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On Stanley Depths of Certain Monomial Factor Algebras

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Abstract. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n-variables over a field K and I a monomial ideal of S. According to one standard primary decomposition of I, we get a Stanley decomposition of the monomial factor algebra S/I. Using this Stanley decomposition, one can estimate the Stanley depth of S/I. It is proved that $\operatorname{sdepth}_S(S/I) \geq \operatorname{size}_S(I)$. When I is squarefree and $\operatorname{bigsize}_S(I) \leq 2$, the Stanley conjecture holds for S/I, *i.e.*, $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$.

1 Introduction

Let K be a field and let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n-variables over K. Let M be a finitely generated multigraded S-module. Then a Stanley decomposition $\mathcal{D}$ of M is a finite direct sum of K-spaces:

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i],$$

where $m_i \in M$ is homogeneous and $Z_i \subseteq \{x_1, \dots, x_n\}$, $i = 1, \dots, r$, and its Stanley depth, $\operatorname{sdepth}_S(\mathcal{D})$, is defined as $\min\{|Z_i| \mid i = 1, \dots, r\}$. The Stanley depth of M is

$$\operatorname{sdepth}_S(M) = \max \operatorname{sdepth}_S(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

Stanley [11] conjectured that $\operatorname{sdepth}_S(M) \geq \operatorname{depth}_S(M)$, which has recently become an interesting topic in commutative algebra.

To estimate the Stanley depth, it is important to find "good" Stanley decompositions. When I is a monomial ideal of S, S/I is said to be a monomial factor algebra. Through one kind of Stanley decomposition of I, several interesting results about the Stanley depth of I were obtained in [3, 6, 7].

In this paper, we are interested in whether the similar results hold for S/I. Our approach is to find a similar Stanley decomposition for S from which one can get a Stanley decomposition for S/I. Such decompositions are presented in Section 2. Two applications are given in the following two sections.

The size of a monomial ideal was introduced by Lyubeznik [5], who also proved the well-known result that $\operatorname{depth}_S(I) \geq 1 + \operatorname{size}_S(I)$. For the Stanley depth, Herzog, Popescu and Vadoiu [3] obtained a similar result that $\operatorname{sdepth}_S(I) \geq 1 + \operatorname{size}_S(I)$. They expected that it holds that $\operatorname{sdepth}_S(S/I) \geq \operatorname{size}_S(I)$. We will show that the result is true in Section 3.

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Let I be a squarefree monomial ideal of S. When the sum of every three different minimal prime ideals of I is the maximal ideal, *i.e.,* $\operatorname{bigsize}_S(I) \le 2$, Popescu [8] proved that the Stanley conjecture holds for I. In this case, we will show that the Stanley conjecture holds also for S/I in Section 4.

2 Stanley Decompositions of Polynomial Rings

Let I be a monomial ideal of $S = K[x_1, \dots, x_n]$. Then I has the standard primary decomposition, *i.e.,* the unique irredundant presentation as an intersection of irreducible monomial ideals:

$$I = Q_1 \cap \cdots \cap Q_s,$$

with $\sqrt{Q_i} = P_i$, $i = 1, \dots, s$, and $\operatorname{Ass}_S(S/I) = \{P_1, \dots, P_s\}$.

Let $r < n$ be a positive integer, $S' = K[x_1, \dots, x_r]$, and $S'' = K[x_{r+1}, \dots, x_n]$. For any $\tau \subseteq [s] = \{1, \dots, s\}$, let I_τ be the K-subspace of S generated by the following set of monomials

$$\{uv \mid u \in S', u \in (\cap_{j \notin \tau} Q_j) \smallsetminus \big(\sum_{j \in \tau} Q_j\big), v \in S'', v \in \cap_{j \in \tau} Q_j\}.$$

Then by [3, Proposition 2.1], I has a decomposition as K-spaces:

$$I = \bigoplus_{\tau \subseteq [s]} I_\tau.$$

This kind of decomposition was first introduced by Popescu [6], where the Stanley conjecture for squarefree monomial ideals of intersections of three monomial prime ideals was verified. The generalized forms of the decomposition were used to estimate the Stanley depth of I in [3,7].

We plan to consider the Stanley depth of S/I. To obtain a suitable Stanley decomposition for S/I, it is necessary to decompose S.

For any $\tau \subseteq [s]$, set S_τ as the K-subspace of S generated by the following set of monomials

$$\{uv \mid u \in S', u \in (\cap_{j \notin \tau} Q_j) \smallsetminus \big(\sum_{j \in \tau} Q_j\big), v \in S''\}.$$

Note that $S_\varnothing = (I \cap S')S$ and $S_{[s]} = ((S \smallsetminus (\sum_{j=1}^s Q_j)) \cap S')[x_{r+1}, \dots, x_n]$.

