{s=1}^{n} x_s^{\ell_s - R_s(\mathrm{m}) + R_{s-1}(\mathrm{m})} \otimes \zeta(\mathrm{m})\Big) = \ell_r\Big(\prod_{s=1}^{n} x_s^{\ell_s - R_s(\mathrm{m}) + R_{s-1}(\mathrm{m})} \otimes \zeta(\mathrm{m})\Big)$$

and therefore $\prod_{s=1}^{n} x_s^{\ell_s - R_s(\mathrm{m}) + R_{s-1}(\mathrm{m})} \otimes \zeta(\mathrm{m})$ belongs to the $\lambda^{(\ell)}$ weight space. Since $\{\zeta(\mathrm{m}) \mid \mathrm{m} \in I(\pi)\}$ is a linear independent set, it is clear that $B_{\lambda^{(\ell)}}$ forms a basis for $T(\ell;\pi)_{\lambda^{(\ell)}}$. This completes part (i). Part (ii) follows from (1.3), since $\zeta([\ell;\mathrm{m}])$ are the only basis vectors of $V(K - \sum_{i=1}^{n-1} p_i, p_1, \ldots, p_{n-1})$ having weight $\lambda^{(\ell)}$. ■

(b) GEL'FAND-ZETLIN ALGEBRA. Let $U_r = U(G_r)$ denote the universal enveloping algebra of G_r, U_r^0 denote the centralizer of H_r in U_r and Z_r denote the center of U_r. Clearly $Z_r \subset U_r^0$. By the universal mapping property of the universal enveloping algebra, every G_r-module is a $U(G_r)$-module and by the inclusion $G_r \subset U(G_r)$ the converse is true.

From [Z], we know that Z_r is generated by the set of all elements

$$c_{rk} = \sum e_{j_1 j_2} e_{j_2 j_3} \cdots e_{j_k j_1}$$

where $k = 1, 2, \ldots, r$ and the sum ranges over all distinct sequences of integers $\{j_1, \ldots, j_k\}$ with $1 \le j_i \le r$.

In [L2], it is shown that if one wants to study simple H_r-diagonalizable modules, then it suffices, in the following sense, to study simple U_r^0-modules.

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THEOREM 1.10. *(i) If M is a simple H_r-diagonalizable module, then each weight space M_ν of M is a simple U_r^0-module.*

(ii) If M_ν is a simple U_r^0-module, then there exist a unique simple H_r-diagonalizable module having M_ν as one of its weight spaces.

Although U_n^0 is important to us there is an abelian subalgebra of U_n^0—which is equally important. Following Drozd-Ovienko-Futorny [DOF], we define the *Gel'fand-Zetlin subalgebra* Γ of U_n to be the subalgebra generated by $\{c_{rk} \mid 1 \le k \le r \le n\}$. From [DOF], Γ is an abelian subalgebra of U_n which is isomorphic to the polynomial ring in $\frac{n(n+1)}{2}$ variables c_{rk} over $\mathbb{C}$. We wish to address the Γ-*diagonalizability* of simple finite dimensional G_n-modules. By this we mean that there exists a basis of simultaneous eigenvectors for the operators in Γ. Towards this end, we let N_r^+ denote the subalgebra of G_r consisting of all strictly upper triangular matrices and throughout we choose a description of U_r^0 which makes computations on maximal vectors easy.

$$U_r^0 = U(H_r) \oplus (U_r^0 \cap U_r N_r^+). \tag{1.11}$$

Let $z = p_z + u_z \in U_r^0$ with $p_z \in U(H_r)$ and $u_z \in U_n^0 \cap U_r N_r^+$. If v^+ is a maximal vector in a G_r-module, then $N_r^+ v^+ = 0$ and so we have $zv^+ = p_z v^+$. $U(H_r)$ is isomorphic to the polynomial ring $\mathbb{C}[e_{11}, \ldots, e_{rr}]$. We identify these rings and write $U(H_r) = \mathbb{C}[e_{11}, \ldots, e_{rr}]$. Let $\eta\colon U_n^0 \longrightarrow U_n^0$ denote the projection of U_n^0 onto $U(H_n)$ along $U^0 \cap U_n N_n^+$ which gives $\eta(z) = p_z \in U(H_r) = \mathbb{C}[e_{11}, \ldots, e_{rr}]$. For $\mathrm{m} \in I(\pi)$, let

$$\begin{gathered} \Lambda_r^{\mathrm{m}}\colon U(H_r) \longrightarrow \mathbb{C}; \\ \Lambda_r^{\mathrm{m}}\big(p(e_{11}, \ldots, e_{rr})\big) = p(m_{1r}, \ldots, m_{rr}). \end{gathered} \tag{1.12}$$

Λ_r^{m} is the natural extension of the weight $\sum_{i=1}^r m_{ir}\epsilon_i$ to an algebra homomorphism on $U(H_r)$. As we did for z above, we write $c_{rk} = p_{rk}(e_{11}, \ldots, e_{nn}) + u_{rk} \in U(H_r) \oplus (U_r^0 \cap U_r N_r^+)$. Since Γ is the polynomial ring in the $\frac{n(n+1)}{2}$ variables c_{rk} over $\mathbb{C}$, an algebra homomorphism $\gamma_{\mathrm{m}}\colon \Gamma \longrightarrow \mathbb{C}$ is determined by its action on each c_{rk} given by

$$\gamma_{\mathrm{m}}(c_{rk}) = \Lambda_r^{\mathrm{m}} \circ \eta(c_{rk}) = \Lambda_r^{\mathrm{m}}\big(p_{rk}(e_{11}, \ldots, e_{rr})\big) = p_{rk}(m_{1r}, \ldots, m_{rr}) \tag{1.13}$$

An algebra homomorphism $\gamma\colon \Gamma \longrightarrow \mathbb{C}$ is called a *GZ-character*.

LEMMA 1.14. *Let* $\mathrm{m}, \mathrm{m}' \in I(\pi)$. *Then*

(i) if $z \in Z_r$, *then* $z\zeta(\mathrm{m}) = (\Lambda_r^{\mathrm{m}} \circ \eta)(z)\zeta(\mathrm{m})$, *and the restriction to* Z_r, $\Lambda_r^{\mathrm{m}} \circ (\eta_{\downarrow Z_r})$, *is the central character of the simple* G_r*-module* $V(\mathrm{m}_r)$,

(ii) if $z \in \Gamma$, *then* $z\zeta(\mathrm{m}) = \gamma_{\mathrm{m}}(z)\zeta(\mathrm{m})$,

(iii) if $\mathrm{m}_r \neq \mathrm{m}'_r$ *then* $\gamma_{\mathrm{m}'} \downarrow_{Z_r} \neq \gamma_{\mathrm{m}} \downarrow_{Z_r}$,

(iv) $V(\pi)$ *is* Γ*-diagonalizable with* 1*-dimensional* Γ*-weight spaces* $\mathbb{C}\zeta(\mathrm{m})$.

PROOF. First, we notice that (i) implies (ii) and so we prove (i). From 1.3–1.5, we see that if we ignore the top $n - r$ rows of the labeling patterns, then the action of $z \in Z_r$ on $\zeta(\mathrm{m})$ is the "same" as the action of z on $\zeta(\mathrm{m}_r)$ in the G_r-module $V(\mathrm{m}_r)$ where $\mathrm{m}_r =$

$[m_{1r}, m_{2r}, \ldots, m_{rr}]$. Express z as $z = p_z + u_z$ with $p_z = p_z(e_{11}, \ldots, e_{rr}) \in \mathbb{C}[e_{11}, \ldots, e_{rr}]$ and $u_z \in U_n^0 \cap U_r \mathcal{N}_r^+$. The maximal vector $v^+ \in V(\mathrm{m}_r)$ is labeled by the pattern

$$\hat{\mathrm{m}}_r = \begin{bmatrix} m_{1r} & & m_{2r} & & \cdots & & m_{rr} \\ & m_{1,r} & & m_{2,r} & \cdots & m_{r-1,r} & \\ & & & & \cdots & & \\ & & & & m_{1r} & & \end{bmatrix}$$

and so it belongs to the weight

$$\lambda = \sum_{i=1}^{r} \big(R_i(\hat{\mathrm{m}}_r) - R_{i-1}(\hat{\mathrm{m}}_r)\big)\epsilon_i = \sum_{i=1}^{r} m_{ir}\epsilon_i$$

Thus, for $z \in \mathcal{Z}_r$,

$$zv^+ = (p_z + u_z)v^+ = p_z(e_{11}, \ldots, e_{rr})v^+ = p_z(m_{1r}, \ldots, m_{rr})v^+ = (\Lambda_r^{\mathrm{m}} \circ \eta)(z)v^+.$$

The simplicity of the $\mathcal{G}_r$-module $V(\mathrm{m}_r)$ implies $z\zeta(\mathrm{m}_r') = (\Lambda_r^{\mathrm{m}} \circ \eta)(z)\zeta(\mathrm{m}_r')$ for all $z \in Z_r$ which implies $\Lambda_r^{\mathrm{m}} \circ (\eta_{\downarrow Z_r})$ is the central character of the $\mathcal{G}_r$-module $V(\mathrm{m}_r)$. Also, by our observation above, we have $z\zeta(\mathrm{m}) = p_z(e_{11}, \ldots, e_{rr})v^+ = p_z(m_{1r}, \ldots, m_{rr})\zeta(\mathrm{m})$.