Then, as a K-space, S has the following decomposition.

Proposition 2.1 $S = \bigoplus_{\tau \subseteq [s]} S_\tau$.

Proof Any monomial $w \in S$ can be written as $w = uv$ with $u \in S'$ and $v \in S''$. Set $\tau = \{j \in [s] \mid u \notin Q_j\}$. Then $w \in S_\tau$. Hence $S = \sum_{\tau \subseteq [s]} S_\tau$.

We still need to show that the sum is direct. Suppose that $\tau \neq \sigma \subseteq [s]$, $j \in \tau \smallsetminus \sigma$ and $w = uv \in S_\tau \cap S_\sigma$. By $j \in \tau$, we have that $u \notin Q_j$, but by $j \notin \sigma$, we have that $u \in Q_j$, a contradiction. The result follows. ■

Note that $I_\tau \subseteq S_\tau$. Then from Proposition 2.1, we get a decomposition of S/I.

Corollary 2.2 $S/I = \bigoplus_{\tau \subseteq [s]} S_\tau / I_\tau$.

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Example 2.3 Let $S = K[x_1, x_2, x_3]$, $m = (x_1, x_2, x_3)$ and $I = m^2$. Then I has the following standard primary decomposition:

$$I = (x_1^2, x_2, x_3) \cap (x_1, x_2^2, x_3) \cap (x_1, x_2, x_3^2).$$

Put $r = 1$. Then, according to [3, Example 2.2], I has the decomposition

$$I = x_1^2 K[x_1, x_2, x_3] \oplus (x_1x_2, x_1x_3)K[x_2, x_3] \oplus (x_2^2, x_2x_3, x_3^2)K[x_2, x_3].$$

In this case, the decomposition of S in Proposition 2.1 is

$$S = x_1^2 K[x_1, x_2, x_3] \oplus x_1 K[x_2, x_3] \oplus K[x_2, x_3].$$

For the Stanley depth of an S-module, we have the following lemma.

Lemma 2.4 ([1, Thm. 1.4]) *Let M be a multigraded S-module. If* $\operatorname{depth}_S(M) > 0$, *then* $\operatorname{sdepth}_S(M) > 0$.

Let H be a multigraded S'-module and let L be a multigraded S''-module. Then they have natural multigraded S-module structures. As noted in the proof of [3, Theorem 3.1], one has that

$$\operatorname{sdepth}_S(H \otimes_K L) \geq \operatorname{sdepth}_{S'}(H) + \operatorname{sdepth}_{S''}(L).$$

Now, we use Corollary 2.2 to estimate the Stanley depth of S/I. We will see that each S_τ/I_τ has a structure of multigraded S-module. It turns out that $\operatorname{sdepth}_S(S/I)$ is not less than the minimum of all $\operatorname{sdepth}_S(S_\tau/I_\tau)$, $\tau \subseteq [s]$.

Theorem 2.5 *Let the situation be as above and assume that one of P_i is $(x_1, \dots, x_r)$. Then*

$$\operatorname{sdepth}_S(S/I) \geq \min\{n - r, \operatorname{sdepth}_{S'}(H_\tau) + \operatorname{sdepth}_{S''}(S''/(\cap_{j\in\tau} Q_j) \cap S'')\},$$

where the minimum is taken over all nonempty proper subsets $\tau \subset [s]$ such that $H_\tau \neq 0$ with

$$H_\tau = ((\cap_{j\notin\tau} Q_j) \cap S' + \Big(\sum_{j\in\tau} Q_j\Big) \cap S')/\Big(\sum_{j\in\tau} Q_j\Big) \cap S'.$$

Proof Let $L_\tau = (\cap_{j\in\tau} Q_j) \cap S''$. Then, as K-spaces,

$$S_\tau = H_\tau \otimes_K S'', \quad I_\tau = H_\tau \otimes_K L_\tau,$$

and

$$S_\tau/I_\tau = (H_\tau \otimes_K S'')/(H_\tau \otimes_K L_\tau) = H_\tau \otimes_K (S''/L_\tau).$$

By the natural multigraded S'-module structure of H_τ and S''-module structure of L_τ, we get a multigraded S-module structure on S_τ/I_τ. By virtue of Corollary 2.2, we have

$$\operatorname{sdepth}_S(S/I) \geq \min_{\tau\subseteq[s]}\{\operatorname{sdepth}_S(S_\tau/I_\tau) \mid S_\tau/I_\tau \neq 0\}.$$

Note that $S_{\varnothing} = (I \cap S')S = I_{\varnothing}$, $S_{[s]} = ((S \smallsetminus (\sum_{j=1}^{s} Q_j)) \cap S')[x_{r+1}, \dots, x_n]$, and $I_{[s]} = 0$, as $L_{[s]} = 0$ by the assumption that one of P_i is $(x_1, \dots, x_r)$. It follows that

$$\begin{aligned} \operatorname{sdepth}_S(S_{[s]}/I_{[s]}) &= \operatorname{sdepth}_S\Big(\Big(\Big(S \smallsetminus \Big(\sum_{j=1}^{s} Q_j \Big) \Big) \cap S' \Big) [x_{r+1}, \dots, x_n] \Big) \\ &\geq n - r. \end{aligned}$$

For any nonempty proper subset $\tau \subset [s]$ such that $H_\tau \neq 0$,

$$\begin{aligned} \operatorname{sdepth}_S(S_\tau/I_\tau) &= \operatorname{sdepth}_S(H_\tau \otimes_K (S''/L_\tau)) \\ &\geq \operatorname{sdepth}_{S'}(H_\tau) + \operatorname{sdepth}_{S''}(S''/L_\tau). \end{aligned}$$

The result follows. ∎

3 Stanley Depth and Size

Let I be a monomial ideal of S and notations as in Section 2. The size of I is defined as

$$\operatorname{size}_S(I) = v + n - \operatorname{height}\Big(\sum_{j=1}^{s} P_j \Big) - 1,$$

where

$$v = \min\{ t \mid \exists i_1 < i_2 < \cdots < i_t \text{ such that } \sum_{k=1}^{t} P_{i_k} = \sum_{j=1}^{s} P_j \}.$$

It was showed by Lyubeznik [5] that $\operatorname{depth}_S(S/I) \geq \operatorname{size}_S(I)$. Let us discuss the Stanley depth of S/I by using Theorem 2.5. We need the following lemma.

Lemma 3.1 *([3, Lemma 3.2]) Assume that $P_1 = (x_1, \dots, x_r)$ and $(\cap_{j \in \tau} Q_j) \cap S'' \neq 0$. Then*

$$\operatorname{size}_S(I) \leq \operatorname{size}_{S''}((\cap_{j \in \tau} Q_j) \cap S'') + 1.$$

Furthermore, if $P_1 \subseteq \sum_{j \in \tau} P_j$, then

$$\operatorname{size}_S(I) \leq \operatorname{size}_{S''}((\cap_{j \in \tau} Q_j) \cap S'').$$

Herzog, Popescu, and Vadoiu [3] showed that $\operatorname{sdepth}_S(I) \geq 1 + \operatorname{size}_S(I)$ for any monomial ideal I of S. They expected that $\operatorname{sdepth}_S(S/I) \geq \operatorname{size}_S(I)$ also holds. Let us prove this result. Note that its squarefree case was shown in [12].

Theorem 3.2 *Let I be a monomial ideal of S. Then*

$$\operatorname{sdepth}_S(S/I) \geq \operatorname{size}_S(I).$$

Proof Let $I = Q_1 \cap \cdots \cap Q_s$ be the standard primary decomposition of I. Set $\sqrt{Q_i} = P_i$, $i = 1, \dots, s$. In order to prove the result, we may assume that $\sum_{i=1}^{s} P_i = m = (x_1, \dots, x_n)$.

We use induction on s. If $s = 1$, then I is m-primary. By definition, we see immediately that $\operatorname{size}_S(I) = 0$, so the result is clear. Now assume that $s > 1$, and the result is true for any positive integer less than s. We may assume that $P_1 = (x_1, \dots, x_r)$. If $r = n$, then the result is also clear, as $\operatorname{size}_S(I) = 0$ by definition. Let us assume that $r < n$ and use this r to estimate $\operatorname{sdepth}_S(S/I)$ by Theorem 2.5.

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Note that
$$n-r=\dim(S/P_1)\geq \operatorname{depth}_S(S/I)\geq \operatorname{size}_S(I).$$
For any nonempty proper subset $\tau\subset[s]$ such that $H_\tau\neq 0$, let us show that
$$\operatorname{sdepth}_{S'}(H_\tau)+\operatorname{sdepth}_{S''}(S''/(\cap_{j\in\tau}Q_j)\cap S'')\geq \operatorname{size}_S(I).$$
By induction hypothesis, $\operatorname{sdepth}_{S''}(S''/(\cap_{j\in\tau}Q_j)\cap S'')\geq \operatorname{size}_{S''}((\cap_{j\in\tau}Q_j)\cap S'')$. Then, by Lemma 3.1, it is enough to show that $\operatorname{sdepth}_{S'}(H_\tau)>0$ provided $P_1\notin\sum_{j\in\tau}P_j$. According to Lemma 2.4, it is enough to show that $\operatorname{depth}_{S'}(H_\tau)>0$ in this case.