Since (ii) and (iii) imply (iv) and (ii) is established, we prove (iii). Assume that $\mathrm{m}_r \neq \mathrm{m}_r'$ are distinct. If $\gamma_{\mathrm{m}} \downarrow_{Z_r} = \gamma_{\mathrm{m}'} \downarrow_{Z_r}$ then we have $\Lambda_r^{\mathrm{m}} \downarrow_{H_r} = \Lambda_r^{\mathrm{m}'} \downarrow_{H_r}$ where $\mathcal{H}_r$ denotes the Cartan subalgebra of diagonal matrices in $s\ell(r, \mathbb{C})$. By the Harish-Chandra Theorem, this in turn implies that the highest weights $\sum_{i=1}^r m_{ir}\epsilon_i \downarrow_{H_r}$ and $\sum_{i=1}^r m_{ir}'\epsilon_i \downarrow_{H_r}$ are linked. Thus, for $\delta = \sum_{i=1}^r (\frac{r+1}{2} - i)\epsilon_i$, half the sum of the positive roots of $s\ell(r, \mathbb{C})$, there exists an element σ in the Weyl group S_r such that

$$\sum_{i=1}^{r} \Big(m_{ir} + \frac{r+1}{2} - i\Big)\epsilon_{\sigma(i)} = \sum_{i=1}^{r} \Big(m_{ir}' + \frac{r+1}{2} - i\Big)\epsilon_i.$$

Thus, $m_{\sigma^{-1}(i)r} - \sigma^{-1}(i) = m_{ir}' - i$ for all i. However since $m_{1r} \geq \cdots \geq m_{rr}$ and $m_{1r}' \geq \cdots \geq m_{rr}'$ it follows that $m_{ir} = m_{ir}'$, contradicting the assumption that $\mathrm{m}_r \neq \mathrm{m}_r'$. ∎

(c) TORSION FREE MODULES. An $\mathcal{H}_n$-diagonalizable $\mathcal{G}_n$-module $\mathcal{M}$ is said to be *torsion free* provided each root vector e_{ij}, $i \neq j$, acts injectively on $\mathcal{M}$. It is shown in [F] that we need only require that $e_{k-1,k}$ and $e_{k,k-1}$ act injectively for $k = 2, \ldots, n$. Throughout, $\mathcal{M}$ is an $\mathcal{H}_n$-diagonalizable torsion free module having weights $\Phi = \lambda + \{\sum_{i=1}^n k_i\epsilon_i \mid k_i \in \mathbb{Z}, \sum_{i=1}^n k_i = 0\}$ for some fixed weight λ. The torsion free assumption and the condition on the weights of $\mathcal{M}$ imply that we may move from one weight space to another by the repeated action of root vectors of the form $e_{k-1,k}$ and $e_{k,k-1}$. Thus, the injectivity assumption implies that the weight spaces of $\mathcal{M}$ all have the same dimension called the *degree* of $\mathcal{M}$. Throughout, we assume that $\mathcal{M}$ is of finite degree.

THEOREM 1.15. *Let $\mathcal{M}$ be a torsion free $\mathcal{G}_n$-module of degree d.*

(i) Every submodule of $\mathcal{M}$ is torsion free.

(ii) Every quotient module of $\mathcal{M}$ is torsion free.

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(iii) If F is a U_n-module of dimension m, then the tensor product module $M \otimes F$ is torsion free of degree md.

PROOF. (i) is clear. For (ii), let N be a torsion free submodule of M. Then

$$M/N = \bigoplus \sum_{\nu\in\Phi}(M_\nu + N)/N$$

is the weight space decomposition of M/N. Clearly, e_{ij} is injective on M/N if and only if it is injective on each $(M_\nu + N)/N$. Suppose (ii) is false then, for some choice of $i \neq j$, $e_{ji}e_{ij} \in U_n^0$ is not injective on the finite dimensional U_n^0-module

$$(M_\nu + N)/N \simeq M_\nu/N_\nu.$$

But this implies that $\dim e_{ji}e_{ij}M_\nu < \dim M_\nu$, contrary to M being torsion free.

To prove (iii) we first note that by the complete reducibility of finite dimensional U_n-modules, we may assume that F is simple. Let v^+ be a maximal vector of F of weight α and let μ be any weight of F. Then $\alpha - \mu = \sum_{i=1}^n k_i\epsilon_i$ with $k_i \in \mathbb{Z}$ and since it is in the root lattice of G_n, $\sum_{i=1}^n k_i = 0$. Suppose the weight space F_μ belonging to μ has basis $\{v_1^{(\mu)}, \ldots, v_t^{(\mu)}\}$ and the weight space M_ν belonging to $\nu = \lambda + \sum_{i=1}^n k_i\epsilon_i$ has basis $\{m_1^{(\nu)}, \ldots, m_d^{(\nu)}\}$. Then $m_i^{(\nu)} \otimes v_j^{(\mu)}$ is in the weight space of M belonging to $\lambda + \alpha$. Since this can be done for every weight space of F, we see that $(M \otimes F)_{\lambda+\alpha}$ has dimension greater than or equal to md. The reverse inequality follows also by noting that we have accounted for all simple tensors which lie in $(M \otimes F)_{\lambda+\alpha}$. Thus, $\deg(M \otimes F) = md$.

Finally suppose $M \otimes F$ is not torsion free. Then there is some element e_{ij} with $i \neq j$ which annihilates a nonzero weight vector v. We can write $v = \sum_{k=1}^h m_k \otimes v_k$ where the m_k are linearly independent weight vectors with weights λ_k in M. Since M is torsion free, we have that $\{e_{ij}m_k \mid k = 1, \ldots, h\}$ is linearly independent set of weight vectors with weights $\lambda_k + \epsilon_i - \epsilon_j$. Without loss of generality we may assume that $\lambda_1 + \epsilon_i - \epsilon_j \notin \{\lambda_k \mid k = 1, \ldots, h\}$, which implies that $e_{ij}m_1 \notin \{m_k \mid k = 2, \ldots, h\}$. It follows then that $(e_{ij}m_1) \otimes v_1$ is not in the span of the set $\{(e_{ij}m_k) \otimes v_k \mid k = 2, \ldots, h\} \cup \{m_k \otimes (e_{ij}v_k) \mid k = 1, \ldots, h\}$ and therefore $e_{ij}v \neq 0$. This contradiction establishes that $M \otimes F$ is torsion free as required. ∎

LEMMA 1.16. *Suppose that M has degree d with weights $\Phi = \lambda + \{\sum_{i=1}^n k_i\epsilon_i \mid k_i \in \mathbb{Z}, \sum_{i=1}^n k_i = 0\}$ for some fixed weight λ. Let M_ν denote the weight space of M belonging to ν. Then*

(i) M is a simple G_n-module if and only if M_ν is a simple U_n^0-module for some weight $\nu \in \Phi$ (or equivalently for all $\nu \in \Phi$),

(ii) M has a composition series $0 \subset M_1 \subset M_2 \subset \cdots \subset M_m = M$ of G_n-modules if and only if the U_n^0-module M_ν has a composition series

$$0 \subset (M_1)_\nu \subset (M_2)_\nu \subset \cdots \subset (M_m)_\nu = M_\nu$$

for some $\nu \in \Phi$ (or equivalently for all $\nu \in \Phi$).

(iii) For $1 \le i \le q$, let M_i be U_n-modules. Then $M = \oplus \sum_{i=1}^{q} M_i$ if and only if each M_i is torsion free with weight lattice Φ and $M_\nu = \oplus \sum_{i=1}^{q} (M_i)_\nu$ for some $\nu \in \Phi$ (or equivalently for all $\nu \in \Phi$).

PROOF. By Theorem 1.10, if M is simple, then each of its weight spaces is a simple U_n^0-module. Suppose some weight space M_ν is a simple U_n^0-module and yet M is not a simple U_n-module having a proper submodule N. Since the weights of M are $\Phi = \lambda + \{\sum_{i=1}^{n} k_i \epsilon_i \mid k_i \in \mathbb{Z}, \sum_{i=1}^{n} k_i = 0\}$ for some fixed weight λ and N is torsion free, it too has this set of weights. The degree of N must be less than d but this implies that M_ν has a proper U_n^0-submodule, contrary to assumption. Also, we now have that if any weight space of M is simple then they all are. This gives (i).

Since

$$\left(M_i / M_{i-1}\right)_\nu \simeq (M_i)_\nu / (M_{i-1})_\nu$$

as U_n^0-modules, (ii) follows from (i). Part (iii) is clear. ∎

Before closing this part we point out that, by [BL1], if M is multiplicity free, then for some fixed positive integer N and some choice of noninteger complex numbers $a_1, \dots, a_n$,

$$M \simeq M(a;N) \equiv \operatorname{span}_{\mathbb{C}} \left\{ x^{a+h} \mid h_i \in \mathbb{Z} \text{ with } \sum_{i=1}^{n} h_i = -N \right\} \tag{1.17}$$

under the action of multiplication and partial differentiation

$$e_{ij} x^b = b_j x^{b+\delta_i - \delta_j}.$$

Compare (1.17) with (1.6).

(d) THE POLYNOMIAL LEMMAS. Let $t_1, \dots, t_n$ be algebraically independent over the complex numbers $\mathbb{C}$, $\mathbb{C}[t] = \mathbb{C}[t_1, \dots, t_n]$ be the polynomial ring in $t_1, \dots, t_n$, $\mathbb{C}(t) = \mathbb{C}(t_1, \dots, t_n)$ be the transcendental field extension. There are two results concerning the zeroes of polynomials which we require.

LEMMA 1.18. *Let B be a positive integer and $k_0 = 0$.*

(i) Suppose $\phi_1(t), \phi_2(t) \in \mathbb{C}(t)$ with $\phi_i(t) = \frac{f_i(t)}{g_i(t)}$ where $f_i(t), g_i(t) \in \mathbb{C}[t]$ and $\phi_1(q) = \phi_2(q)$ for all $q = (q_1, \dots, q_n) \in \mathbb{Z}^n_{\ge 0}$ with $q_i - q_{i-1} \ge B$ for $i = 1, \dots, n$. Then $\phi_1(t) = \phi_2(t)$.

(ii) Suppose $f_1(t), f_2(t) \in \mathbb{C}[t]$ and $f_1(\ell) = f_2(\ell)$ for all $\ell = (\ell_1, \dots, \ell_n) \in \mathbb{Z}^n_{\ge 0}$ with $\ell_i \ge B$ for $i = 1, \dots, n$. Then $f_1(t) = f_2(t)$.

PROOF. Let $h(t) = f_1(t) g_2(t) - f_2(t) g_1(t)$. By assumption $h(q) = 0$ for all n-tuples of integers $q = (q_1, \dots, q_n)$ with $q_i - q_{i-1} \ge B$. It suffices to prove that $h(t) = 0$.

Proceed by induction on n. For $n = 1$ the result is obvious since any nontrivial polynomial in one variable can have only finitely many zeroes.

Assume $n > 1$ and write

$$h(t) = \sum_{j=0}^{N} \theta_j(t_1, \dots, t_{n-1}) t_n^j$$

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For any $n-1$ tuple of integers $(q_1,\ldots,q_{n-1})$ satisfying $q_i - q_{i-1} \geq B$, $h(q_1,\ldots,q_{n-1},t_n)$ is a polynomial in one variable with infinitely many zeroes, namely all integers $q_n \geq q_{n-1}+B$ and hence must be identically zero. It follows that $\theta_j(q_1,\ldots,q_{n-1}) = 0$ for any sequence of integers $\{q_1,\ldots,q_{n-1}\}$ satisfying $q_i - q_{i-1} \geq B$. By our inductive hypothesis then we have that $\theta_j(t_1,\ldots,t_{n-1}) = 0$ and hence $h(t) = 0$ as required for (i).