Assume, on the contrary, that $\operatorname{depth}_{S'}(H_\tau)=0$; then H_τ is a torsion S'-module. By $P_1\notin\sum_{j\in\tau}P_j$, we may assume that $(\sum_{j\in\tau}Q_j)\cap S'=(x_k^{a_k},\dots,x_r^{a_r})$ with $k>1$. Then
$$H_\tau=((\cap_{j\notin\tau}Q_j)\cap S'+(x_k^{a_k},\dots,x_r^{a_r}))/(x_k^{a_k},\dots,x_r^{a_r}).$$
Since H_τ is torsion, $0:_{H_\tau}x_1^t\neq 0$ for some $t>0$. But this is impossible by the above presentation of H_τ, a contradiction. This completes the proof. ∎

4 The Stanley Conjecture on Monomial Factor Algebras

We will show that the Stanley conjecture holds for certain monomial factor algebras S/I. Notice that, by [4, Corollary 4.5], one can reduce the situation to the case where I is squarefree.

Suppose that I is a squarefree monomial ideal of S. Then $I=P_1\cap\cdots\cap P_s$, where the monomial prime ideals P_k, $k=1,\dots,s$ have the form $(x_{i_1},\dots,x_{i_t})$ and are not included one in other. In this case, Theorem 2.5 has a simpler form.

Suppose that $\sum_{i=1}^s P_i=m$ and one of the monomial prime ideals in the decomposition of I is $(x_1,\dots,x_r)$. For any nonempty proper subset $\tau\subset[s]$, set
$$\overline{S}_\tau=K\Big[\Big\{x_i\mid 1\leq i\leq r, x_i\notin\sum_{j\in\tau}P_j\Big\}\Big]$$
and $H_\tau=(\cap_{j\in[s]\setminus\tau}P_j)\cap\overline{S}_\tau$, then
$$S=S''\oplus(\oplus_{\tau\subset[s]}H_\tau\overline{S}_\tau[x_{r+1},\dots,x_n]),$$
$$S/I=S''\oplus\Big(\oplus_{\tau\subset[s]}\frac{H_\tau\overline{S}_\tau[x_{r+1},\dots,x_n]}{H_\tau\overline{S}_\tau[x_{r+1},\dots,x_n]\cap((\cap_{j\in\tau}P_j)\cap S'')\overline{S}_\tau[x_{r+1},\dots,x_n])}\Big),$$
and Theorem 2.5 has the form
$$\operatorname{sdepth}_S(S/I)\geq$$
$$\min\Big\{n-r,\operatorname{sdepth}_{\overline{S}_\tau}\big((\cap_{j\in[s]\setminus\tau}P_j)\cap\overline{S}_\tau\big)+\operatorname{sdepth}_{S''}\big(S''/(\cap_{j\in\tau}P_j)\cap S''\big)\Big\},$$
where the minimum is taken over all nonempty proper subsets $\tau\subset[s]$ such that $(\cap_{j\in[s]\setminus\tau}P_j)\cap\overline{S}_\tau\neq 0$.

Note that, here, there are also decompositions for $(x_1,\dots,x_r)$ and $(x_1,\dots,x_r)/I$:
$$(x_1,\dots,x_r)=\bigoplus_{\tau\subset[s]}H_\tau\overline{S}_\tau[x_{r+1},\dots,x_n]),$$
$$(x_1,\dots,x_r)/I=\bigoplus_{\tau\subset[s]}\frac{H_\tau\overline{S}_\tau[x_{r+1},\dots,x_n]}{H_\tau\overline{S}_\tau[x_{r+1},\dots,x_n]\cap((\cap_{j\in\tau}P_j)\cap S'')\overline{S}_\tau[x_{r+1},\dots,x_n])}.$$

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Then similarly, we have

$$\operatorname{sdepth}_S((x_1,\dots,x_r)/I) \geq \min_{\tau\subset[s]}\Big\{\operatorname{sdepth}_{\overline{S}_\tau}\Big(\Big(\bigcap_{j\in[s]\setminus\tau} P_j\Big)\cap \overline{S}_\tau\Big) + \operatorname{sdepth}_{S''}\Big(S''/\Big(\bigcap_{j\in\tau} P_j\Big)\cap S''\Big)\Big\},$$

where the minimum is taken over all nonempty proper subsets $\tau \subset [s]$ such that $(\bigcap_{j\in[s]\setminus\tau} P_j)\cap \overline{S}_\tau \neq 0$.