For (ii), set $q_j = \ell_j + q_{j-1}$ for $1 \leq j \leq n$ and $\phi_i(t_1, t_2 - t_1, \ldots, t_n - t_{n-1}) = f_i(t)$. Then $f_i(\ell) = \phi_i(\ell_1, \ell_2 - \ell_1, \ldots, \ell_n - \ell_{n-1}) = \phi_i(q)$ for all $q_j - q_{j-1} = \ell_j \geq B$ and so this part follows from (i). ∎

LEMMA 1.19. *For $n \geq 2$, let $a_1,\ldots,a_n$ be fixed complex numbers. Let B be a fixed positive integer. Let $p(t)$ be a polynomial in $\mathbb{C}[t]$. If $p(a_1+h_1,\ldots,a_n+h_n) = 0$ for all $h_1,\ldots,h_{n-1} \geq B$ and $h_n = -\sum_{i=1}^{n-1} h_i$, then*

$$p(t_1,\ldots,t_n) = \big(t_1+\cdots+t_n - (a_1+\cdots+a_n)\big) g(t_1,\ldots,t_n)$$

for some $g(t_1,\ldots,t_n) \in \mathbb{C}[t]$.

PROOF. Replacing t_i by $t_i + a_i$ in $p(t_1,\ldots,t_n)$, we obtain a polynomial $p'(t_1,\ldots,t_n)$ such that $p'(h_1,\ldots,h_n) = 0$ for all $h_1,\ldots,h_{n-1} \geq B$ and $h_n = -\sum_{i=1}^{n-1} h_i$. Set $s_i = t_1+\cdots+t_i$. Then we may express $p'(t_1,\ldots,t_n)$ as a polynomial $p''(s_1,\ldots,s_n)$. Now, if we set $q_i = h_1+\cdots+h_i$and $q_0 = 0$, then $p''(q_1,\ldots,q_{n-1},0) = 0$ for all $q_i - q_{i-1} \geq B$ and $i = 1,\ldots,n-1$. It suffices to prove that $p''(s_1,\ldots,s_n)$ is divisible by s_n.

Our proof is by induction on $n \geq 2$.

If $n = 2$ then $p''(s_1,s_2) = \sum_j \theta_j(s_2)s_1^j$ has the property that $p''(s_1,0) = \sum_j \theta_j(0)s_1^j$ has infinitely many roots, namely $q_1 \geq B$ and so this is the zero polynomial. Thus each $\theta_j(s_2)$ is divisible by s_2.

Assume now that $n > 2$ and set $p''(s_1,\ldots,s_n) = \sum_j \theta_j(s_1,\ldots,s_{n-2},s_n)s_{n-1}^j$. By our inductive assumption it suffices to show that $\theta_j(q_1,\ldots,q_{n-2},0) = 0$ for all $q_i - q_{i-1} \geq B$. Clearly the polynomial

$$p''(q_1,\ldots,q_{n-2},s_{n-1},0) = \sum_j \theta_j(q_1,\ldots,q_{n-2},0)s_{n-1}^j$$

is zero whenever s_{n-1} is replaced by any value $q_{n-1} \geq B + q_{n-2}$ and hence each $\theta_j(q_1,\ldots,q_{n-2},0)$ is zero for all $q_i - q?_{i-1} \geq B$ with $i = 1,\ldots,n-2$. ∎

2. **Generic modules.** As noted in the introduction our primary goal is to study the decomposition of tensor product modules of the form $M(a;N) \otimes V(\pi)$ where $a = (a_1\ldots,a_n)$ is an n-tuple of complex noninteger scalars. However when computing in such modules we are continually plagued with the problem of knowing when certain coefficients depending on the scalars a_i might be zero. In order to overcome this difficulty we replace the scalars a_i by algebraically independent variables t_i and study the analogous decomposition problem over the transcendental field extension $\mathbb{C}(t)$. We first need to formulate this problem precisely.

Let $G_r(t) = \mathrm{gl}(r,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}(t)$ denote the Lie algebra obtained from G_r by extension of the base field. This construction carries along with it the Cartan subalgebra $H_r(t) =$

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$H_r \otimes_{\mathbb{C}} \mathbb{C}(t)$ consisting of all diagonal matrices in $G_r(t)$, the universal enveloping algebra $U_r(t) = U_r \otimes_{\mathbb{C}} \mathbb{C}(t)$ of $G_r(t)$, the center $Z_r(t) = Z_r \otimes_{\mathbb{C}} \mathbb{C}(t)$ of $U_r(t)$, and the centralizer $U_r^0(t) = U_r^0 \otimes \mathbb{C}(t)$ of the Cartan subalgebra $H_r(t)$ in $U_r(t)$. $Z_r(t)$ is generated by the elements $\{c_{rk} \otimes 1 \mid 1 \leq k \leq r\}$. Finally the Gel'fand-Zetlin subalgebra of $U_n(t)$ is given by $\Gamma(t) = \Gamma \otimes_{\mathbb{C}} \mathbb{C}(t)$.

It is clear that if V is a G_n-module then $V \otimes_{\mathbb{C}} \mathbb{C}(t)$ is a $G_n(t)$-module under the action

$$\big(g \otimes f_1(t)\big)\big(v \otimes f_2(t)\big) = gv \otimes f_1(t)f_2(t)$$

where $f_1(t), f_2(t) \in \mathbb{C}(t)$, $g \in G_n$ and $v \in V$. Moreover, the simplicity of V implies the simplicity of $V \otimes_{\mathbb{C}} \mathbb{C}(t)$. In this section, we establish the analogues of (1.8), Proposition 1.9 and Lemma 1.14 as formulated in the setting of $G_n(t)$-modules.

REMARK 2.1. Evidently, the analogues of Theorem 1.10, Theorem 1.15, and Lemma 1.16 hold in the setting of $H_n(t)$ diagonalizable $G_n(t)$-modules.

The following notion of specialization allows us to transport information for one setting to another.

DEFINITION 2.2. Let $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. Let A be an associative algebra over the complex numbers which is generated by $\{g_\alpha \mid \alpha \in \Omega\}$ and let $M^{(a)}$ be an A-module with basis $B(a) = \{m_i^{(a)} \mid i \in I\}$. Assume $M^{(t)}$ is an $A \otimes \mathbb{C}(t)$-module with basis $B(t) = \{m_i^{(t)} \mid i \in I\}$, also index by I. Then $M^{(a)}$ is said to be a specialization of $M^{(t)}$ by a provided for each g_α:

$$(g_\alpha \otimes 1)m_i^{(t)} = \sum_{j \in I} q_{i,j,\alpha}(t) m_j^{(t)} \quad \text{for } q_{i,j,\alpha}(t) \in \mathbb{C}(t), \text{ and}$$

$$g_\alpha m_i^{(a)} = \sum_{j \in I} q_{i,j,\alpha}(a) m_j^{(a)}$$

where $q_{i,j,\alpha}(a)$ is obtained from the rational function $q_{i,j,\alpha}(t)$ by substituting a_i for t_i. Implicitly, we are assuming that $q_{i,j,\alpha}(a)$ are well defined.

We now present an example which illustrates the notion of specialization.

EXAMPLE 2.3. Every multiplicity free torsion free G_n-module $M(a; N)$ defined in Section 1 part (c) can be obtained through specialization of the $G_n(t)$-module $M(t; N)$ by a where $M(t; N)$ is defined as follows. Let $\{x_1, \ldots, x_n\}$ be a set of n commuting variables. Let

$$M(t; N) = \operatorname{span}_{\mathbb{C}(t)}\Big\{x^{t+h} \mid h = (h_1, \ldots, h_n) \in \mathbb{Z}^n, \sum_{i=1}^n h_i = -N\Big\}. \tag{2.4}$$

For $1 \leq i, j \leq n$ the action of e_{ij} on $M(t; N)$ is defined by

$$e_{ij} x^{t+h} = (t_j + h_j) x^{t+h-\delta_j+\delta_i}.$$

Compare (2.4) with (1.6) and (1.17). It is clear that x^{t+h} is a weight vector belonging to the weight $\sum_{i=1}^n (t_i + h_i)\epsilon_i$.

According to [BL2], the algebras U_r^0 and $U_r^0(t)$ are finitely generated associative algebras. In fact, the generators of U_r^0 are given by

$$\{e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_k i_1} \mid 1 \leq i_j \leq r, i_1, \ldots, i_k \text{ distinct}\}$$

We now establish some general results on specializations for finitely generated algebras.

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LEMMA 2.5. *Let A be an associative algebra over $\mathbb{C}$ with a finite generating set $\{g_k \mid 1 \leq k \leq m\}$ and let $M^{(t)}$ be a finite dimensional $A \otimes \mathbb{C}(t)$-module with basis $\mathcal{B}(t) = \{m_i^{(t)} \mid 1 \leq i \leq d\}$. Assume that there exists a $B \in \mathbb{Z}$ such that $M^{(\ell)}$ with basis $\mathcal{B}(\ell) = \{m_i^{(\ell)} \mid 1 \leq i \leq d\}$ is a specialization of $M^{(t)}$ by ℓ for all ℓ with $\ell_i \in \mathbb{Z}$, and $\ell_i \geq B$. If each specialization $M^{(\ell)}$ is simple, then $M^{(t)}$ is simple.*

PROOF. Suppose that $M^{(t)}$ is a reducible $A \otimes \mathbb{C}(t)$-module. It follows that we can select a basis $B_0^{(t)} = \{v_i(t) = \sum_{j=1}^{p} p_{ij}(t) m_j^{(t)} \mid i = 1, \ldots, d\}$ of $M^{(t)}$ such that for some $1 \leq d_1 < d$ the elements $\{v_i(t) \mid i = 1, \ldots, d_1\}$ span a proper submodule. Thus, each generator $g_h k \otimes 1$ has a matrix representation of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$$

with respect to the ordered basis $B_0^{(t)}$. Without loss of generality, we may assume that the coefficients $p_{ij}(t)$ have been selected to be polynomials. Clearly, $\det[p_{ij}(t)] \neq 0$. By Lemma 1.18, there exists a vector ℓ such that $\ell_i \geq B$ and $\det[p_{ij}(\ell)] \neq 0$ and all the rational functions are well defined. Hence, $B_0^{(\ell)} = \{v_i(\ell) \mid i = 1, \ldots, d\}$ is a basis of $M^{(\ell)}$ and the matrix representation of each generators g_k of A with respect to the basis $\mathcal{B}^{(\ell)}$ is obtained by substituting ℓ_i for t_i in the matrix representation of $g_k \otimes 1$ with respect to $\mathcal{B}_0^{(t)}$. However this implies that $\{v_i(\ell) \mid i = 1, \ldots, d_1\}$ spans a proper submodule of $M^{(\ell)}$, contrary to our assumption. ∎

LEMMA 2.6. *Let A be an associative algebra over $\mathbb{C}$ with finite generating set $\{g_k \mid 1 \leq k \leq m\}$ and let $M^{(t)}$ and $M'^{(t)}$ be two $A \otimes \mathbb{C}(t)$-modules with bases $\mathcal{B}(t) = \{m_i^{(t)} \mid 1 \leq i \leq d\}$ and $\mathcal{B}'(t) = \{m_i'^{(t)} \mid 1 \leq i \leq d\}$, respectively. Assume there exists a $B \in \mathbb{Z}$ such that $M^{(\ell)}$ is a specialization of both $M^{(t)}$ and $M'^{(t)}$ by ℓ for all ℓ with $\ell_i \in \mathbb{Z}$, and $\ell_i \geq B$ with $\mathbb{C}m_1^{(\ell)} = \mathbb{C}m_1'^{(\ell)}$ for all ℓ. Then, if $M^{(t)}$ is simple so is $M'^{(t)}$ and $M^{(t)} \simeq M'^{(t)}$.*

PROOF. We claim that the annihilator of $m_1^{(t)}$ is equal to the annihilator of $m_1'^{(t)}$. In fact let $a = \sum_{i=1}^{q} a_i \otimes \frac{f_i(t)}{g_i(t)}$ where each a_i is a product of the generators $\{g_k \mid 1 \leq k \leq m\}$ and $f_i(t), g_i(t) \in \mathbb{C}[t]$. Then

$$\begin{aligned} am_1^{(t)} = 0 &\iff \Big[\sum_{i=1}^{q}\Big(a_i \otimes f_i(t) \prod_{j\neq i} g_j(t)\Big)\Big]\mathbb{C}(t) m_1^{(t)} = 0 \\ &\iff \Big[\sum_{i=1}^{q}\Big(f_i(\ell) \prod_{j\neq i} g_j(\ell) a_i\Big)\Big]\mathbb{C} m_1^{(\ell)} = 0 \forall \ell \text{ with } \ell_i \geq B \\ &\iff \Big[\sum_{i=1}^{q}\Big(f_i(\ell) \prod_{j\neq i} g_j(\ell) a_i\Big)\Big]\mathbb{C} m_1'^{(\ell)} = 0 \forall \ell \text{ with } \ell_i \geq B \\ &\iff \Big[\sum_{i=1}^{q}\Big(a_i \otimes f_i(t) \prod_{j\neq i} g_j(t)\Big)\Big]\mathbb{C}(t) m_1'^{(t)} = 0 \\ &\iff am_1'^{(t)} = 0. \end{aligned}$$

Thus, the annihilators of $m_1^{(t)}$ and $m_1'^{(t)}$ in $A \otimes \mathbb{C}(t)$ are equal. This means that the submodule of $M'^{(t)}$ generated by $m_1'^{(t)}$ is isomorphic to the submodule of $M^{(t)}$ generated by $m_1^{(t)}$. By simplicity, the submodule generated by $m_1^{(t)}$ is $M^{(t)}$ and has dimension d. Therefore, the submodule generated by $m_1'^{(t)}$ is $M'^{(t)}$ because each has dimension d. ∎

We now construct a family of torsion free simple $G_n(t)$-modules which will be proven to be isomorphic to the direct summands in the decomposition of the tensor product module $\mathcal{M}(t;N)\otimes V(\pi)$.

EXAMPLE 2.7. For $p \prec \pi$, construct an indexing set $\mathcal{I}(t;p)$ consisting of all triangular patterns $[t;l;\mathrm{m}]$ defined using the bottom $n-1$ rows of $\mathrm{m}\in\mathcal{I}(\pi)$ which have $\mathrm{m}_{n-1}=p$:

(2.8)
$$[t;l;\mathrm{m}] = \begin{bmatrix} s_n - P & & p_1 & & \cdots & & p_{n-2} & & p_{n-1} \\ & s_{n-1}-l_{n-1} & & m_{1,n-2} & \cdots & & & m_{n-2,n-2} & \\ & & & & \cdots & & & & \\ & & & & s_1 - l_1 & & & & \end{bmatrix}$$

where $s_r = \sum_{i=1} t_i$, $P=\sum_{i=1}^{n-1} p_i$, and $l_i \in \mathbb{Z}$.

Let $V(t;p)$ denote the $\mathbb{C}(t)$ linear space having a formal basis consisting of vectors $\zeta([t;l;\mathrm{m}])$ indexed by the elements $[t;l;\mathrm{m}] \in \mathcal{I}(t;p)$. We define an action of $G_n(t)$ on $V(t;p)$ by abuse of notation, writing $e_{ij}\otimes 1$ as e_{ij} and $[t;l;\mathrm{m}]$ as m and using (1.3)–(1.5) to define our action by replacing the pattern entries appearing in these formulas by the corresponding entries from $[t;l;\mathrm{m}]$. Once we have shown that $V(t;p)$ is a $G_n(t)$-module, then we shall focus on its $\lambda^{(t)} = \sum_{i=1}^n t_i\epsilon_i$ weight space and so we give the patterns indexing these weight vectors a special designation as $[t;\mathrm{m}]$ which denotes:

$$\begin{bmatrix} s_n - \sum p_i & & p_1 & & & \cdots & & p_{n-1} \\ & s_{n-1}-R_{n-2}(\mathrm{m}) & & m_{1,n-2} & & \cdots & m_{n-2,n-2} & \\ & & & & & \cdots & & \\ & & & & s_2 - R_1(\mathrm{m}) & & m_{1,1} & \\ & & & & & s_1 & & \end{bmatrix}$$

and so we have the analogue of the finite situation described in Proposition 1.9.

REMARK 2.9. For a given $[t;l;\mathrm{m}] \in \mathcal{I}(t;p)$, let $B([t;l;\mathrm{m}])$ be the maximum value in the set $\{0, p_1+l_1, l_2-l_1, \ldots, l_{n-1}-l_{n-2}, P-l_{n-1}\}$. Let $\mathcal{A}$ be any finite set of indexing patterns in $\mathcal{I}(t;\pi)$. Define $B(\mathcal{A})$ to be the maximum value in $\{B([t;l;\mathrm{m}]) \mid [t;l;\mathrm{m}]\in\mathcal{A}\}$. Let $q=(q_1,\ldots,q_n)\in\mathbb{Z}^n_{>0}$ and $q_0=0$ be such that $\ell_i = q_i - q_{i-1} \geq B(\mathcal{A})$. Then substituting q_i for s_i (or equivalently ℓ_i for t_i) into any indexing pattern in $\mathcal{A}$ yields an indexing pattern in $\mathcal{I}(q_n - P, p_1, \ldots, p_{n-1})$.

THEOREM 2.10. *$V(t;p)$ is a simple torsion free $G_n(t)$-module of dimension $|\mathcal{I}(p)|$ which is the dimension of the G_{n-1}-module $V(p)$.*

PROOF. Since the algebra structure of $G_n(t)$ is determined by the commutation relations
$$[e_{ij}, e_{kl}] = (\delta_{jk}e_{il} - \delta_{il}e_{kj}),$$
$V(t;p)$ is a $G_n(t)$-module provided

$$[e_{ij}, e_{kl}]\zeta([t;l;\mathrm{m}]) = (\delta_{jk}e_{il} - \delta_{il}e_{kj})\zeta([t;l;\mathrm{m}]). \tag{2.11}$$

We express both sides of (2.11) in terms of our basis using (1.3)–(1.5) as described above:

$$[e_{ij}, e_{kl}]\zeta([t; l; \mathrm{m}]) = \sum \phi_h(t)\zeta([t; l; \mathrm{m}]_h) \tag{2.12}$$

$$(\delta_{jk}e_{il} - \delta_{il}e_{kj})\zeta([t; l; \mathrm{m}]) = \sum \phi'_h(t)\zeta([t; l; \mathrm{m}]_h). \tag{2.13}$$

Let A be the finite set of indexing patterns appearing in (2.12) and (2.13). Let $q = (q_1, \ldots, q_n) \in \mathbb{Z}^n_{\geq 0}$ and $q_0 = 0$ be such that $q_i - q_{i-1} \geq B(A)$. Then substituting q_i for s_i into any indexing pattern in A yields an indexing pattern in $I(q_n - P, p_1, \ldots, p_{n-1})$ and so the complex numbers $\phi_h(q) = \phi'_h(q)$. It now follows from Lemma 1.18 that $\phi_h(t) = \phi'_h(t)$ and so (2.11) holds.

In view of the definition of the action of e_{ii} on the basis vectors of $V(t; p)$ it is clear that this module is an $H_n(t)$ diagonalizable module and in fact its weights are given by

$$\Phi(t) = \lambda^{(t)} + \Big\{ \sum_{i=1}^{n} \big(l_{i-1} - l_i + R_{i-1}(\mathrm{m}) - R_{i-2}(\mathrm{m}) \big)\epsilon_i \mid l_i \in \mathbb{Z}, \sum_{i=1}^{n} l_i = 0 \text{ and } \mathrm{m} \in I(\pi) \Big\}$$

As noted in Remark 2.1, in order to show that $V(t; p)$ is simple it suffices to show that any weight space is a simple $U^0_n(t)$-module.

Consider the $\lambda^{(t)} = \sum_{i=1}^n t_i\epsilon_i$ weight space $V(t; p)_{\lambda^{(t)}}$. As noted earlier, a basis for this weight space is indexed by the finite set of patterns $A = \{[t; \mathrm{m}] \mid \mathrm{m} \in I(\pi), m_{n-1} = p\}$.

Fix K, ℓ, q, and $\lambda^{(\ell)}$ satisfying the conditions of Proposition 1.9. Then substituting q_i for s_i in the patterns of A yields exactly the patterns labeling a basis for the simple U^0_n-module $V(q_n - P, p_1, \ldots, p_{n-1})_{\lambda^{(\ell)}}$. U^0_n and $U^0_n(t)$ are finitely generated and it is clear that $V(k_n - P, p_1, \ldots, p_{n-1})_{\lambda^{(\ell)}}$ is a specialization of $V(t; p)_{\lambda^{(t)}}$. Therefore, by Lemma 2.5, it follows that $V(t; p)_{\lambda^{(t)}}$ is a simple $G_n(t)$-module.

Finally we show that $V(t; p)$ is a torsion free $G_n(t)$-module. It suffices to show that $e_{k-1,k}$ and $e_{k,k-1}$ act injectively on $V(t; p)$. We restrict attention to the action of $e_{k-1,k}$ on $V(t; p)$ since the case of $e_{k,k-1}$ is similar. By definition of the action of $e_{k-1,k}$ it is clear that the coefficient of $\zeta([t; \ell; \mathrm{m}] + \delta_{1,k-1})$ in the expansion of $e_{k-1,k}\zeta([t; \ell; \mathrm{m}])$ is a nonzero rational function. Now consider any nonzero element $x \in V(t; p)$ and assume that $\zeta([t; \ell; \mathrm{m}])$ is a basis vector with a nonzero coefficient in the expansion of x such that l_{k-1} is minimal then by the minimality condition the only contribution to the coefficient of $\zeta([t; \ell; \mathrm{m}] + \delta_{1,k-1})$ arises from the action of $e_{k-1,k}$ on $\zeta([t; \ell; \mathrm{m}])$ we have that $e_{k-1,k}x \neq 0$ as required. ∎

Let

$$T(t; \pi) = M(t; N) \otimes_{\mathbb{C}(t)} \Big(V(\pi) \otimes_{\mathbb{C}} \mathbb{C}(t) \Big)$$

and $\lambda^{(t)} = \sum_{i=1}^n t_i\epsilon_i$. Then the weight space $T(t; \pi)_{\lambda^{(t)}}$ of $T(t; \pi)$ has basis:

$$B_{\lambda^{(t)}} = \Big\{ \Big(\prod_{i=1}^{n} x_i^{t_i - R_i(\mathrm{m}) + R_{i-1}(\mathrm{m})} \Big) \otimes \zeta(\mathrm{m}) \mid \mathrm{m} \in I(\mathrm{m}_n) \Big\}$$

and so $\dim T(t; \pi)_{\lambda^{(t)}} = |I(\pi)|$, *i.e.* this torsion free tensor product module has degree equal to the dimension of $V(\pi)$. Since $\dim T(t; \pi)_{\lambda^{(t)}} = |I(\pi)|$ and $\dim V(t; p)_{\lambda^{(t)}} = |I(p)|$, we see that

$$\dim T(t; \pi)_{\lambda^{(t)}} = \sum_{p \prec \pi} \dim V(t; p)_{\lambda^{(t)}}. \tag{2.14}$$

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Our aim now is to improve (2.14) by showing

$$T(t;\pi) \simeq \bigoplus \sum_{p \prec \pi} V(t;p) \quad \text{or equivalently} \tag{2.15}$$

$$T(t;\pi)_{\lambda(t)} \simeq \bigoplus \sum_{p \prec \pi} V(t;p)_{\lambda(t)}. \tag{2.16}$$

To this end we prove first that, in analogy with the finite dimensional case, the weight space $T(t;\pi)_{\lambda(t)}$ can be decomposed into one dimensional eigenspaces with respect to the action of the elements of the Gel'fand-Zetlin subalgebra $\Gamma(t)$. In fact for each $r = 1, \dots, n$ we observe that the universal enveloping algebra $U\big(H_r(t)\big) = U(H_r) \otimes \mathbb{C}(t) \simeq \mathbb{C}(t)[e_{11}, \dots, e_{rr}]$. Then for each pattern $\mathrm{m} \in I(\pi)$ we define the map

$$\Lambda_r^{\mathrm{m},t} \colon U\big(H_r(t)\big) \longrightarrow \mathbb{C}(t); \tag{2.17}$$
$$\Lambda_r^{\mathrm{m},t}\big(p(e_{11}, \dots, e_{rr})\big) = p\big(s_r - R_{r-1}(\mathrm{m}), m_{1,r-1}, \dots, m_{r-1,r-1}\big)$$

and the map

$$\gamma_{[t;\mathrm{m}]} \colon \Gamma(t) \longrightarrow \mathbb{C}(t) \quad \text{by defining its action on each } c_{rk};$$
$$\gamma_{[t;\mathrm{m}]}(c_{rk}) = \Lambda_r^{\mathrm{m},t}\big(\eta(c_{rk})\big)$$

and extending as an algebra homomorphism to all of $\Gamma(t)$. (Compare with (1.12) and (1.13).)

PROPOSITION 2.18. *(i) For any pattern* $\mathrm{m} \in I(\pi)$ *and any element* $z \in Z_r$ *we have that* $\gamma_{[t;\mathrm{m}]}(z)$ *is a polynomial over* $\mathbb{C}$ *in the single variable* s_r.

(ii) If $\mathrm{m}, \mathrm{m}' \in I(\pi)$ *with* $\mathrm{m}_{r-1} \neq \mathrm{m}'_{r-1}$ *then* $\gamma_{[t;\mathrm{m}]} \downarrow_{Z_r} \neq \gamma_{[t;\mathrm{m}']} \downarrow_{Z_r}$.

(iii) For any element $z \in Z_r$ *we have* $z\zeta([t;\mathrm{m}]) = \gamma_{[t;\mathrm{m}]}(z)\zeta([t;\mathrm{m}])$.

PROOF. Since Z_r is generated by the elements c_{rk} part (i) follows directly from the definition of the map $\gamma_{[t;\mathrm{m}]}$.

For (ii) it is clear that if $\mathrm{m}, \mathrm{m}' \in I(\pi)$ with $\mathrm{m}_{r-1} \neq \mathrm{m}'_{r-1}$ then $\Lambda_r^{\mathrm{m},t}$ is not linked to $\Lambda_r^{\mathrm{m}',t}$ and hence by the Harish-Chandra Theorem, the maps $\gamma_{[t;\mathrm{m}]} \downarrow_{Z_r}$ and $\gamma_{[t;\mathrm{m}']} \downarrow_{Z_r}$ are not equal.

Finally for part (iii) let $A = \{[t;\mathrm{m}] \mid \mathrm{m} \in I(\pi)\}$. Following the technique described in Remark 2.9, we take any vector $q \in \mathbb{Z}^n_{>0}$ such that $\ell_i = q_i - q_{i-1} \geq B(A)$. Then substituting q_i for s_i (or equivalently ℓ_i for t_i) into any indexing pattern in A yields an indexing pattern in $I\big(q_n - R_{r-1}(\mathrm{m}), m_{1,r-1}, \dots, m_{r-1,r-1}\big)$. Now in each of these finite dimension representations we have that $z\zeta([\ell;\mathrm{m}]) = \gamma_{[\ell;\mathrm{m}]}(z)\zeta([\ell;\mathrm{m}])$. Therefore by part (i) of this proposition and Lemma 1.18(i), we may conclude that $z\zeta([t;\mathrm{m}]) = \gamma_{[t;\mathrm{m}]}(z)\zeta([t;\mathrm{m}])$. ∎

By Proposition 2.18 the right hand side of (2.16) decomposes as a $\Gamma(t)$-module into inequivalent one dimensional submodules with scalar action given by the maps $\gamma_{[t;\mathrm{m}]}$ for each $\mathrm{m} \in I(\pi)$. We now show that the same is true for the left hand side of (2.16).

Fix $\mathrm{m} \in I(\pi)$. For each $\mathrm{m}' \in I(\pi)$ with $\mathrm{m}' \neq \mathrm{m}$, there exists a maximal index r such that $\mathrm{m}'_r \neq \mathrm{m}_r$ and hence by Proposition 2.18(ii) an element $z_{\mathrm{m}'} \in Z_r$ such that

$$\gamma_{[t;\mathrm{m}]}(z_{\mathrm{m}'}) \neq \gamma_{[t;\mathrm{m}']}(z_{\mathrm{m}'})$$

In fact $\gamma_{[t;\mathrm{m}]}(z_{\mathrm{m}'}) - \gamma_{[t;\mathrm{m}']}(z_{\mathrm{m}'})$ is a nonzero polynomial over $\mathbb{C}$ in the single variable s_r. Now we define the operator

$$P^{(t)}_{\mathrm{m}} = \prod_{\mathrm{m}' \in I(\pi); \mathrm{m}' \neq \mathrm{m}} \big(z_{\mathrm{m}'} - \gamma_{[t;\mathrm{m}']}(z_{\mathrm{m}'})\big).$$

PROPOSITION 2.19. *For each $z \in \Gamma(t)$ and each $v \in P^{(t)}_{\mathrm{m}} T(t;\pi)_{\lambda^{(t)}}$ we have*

$$zv = \gamma_{[t;\mathrm{m}]}(z)v$$

and moreover $\dim P^{(t)}_{\mathrm{m}} T(t;\pi)_{\lambda^{(t)}} = 1$.

PROOF. This result follows by considering various specializations of $T(t;\pi)_{\lambda^{(t)}}$. In particular select any integral n-tuple ℓ with sufficiently large components so that $T(\ell,\pi)_{\lambda^{(\ell)}}$ is a specialization of $T(t;\pi)_{\lambda^{(t)}}$ and

$$\prod_{\mathrm{m}' \in I(\pi); \mathrm{m}' \neq \mathrm{m}} \big(z_{\mathrm{m}'} - \gamma_{[\ell;\mathrm{m}']}(z_{\mathrm{m}'})\big)\Big(\bigoplus \sum_{p \prec \pi} V(\ell;p)_{\lambda^{(\ell)}}\Big) = \mathbb{C}\zeta([\ell;\mathrm{m}]) \neq 0.$$

From (1.8), we know that

$$T(\ell,\pi)_{\lambda^{(\ell)}} \simeq \bigoplus \sum_{p \prec \pi} V(\ell;p)_{\lambda^{(\ell)}}$$

Therefore, $P^{(\ell)}_{\mathrm{m}} T(\ell,\pi)_{\lambda^{(\ell)}}$ is 1-dimension and for any $z \in \Gamma$ we have

$$\big(z - \gamma_{[\ell;\mathrm{m}]}(z)\big) P^{(\ell)}_{\mathrm{m}} T(\ell;\pi)_{\lambda^{(\ell)}} = (0)$$

It follows then that

$$P^{(t)}_{\mathrm{m}} T(t;\pi)_{\lambda^{(t)}} \neq (0)$$

and for any $z \in \Gamma$ we have

$$\big(z - \gamma_{[t;\mathrm{m}]}(z)\big) P^{(t)}_{\mathrm{m}} T(t;\pi)_{\lambda^{(t)}} = (0).$$

Finally, since $\dim T(t;\pi)_{\lambda^{(t)}} = |I(\pi)|$ and the maps $\gamma_{[t;\mathrm{m}]}$ for $\mathrm{m} \in I(\pi)$ are distinct, we have that $\dim P^{(t)}_{\mathrm{m}} T(\ell;\pi)_{\lambda^{(\ell)}} = 1$. ∎

We now summarize the main results of this section in the following theorem.

THEOREM 2.20. *For each $p \prec \pi$ let $J(p) = \{\mathrm{m} \in I(\pi) \mid \mathrm{m}_{n-1} = p\}$ and let $X^{(t)}_p = \gamma_{[t,\mathrm{m}]} \downarrow_{Z_n}$ for $\mathrm{m} \in J(p)$. Then*

(i)

$$\begin{aligned}\bigoplus \sum_{\mathrm{m}' \in J(p)} P^{(t)}_{\mathrm{m}'} T(t;\pi)_{\lambda^{(t)}} &= \big\{v \in T(t;\pi)_{\lambda^{(t)}} \mid \big(z - \gamma_{[t;\mathrm{m}]}(z)\big)v = 0 \text{ for all } z \in Z_n\big\} \\ &\simeq \bigoplus \sum_{\mathrm{m}' \in J(p)} \mathbb{C}(t)\zeta([t;\mathrm{m}']) = V(t;p)_{\lambda^{(t)}}\end{aligned}$$

is a $U^0_n(t)$-module isomorphism where these modules have dimension $|I(p)|$ equal to the dimension of the G_{n-1}-module $V(p)$, *(ii) the module $T(t;\pi)$ is completely reducible*

$$T(t;\pi) \simeq \bigoplus \sum_{p\prec\pi} V(t;p) \simeq \bigoplus \sum_{p\prec\pi} T_{X_p^{(t)}},$$

with simple constituents $V(t;p) \simeq T_{X_p^{(t)}} = \{v \in T(t;\pi) \mid \big(z - X_p^{(t)}(z)\big)v = 0$ *for all* $z \in Z_n(t)\}$*,*

(iii) the central character $X_p^{(t)}$ is determined by the pseudo-highest weight

$$\Big(s_n - \sum_{i=1}^{n-1} p_i\Big)\epsilon_1 + p_1\epsilon_2 + \cdots + p_{n-1}\epsilon_n,$$

(iv) $\mathrm{Ch}(t;N) = \{X_p^{(t)} \mid p \prec \pi\}$ *is the set of central characters of the decomposition of the tensor product module $T(t;\pi)$.*

PROOF. First, we notice that $\gamma_{[t;\mathrm{m}]}$ restricted to Z_n is a central character and that these central characters are different for $\mathrm{m} \in I(\pi)$ having distinct $(n-1)^{st}$ rows. In particular the definition of $X_p^{(t)}$ is independent of our choice of $\mathrm{m} \in J(p)$. Also we see that

$$\bigoplus \sum_{\mathrm{m}'\in J(p)} P_{\mathrm{m}'}^{(t)} T(t;\pi)_{\lambda^{(t)}} = \big\{v \in T(t;\pi)_{\lambda^{(t)}} \mid \big(z - \gamma_{[\ell,\mathrm{m}]}(z)\big)v = 0 \text{ for all } z \in Z_n\big\}.$$

Since the maps $\gamma_{[t;\mathrm{m}']}$ are distinct, $\sum_{\mathrm{m}'\in J(p)} P_{\mathrm{m}'}^{(t)} T(t;\pi)_{\lambda^{(t)}}$ is direct and hence as a $U_n^0(t)$-submodule has dimension $|I(p)|$.

For all ℓ with ℓ_i large, we have that $V(\ell;p)_{\lambda^{(\ell)}}$ is a specialization of both $V(t;p)_{\lambda^{(t)}}$ and $\sum_{\mathrm{m}'\in J(p)} P_{\mathrm{m}'}^{(t)} T(t;\pi)_{\lambda^{(t)}}$. Moreover, our isomorphism

$$\sum_{\mathrm{m}'\in J(p)} P_{\mathrm{m}'}^{(\ell)} T(\ell;\pi)_{\lambda^{(\ell)}} \simeq V(\ell;p)_{\lambda^{(\ell)}}$$

carries $P_{\mathrm{m}'}^{(\ell)} T(\ell;\pi)_{\lambda^{(\ell)}}$ to $\mathbb{C}\zeta([\ell;\mathrm{m}'])$ and so by Lemma 2.6 we have a $\Gamma(t)$-module direct sum

$$\bigoplus \sum_{\mathrm{m}'\in J(p)} P_{\mathrm{m}'}^{(t)} T(t;\pi)_{\lambda^{(t)}} \simeq V(t;p)_{\lambda^{(t)}}.$$

This completes the proof of part (i).

Part (ii) follows from part (i) and Lemma 1.16, part (iii) is merely the definition of $X_p^{(t)}$ and part (iv) is clear. ∎

3. **Main Theorem.** Let M be an arbitrary simple multiplicity free torsion free G_n-module. Our goal in this section is to study the torsion free G_n-module $M \otimes V(\pi)$ or equivalently, by (1.17), the G_n-module $T(a;\pi) = M(a;N) \otimes V(\pi)$ where a is a given n-tuple of complex noninteger scalars.

As a first step we determine the central characters which can occur in the decomposition of $T(a;\pi)$. For each $p \prec \pi$, the central character $X_p^{(t)}$ of the module $V(t;p)$ is determined by the pseudo-highest weight

$$\Lambda_p^{(t)} = \Big(s_n - \sum_{i=1}^{n-1} p_i\Big)\epsilon_1 + \sum_{i=1}^{n-1} p_i\epsilon_{i+1}.$$

By Proposition 2.18, if $p, q \prec \pi$ with $p \neq q$ we have that $X_p^{(t)} \neq X_q^{(t)}$. Now for each $p \prec \pi$ we define $X_p^{(a)}$ to be the central character determined by the pseudo-highest weight

$$\Lambda_p^{(a)} = \Big(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} p_i\Big)\epsilon_1 + \sum_{i=1}^{n-1} p_i \epsilon_{i+1}$$

and define

$$\mathrm{Ch}(a, \pi) = \{X_p^{(a)} \mid p \prec \pi\}.$$

PROPOSITION 3.1. *The central characters which occur in the decomposition of $T(a; \pi)$ are contained in the set* $\mathrm{Ch}(a; \pi)$.

PROOF. By Proposition 2.18, for each $p \prec \pi$ and any element $z \in Z_n$ we have that $\big(z - X_p^{(t)}(z)\big)V(t; p) = 0$. Since by Theorem 2.20 we have that

$$T(t; \pi) \simeq \sum_{p \prec \pi} \oplus V(t; p),$$

we may conclude that for any elements $z_p \in Z_n$

$$\prod_{p \prec \pi} \big(z_p - X_p^{(t)}(z_p)\big) T(t; \pi) = 0.$$

It follows that if we substitute $t_i = a_i$ we have that

$$\prod_{p \prec \pi} \big(z_p - X_p^{(a)}(z_p)\big) T(a; \pi) = 0. \qquad \blacksquare$$

Although the central characters $X_p^{(t)}$ are distinct for distinct $p \prec \pi$, this property may be lost when we substitute a for t.

PROPOSITION 3.2. *Let $p, q \prec \pi$ with $p \neq q$. Then $X_p^{(a)} = X_q^{(a)}$ if and only if there exists an index i such that $p_j = q_j$ for $j \neq i$, and $\sum_{k=1}^n a_k = \sum_{j=1}^{n-1} p_j + q_i - i$.*

PROOF. By the Harish-Chandra Theorem, $X_p^{(a)} = X_q^{(a)}$ if and only if

$$\Lambda_p^{(a)} = \Big(\sum_{i=1}^{n} a_i - \sum_{j=1}^{n-1} p_j\Big)\epsilon_1 + \sum_{i=1}^{n-1} p_i \epsilon_{i+1} \quad \text{and} \quad \Lambda_q^{(a)} = \Big(\sum_{i=1}^{n} a_i - \sum_{j=1}^{n-1} q_j\Big)\epsilon_1 + \sum_{i=1}^{n-1} q_i \epsilon_{i+1}$$

are linked which is equivalent to the sequences $(\sum_{i=1}^n a_i - \sum_{j=1}^{n-1} p_j - 1, p_1 - 2, \ldots, p_{n-1} - n)$ and $(\sum_{i=1}^n a_i - \sum_{j=1}^{n-1} q_j - 1, q_1 - 2, \ldots, q_{n-1} - n)$ being permutations of one another. Since $p, q \prec \pi$, this permutation cannot map $p_i - i - 1$ to $q_j - j - 1$ with $i \neq j$. Therefore since $p \neq q$ we must have $\sum_{i=1}^n a_i - \sum_{j=1}^{n-1} p_j - 1 = q_i - i - 1$ and $p_j - j - 1 = q_j - j - 1$ for $j \neq i$. $\blacksquare$

REMARK 3.3. Proposition 3.2 implies that for any fixed π there exist only finitely many values (necessarily integer) for $\sum_{i=1}^n a_i$ which will permit $X_p^{(a)} = X_q^{(a)}$ for $p, q \prec \pi$ with $p \neq q$.

We are now in position to state the main theorem of this paper.

MAIN THEOREM 3.4. *For any n-tuple $a = (a_1, \ldots, a_n)$ of complex noninteger scalars such that the central characters $X_p^{(a)}$ are distinct for $p \prec \pi$ the module $T(a;\pi)$ is completely reducible. In fact*

$$T(a;\pi) = \bigoplus \sum_{p \prec \pi} T(a;\pi)_{X_p^{(a)}}$$

where, for each $p \prec \pi$, $T(a;\pi)_{X_p^{(a)}} = \{v \in T(a;\pi) \mid zv = X_p^{(a)}(z)v \text{ for all } z \in Z_n\}$ is a nonzero simple module.

Since the central characters $X_p^{(a)}$ are assumed to be distinct, it follows from the proof of Proposition 3.1 that

$$T(a;\pi) = \bigoplus \sum_{p \prec \pi} T(a;\pi)_{X_p^{(a)}}.$$

To complete the proof of the Theorem 3.4, it suffices to show that each summand $T(a;\pi)_{X_p^{(a)}}$ is a simple U_n-module or equivalently, by Lemma 1.16, that there exists a weight space of this module which is a simple U_n^0-module. We now develop some preliminary results aimed at selecting a convenient weight space with which to work.

The set of weights of $T(a;\pi)$, and hence of $T(a;\pi)_{X_p^{(a)}}$, is given by

$$\Phi(a;\pi) = \lambda^{(a)} + \Big\{ \sum_{i=1}^{n} h_i \epsilon_i \mid h_i \in \mathbb{Z}, \sum_{i=1}^{n} h_i = 0 \Big\}.$$

Let $\mathbb{Z}_0^n = \{h = (h_1, \ldots, h_n) \mid h_i \in \mathbb{Z}, \sum_{i=1}^n h_i = 0\}$. For any $h \in \mathbb{Z}_0^n$ and each $\mathrm{m} \in I(\pi)$, we define a map $\gamma_{[a+h;\mathrm{m}]} \colon \Gamma \longrightarrow \mathbb{C}$ such that $\gamma_{[a+h;\mathrm{m}]} \downarrow_{Z_r}$ is the central character of the simple G_r-module of highest weight

$$\Lambda_r^{a+h,\mathrm{m}} = \Big(\sum_{i=1}^{r} (a_i + h_i) - R_{r-1}(\mathrm{m}) \Big) \epsilon_1 + \sum_{i=1}^{r-1} m_{i,r-1} \epsilon_{i+1}.$$

PROPOSITION 3.5. *Assume that $a \in \mathbb{C}^n$ is an n-tuple of noninteger complex scalars such that the central characters $X_p^{(a)}$ are distinct for all $p \prec \pi$. Let $h \in \mathbb{Z}_0^n$ with the real part $\mathrm{Re}(a_i + h_i)$ of $a_i + h_i$ being greater than π_1 for all $i = 1, \ldots, n-1$. Then for $\mathrm{m}, \mathrm{m}' \in I(\pi)$ with $\mathrm{m} \neq \mathrm{m}'$, $\gamma_{[a+h;\mathrm{m}]} \neq \gamma_{[a+h;\mathrm{m}']}$.*

PROOF. Let $\mathrm{m}, \mathrm{m}' \in I(\pi)$ with $\mathrm{m} \neq \mathrm{m}'$. There exists $1 \leq r \leq n-1$ such that $\mathrm{m}_r \neq \mathrm{m}'_r$. If $r = n-1$ then we have by assumption that

$$\gamma_{[a+h;m]} \downarrow_{Z_n} = X_{\mathrm{m}_{n-1}}^{(a)} \neq X_{\mathrm{m}'_{n-1}}^{(a)} = \gamma_{[a+h;m']} \downarrow_{Z_n}.$$

On the other hand, if $r < n-1$ then for any $j = 1, \ldots, r-1$

$$\mathrm{Re}\Big(\sum_{i=1}^{r} (a_i + h_i) - R_{r-1}(\mathrm{m}) - 1 \Big) > \pi_1 - 1 > m'_{j,r-1} - j - 1.$$

This implies that $\Lambda_r^{(a+h;\mathrm{m})}$ is not linked to $\Lambda_r^{(a+h;\mathrm{m}')}$ and hence by the Harish-Chandra Theorem $\gamma_{[a+h;\mathrm{m}]} \downarrow_{Z_r} \neq \gamma_{[a+h;\mathrm{m}']} \downarrow_{Z_r}$. ∎

Fix $p \prec \pi$ and a pattern $\mathrm{m} \in I(\pi)$ such that $\mathrm{m}_{n-1} = p$. Then by Proposition 2.19, we know that $P_m^{(t)}T(t;\pi)_{\lambda^{(t)}}$ is a 1-dimensional $\mathbb{C}(t)$ vector space. Certainly, it is possible to select a vector $v_{[t;\mathrm{m}]} \in P_m^{(t)}T(t;\pi)_{\lambda^{(t)}}$ such that when expanded in terms of the basis $B_{\lambda^{(t)}}$ the coefficients are polynomials in t and $(s_n - \sum_{i=1}^n a_i)$ is not a common factor of these polynomials. By Lemma 1.19, there exists a vector $h \in \mathbb{Z}_0^n$ such that $\mathrm{Re}(a_i + h_i) > \pi_1$ for all $i = 1, \ldots, n-1$ and at least one of the $B_{\lambda^{(t)}}$ coefficients of $v_{[t;\mathrm{m}]}$ evaluated at $t = a + h$ is nonzero. In other words, $v_{[a+h;\mathrm{m}]}$ is a nonzero vector. Also since $zv_{[t;m]} = \gamma_{[t;\mathrm{m}]}(z)v_{[t;\mathrm{m}]}$ for all $z \in \Gamma$, it follows that $zv_{[a+h;m]} = \gamma_{[a+h;\mathrm{m}]}(z)v_{[a+h;\mathrm{m}]}$ for all $z \in \Gamma$. The $\sum_{i=1}^n (a_i + h_i)\epsilon_i$ weight space of $T(a;\pi)$ is the weight space on which we will focus.

REMARK 3.6. We note that for any $h \in \mathbb{Z}_0^n$, $M(a+h;N) \simeq M(a;N)$ and hence $T(a+h;\pi) \simeq T(a;\pi)$. Therefore, in order to simplify our notation and without loss of generality, we assume that the vector h, selected above, is 0. In other words, we are assuming that $\mathrm{Re}(a_i) > \pi_1$ for all $i = 1, \ldots, n-1$, $v_{[a;\mathrm{m}]}$ is a nonzero vector in the $\lambda^{(a)} = \sum_{i=1}^n a_i\epsilon_i$ weight space of $T(a;\pi)$ and $zv_{[a;\mathrm{m}]} = \gamma_{[a;\mathrm{m}]}(z)v_{[a;\mathrm{m}]}$ for all $z \in \Gamma$.

Recall that $V(t;p)_{\lambda^{(t)}}$ is a simple $U_n^0(t)$-module having a linear basis $\{\zeta([t;\mathrm{m}]) \mid m \in J(p)\}$. It follows that for any distinct patterns $\mathrm{m}', \mathrm{m}'' \in J(p)$ there exists an element $u_{\mathrm{m}'\mathrm{m}''}(t) \in U_n^0(t)$ such that

$$u_{\mathrm{m}'\mathrm{m}''}(t)\zeta([t;\mathrm{m}']) = Q_{\mathrm{m}'\mathrm{m}''}(t)\zeta([t;\mathrm{m}''])$$

where $Q_{\mathrm{m}'\mathrm{m}''}(t)$ is a nonzero rational function in t. We claim a little more.

PROPOSITION 3.7. *There exist a choice of* $u_{\mathrm{m}'\mathrm{m}''}(t) \in U_n^0(t)$ *such that the corresponding rational function* $Q_{\mathrm{m}'\mathrm{m}''}(t)$ *evaluated at* $t = a$ *is well defined and nonzero.*

PROOF. We say that two patterns $\mathrm{m}^{(1)}, \mathrm{m}^{(2)} \in J(p)$ are *adjacent* provided they are equal in all entries but two, these two are both on the row indexed by r, and there the difference is that the p-th coordinate is one greater in $\mathrm{m}^{(2)}$ than in $\mathrm{m}^{(1)}$ while the q-th coordinate is one less in $\mathrm{m}^{(2)}$ than in $\mathrm{m}^{(1)}$. Briefly, using the notation introduced in Section 1, there exists an index $r = 1, \ldots, n-1$ and indices $1 \leq p \neq q \leq r$ such that $[t;\mathrm{m}^{(2)}] = [t;\mathrm{m}^{(1)}] + \delta_{pr} - \delta_{qr}$. Since for any patterns $\mathrm{m}', \mathrm{m}'' \in J(p)$ there exists a sequence of adjacent patterns connecting them, without loss of generality, we may assume that $\mathrm{m}', \mathrm{m}''$ are adjacent. Therefore we assume that, $[t;\mathrm{m}''] = [t;\mathrm{m}'] + \delta_{pr} - \delta_{qr}$ for some choice of indices r, p, q.

The element $u_{\mathrm{m}'\mathrm{m}''}(t)$ that we want will be constructed as the product of two elements of $U_n^0(t)$. The first factor is $e_{r,r+1}e_{r+1,r}$. The coefficient of the basis vector $\zeta([t;\mathrm{m}''])$ in the expansion of the element $e_{r,r+1}e_{r+1,r}\zeta([t;\mathrm{m}'])$ is equal to the product of the coefficient of $\zeta([t;\mathrm{m}'] - \delta_{qr})$ in the expansion of $e_{r+1,r}\zeta([t;\mathrm{m}'])$ and the coefficient of $\zeta([t;\mathrm{m}'']) = \zeta([t;\mathrm{m}'] + \delta_{pr} - \delta_{qr})$ in the expansion of $e_{r,r+1}\zeta([t;\mathrm{m}'] - \delta_{qr})$. We claim that each of these rational functions is well defined and nonzero when evaluated at $t = a$. By definition of the action of $e_{r+1,r}$ on the basis element $\zeta([t;\mathrm{m}'])$ the coefficient of $\zeta([t;\mathrm{m}'] - \delta_{qr})$ is given by

$$\frac{\prod_{k=1}^{r-1}(l_{k,r-1} - l_{q,r})}{\prod_{k\neq j}(l_{k,r} - l_{q,r})}$$

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where $l_{i,j}$ is equal to the (i,j) component of the pattern $[t;\mathrm{m}']$ minus i. As noted earlier this is a nonzero rational function in the variables t_i. We further claim that since each a_i is assumed to be noninteger with $\mathrm{Re}(a_i) > \pi_1$ for $i = 1,\ldots,n-1$ evaluating this function at $t = a$ also yields a nonzero value. This requires an evaluation of each factor $l_{k,r-1} - l_{qr}$ and $l_{k,r} - l_{qr}$ under substitution of $t = a$. Rather than an exhaustive treatment of each possible case we will illustrate this by considering two typical examples and leave the rest to the reader. First assume that $q \geq 2$ and consider the term $l_{1r} - l_{qr} = (s_r - R_{r-1}(\mathrm{m}') - 1) - (\mathrm{m}'_{q-1,r-1} - q)$ which occurs in the denominator. Substituting $t_i = a_i$ and using the assumption that $\mathrm{Re}(a_i) > \pi_1$ we have

$$\begin{aligned}\mathrm{Re}\Big((l_{1r} - l_{qr})\mid_{t=a}\Big) &= \mathrm{Re}\Big(\sum_{i=1}^{r} a_i - R_{r-1}(\mathrm{m}') - 1 - \mathrm{m}'_{q-1,r-1} + q\Big)\\ &> r\pi_1 - (r-1)\pi_1 - 1 - \pi_1 + q = q - 1 > 0\end{aligned}$$

i.e. $(l_{1r} - l_{qr})\mid_{t=a} \neq 0$. As a second example assume that $q = 1$ and consider the factor $l_{1,r-1} - l_{1,r} = \big(s_{r-1} - R_{r-2}(\mathrm{m}') - 1\big) - (s_r - R_{r-1}(\mathrm{m}') - 1) = \big(-t_r - R_{r-2}(\mathrm{m}') + R_{r-1}(\mathrm{m}')\big)$ which occurs in the numerator. In this case since the scalars a_i are assumed to be noninteger it immediately follows that $(l_{1,r-1} - l_{1,r})\mid_{t=a} \neq 0$. A similar analysis of the coefficient of $\zeta([t;\mathrm{m}''])$ occurring in the expansion of the element $e_{r,r+1}\zeta([t;\mathrm{m}'] - \delta_{qr})$ establishes that it is a nontrivial rational function which is well defined and nonzero when evaluated at $t = a$. Combining these two statements yields our claim.

The second factor in the element $u_{\mathrm{m}'\mathrm{m}''}(t)$ is a refinement of the element $P^{(t)}_{\mathrm{m}}$ introduced in Section 2 with $\mathrm{m} = \mathrm{m}''$. Since $\mathrm{Re}(a_i) > \pi_1$ for $i = 1,\ldots,n-1$, by Proposition 3.5 we know that the maps $\gamma_{[a;\mathrm{m}]}$ are distinct for all $\mathrm{m} \in I(\pi)$. Therefore for any pattern $\mathrm{m} \neq \mathrm{m}''$ there exists an element $z_{\mathrm{m}} \in \cup_{r=1}^{n} Z_r$ such that $\gamma_{[a;\mathrm{m}]}(z_{\mathrm{m}}) \neq \gamma_{[a;\mathrm{m}'']}(z_{\mathrm{m}})$ and hence a fortiori $\gamma_{[t;\mathrm{m}]}(z_{\mathrm{m}}) \neq \gamma_{[t;\mathrm{m}'']}(z_{\mathrm{m}})$. Using these elements we define

$$P^{(t)}_{\mathrm{m}''} = \prod_{\mathrm{m}\neq\mathrm{m}''} \Big(z_{\mathrm{m}} - \gamma_{[t;\mathrm{m}]}(z_{\mathrm{m}})\Big).$$

As in Section 2, we note that for any pattern $\mathrm{m}^{(1)} \in I(\pi)$ with $m^{(1)} \neq m''$ we have

$$P^{(t)}_{\mathrm{m}''}\zeta([t;\mathrm{m}^{(1)}]) = 0$$

and

$$P^{(t)}_{\mathrm{m}''}\zeta([t;\mathrm{m}'']) = \prod_{\mathrm{m}\neq\mathrm{m}''}\Big(\gamma_{[t;\mathrm{m}'']}(z_{\mathrm{m}}) - \gamma_{[t;\mathrm{m}]}(z_{\mathrm{m}})\Big)\zeta([t;\mathrm{m}'']) \neq 0.$$

Further by our choice of the elements z_m we also have that $P^{(t)}_{\mathrm{m}''}\zeta([t;\mathrm{m}^{(1)}])\mid_{t=a}$ is nonzero if and only if $\mathrm{m}^{(1)} = \mathrm{m}''$.

The element $u_{\mathrm{m}'\mathrm{m}''}(t) = P^{(t)}_{\mathrm{m}''} e_{r,r+1} e_{r+1,r}$ then has the required properties. ∎

We are now in position to complete the proof of the Main Theorem.

PROOF OF THEOREM 3.4. Fix $\mathrm{m} \in J(p)$ and select a vector $v_{[t;\mathrm{m}]} \in P^{(t)}_{\mathrm{m}'} T(t;\pi)_{\lambda^{(t)}}$ such that $0 \neq v_{[a;\mathrm{m}]} \in P^{(a)}_{\mathrm{m}'} T(a;\pi)_{\lambda^{(a)}}$. Let

$$\Phi\colon V(t;p) \longrightarrow \bigoplus \sum_{\mathrm{m}'\in J(p)} P^{(t)}_{\mathrm{m}'} T(t;\pi)_{\lambda^{(t)}}$$

be the $U_n^0(t)$-module isomorphism determined by setting $\Phi\big(\zeta([t;\mathrm{m}])\big) = v_{[t;\mathrm{m}]}$.

For each $\mathrm{m}' \in J(p)$ with $\mathrm{m}' \neq \mathrm{m}$, we define $v_{[t;\mathrm{m}']} = u_{\mathrm{mm}'}(t)v_{[t;\mathrm{m}]}$. We note that

$$
\begin{aligned}
u_{\mathrm{m}'\mathrm{m}}(t)u_{\mathrm{mm}'}(t)v_{[t;\mathrm{m}]} &= u_{\mathrm{m}'\mathrm{m}}(t)u_{\mathrm{mm}'}(t)\Phi\Big(\zeta([t;\mathrm{m}])\Big)\\
&= \Phi\Big(u_{\mathrm{m}'\mathrm{m}}(t)u_{\mathrm{mm}'}(t)\zeta([t;\mathrm{m}])\Big)\\
&= Q_{\mathrm{m}'\mathrm{m}}(t)Q_{\mathrm{mm}'}(t)v_{[t;\mathrm{m}]}.
\end{aligned}
\tag{3.8}
$$

Since each of the terms on the right hand side is well defined and nonzero when we substitute $t = a$ we have that $u_{\mathrm{m}'\mathrm{m}}(a)u_{\mathrm{mm}'}(a)v([a;\mathrm{m}]) \neq 0$ and therefore $v_{[a;\mathrm{m}']} = u_{\mathrm{mm}'}(a)v_{[a;\mathrm{m}]} \neq 0$. Clearly $v_{[t;\mathrm{m}']} \in P^{(t)}_{\mathrm{m}'}T(t;\pi)_{\lambda^{(t)}}$ and hence $v_{[a;\mathrm{m}']} \in P^{(a)}_{\mathrm{m}'}T(a;\pi)_{\lambda^{(a)}}$.

By Proposition 3.5, the maps $\gamma_{[a;\mathrm{m}']}$ are distinct for distinct patterns $\mathrm{m}' \in J(p)$ and hence the $\lambda^{(a)}$ weight space of $T(a;\pi)_{X_p^{(a)}}$ is a direct sum of inequivalent Γ-submodules $P^{(a)}_{\mathrm{m}'}T(a;\pi)_{\lambda^{(a)}}$. In order to prove that this subspace is an simple U_n^0-module, it suffices to show that for each $\mathrm{m}' \in J(p)$ with $\mathrm{m}' \neq \mathrm{m}$ we have $v_{[a;\mathrm{m}']} \in U_n^0 v_{[a;\mathrm{m}]}$ and $v_{[a;\mathrm{m}]} \in U_n^0 v_{[a;\mathrm{m}']}$. The first result follows from the definition of $v_{[a;\mathrm{m}']} = u_{\mathrm{mm}'}(a)v_{[a,;\mathrm{m}]}$ and the second follows from (3.8), *i.e.* $u_{\mathrm{m}'\mathrm{m}}(a)v_{[a;\mathrm{m}']} = u_{\mathrm{m}'\mathrm{m}}(a)u_{\mathrm{mm}'}(a)v([a;\mathrm{m}])$ which is a nonzero multiple of $v_{[a;\mathrm{m}]}$. ∎

4. **An indecomposable submodule of a tensor product.** To the best of our knowledge, there is no known nonsimple indecomposable torsion free module for the simple finite symplectic algebras $\mathrm{sp}(2n,\mathbb{C})$. In fact, a slight modification of Chen's work [C], shows that no such modules of degree 2 exist for $\mathrm{sp}(2n,\mathbb{C})$. However, nonsimple indecomposable torsion free modules do exist for $s\ell(n,\mathbb{C})$. In this section, such an example is presented. It arises in $T(a;N)$ when the central characters of $\mathrm{Ch}(a;N)$ are not distinct. This then justifies the hypothesis of our Main Theorem. Consider the simple G_4-module $V(1,1,0,0)$ having highest weight $\epsilon_1+\epsilon_2$. Set $K = 1$ and $N = 0$, and recall (1.6). Then $V(1,1,0,0)$ can be realized as the 6 dimensional submodule of $M\big((1,0,0,0);0\big) \otimes M\big((1,0,0,0);0\big)$ as given by:

$$V(1,1,0,0) = \mathrm{span}_{\mathbb{C}}\{x_i \otimes x_j - x_j \otimes x_i \mid 1 \leq i < j \leq 4\}$$

under the usual action on a tensor product module. Fix a 4-tuple of complex scalars $a = (a_1,a_2,a_3,a_4)$ such that $a_1, a_2, a_3, a_4, a_1+a_2, a_1+a_2+a_3$ are nonintegers and $a_1+a_2+a_3+a_4 = 0$. Recall that $M(a;2)$ denotes the pointed torsion free module with basis given by

$$\{x_1^{a_1+h_1}x_2^{a_2+h_2}x_3^{a_3+h_3}x_4^{a_4+h_4} \mid h_i \in \mathbb{Z} \text{ with } h_1+h_2+h_3+h_4 = -2\}.$$

The tensor product module $T\big(a;(1,1,0,0)\big) = M(a;2) \otimes V(1,1,0,0)$ is torsion free of degree 6. By Lemma 1.16, the composition series for $T\big(a;(1,1,0,0)\big)$ as a U_4-module is determined by the composition series for any weight space as a U_4^0-module.

The $\lambda^{(a)} = \sum_{i=1}^4 a_i\epsilon_i$ weight space $T\big(a;(1,1,0,0)\big)_{\lambda^{(a)}}$ of $T\big(a;(1,1,0,0)\big)$ has a linear basis given by

$$w_{ij} = \frac{x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4}}{x_ix_j} \otimes (x_i \otimes x_j - x_j \otimes x_i)$$

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where $1 \le i < j \le 4$ and the ratio expression gives us a short hand way of subtracting 1 from exponents. For $1 \le i \le 6$, define v_i by

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ a_1+a_2 & a_3 & -a_3 & -a_1-a_2-a_3 & a_1+a_2+a_3 & 0 \\ 0 & a_1 & a_2 & -a_1 & -a_2 & a_1+a_2 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & 0 & a_1+a_2 & a_3 \\ 0 & 0 & 0 & 0 & 0 & a_1 \end{bmatrix} \begin{bmatrix} w_{12} \\ w_{13} \\ w_{23} \\ w_{14} \\ w_{24} \\ w_{34} \end{bmatrix}.$$

The determinant of the coefficient matrix is $a_1^2(a_1+a_2)^2(a_1+a_2+a_3) \neq 0$. Thus, $B = \{v_i \mid i = 1, \dots, 6\}$ forms an alternate basis for this $\lambda^{(a)}$ weight space.

Straightforward calculations using the generators of U_4^0

$$\{e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_k i_1} \mid 1 \le i_j \le 4; i_1, i_2, \dots, i_k \text{ distinct}\}$$

shows that the weight space $T\big(a; (1,1,0,0)\big)_{\lambda^{(a)}}$ is a cyclic module generated by v_6 having proper U_4^0-submodules:

$$V_1 = \operatorname{span}_{\mathbb{C}}\{v_2, v_3\}, \quad V_2 = \operatorname{span}_{\mathbb{C}}\{v_1, v_2, v_3\}, \quad V_3 = \operatorname{span}_{\mathbb{C}}\{v_1, v_2, v_3, v_4\}.$$

(These calculations are easily done using Maple.)

$$T\big(a; (1,1,0,0)\big)_{\lambda^{(a)}} > V_3 > V_2 > V_1 > 0$$

is a composition series. We also note that $T_{\lambda^{(a)}}/V_3$ is equivalent to V_1 under the map defined by sending the vectors $v_5 + V_3$, $v_6 + V_3$ to the vectors v_2, v_3 respectively. Finally V_3/V_2 and V_2/V_1 are pointed torsion free modules with the first module isomorphic to $M(a; 0)$ and the second quotient module isomorphic to $M\big((a_1-1, a_2-1, a_3-1, a_4-1); 0\big)$.

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