In this section, we will show two results that the Stanley conjecture hold for S/I under certain conditions, while the results for I have been proved by Popescu [7, 8]. We first show the following theorem.

Theorem 4.1 *Let I be a squarefree monomial ideal of S. Let $I = P_1\cap\cdots\cap P_s$ be a reduced intersection of monomial prime ideals of S. If $P_i \notin \sum_{j\neq i} P_j$, $i = 1,\dots,s$, then* $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$.

Proof Under the assumption of the theorem, it was shown in [7, Theorem 2.3] that $\operatorname{depth}_S(I) = s$ and $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$. We will use some of the arguments from the proof of [7, Theorem 2.3].

Use induction on s. The case $s = 1$ is clear. Now assume that $s > 1$ and $P_1 = (x_1,\dots,x_r)$. As $P_1 \in \operatorname{Ass}_S(S/I)$, we have

$$n - r = \dim_S(S/P_1) \geq \operatorname{depth}_S(S/I).$$

For any nonempty proper subset $\tau\subset[s]$ such that $(\bigcap_{j\in[s]\setminus\tau} P_j)\cap\overline{S}_\tau \neq 0$, it was proved in the proof of [7, Theorem 2.3] that

$$\operatorname{sdepth}_{\overline{S}_\tau}\Big(\Big(\bigcap_{j\in[s]\setminus\tau} P_j\Big)\cap \overline{S}_\tau\Big) \geq s - |\tau| - \dim_S\Big(S/\Big(P_1 + \sum_{i\in\tau} P_i\Big)\Big),$$

$$\operatorname{depth}_{S''}\Big(S''/\Big(\bigcap_{j\in\tau} P_j\Big)\cap S''\Big) \geq |\tau| - 1 + \dim_S\Big(S/\Big(P_1 + \sum_{i\in\tau} P_i\Big)\Big).$$

By induction hypothesis, we get that

$$\operatorname{sdepth}_{S''}\Big(S''/\Big(\bigcap_{j\in\tau} P_j\Big)\cap S''\Big) \geq \operatorname{depth}_{S''}\Big(S''/\Big(\bigcap_{j\in\tau} P_j\Big)\cap S''\Big).$$

Hence,

$$\begin{aligned}\operatorname{sdepth}_{\overline{S}_\tau}\Big(\Big(\bigcap_{j\in[s]\setminus\tau} P_j\Big)\cap \overline{S}_\tau\Big) + \operatorname{sdepth}_{S''}\Big(S''/\Big(\bigcap_{j\in\tau} P_j\Big)\cap S''\Big) &\geq s-1\\ &= \operatorname{depth}_S(S/I).\end{aligned}$$

It follows that $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$. ∎

Let I be a squarefree monomial ideal of S and let $I = P_1\cap\cdots\cap P_s$ be a reduced intersection of monomial prime ideals. Popescu [7] defined the bigsize of I, denoted by $\operatorname{bigsize}_S(I)$, as $t + n - \operatorname{height}(\sum_{j=1}^s P_j) - 1$, where t is the minimum of the integers e such that $\sum_{k=1}^e P_{i_k} = \sum_{j=1}^s P_j$ for all $i_1 < \cdots < i_e$. The main result in [8] states that $\operatorname{sdepth}_S(I) \geq \operatorname{depth}_S(I)$ provided $\operatorname{bigsize}_S(I) = 2$. Our aim is to show that $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$ in this case. We need certain results from [8].

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Lemma 4.2 ([8, Proposition 2.7]) *Let I be a squarefree monomial ideal and let $I = P_1 \cap \cdots \cap P_s$ be a reduced intersection of monomial prime ideals. Suppose that $\sum_{i=1}^{s} P_i = m$,* $\operatorname{bigsize}_S(I) = 2$*,* $\operatorname{size}_S(I) = 1$*, and* $\operatorname{depth}_S(I) > 3$*. Then, after renumbering (P_i), there exists $1 < r \leq s-2$ such that for $I_1 = P_1 \cap \cdots \cap P_r$ and $I_2 = P_r \cap \cdots \cap P_s$, one has that $P_i + P_j = m, 1 \leq i < r, r < j \leq s$, and*

$$\operatorname{depth}_S(I) = \min\{\operatorname{depth}_S(I_1), \operatorname{depth}_S(I_2)\}.$$

We know that $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$ for $s \leq 4$; *cf.* [10,12]. In the proof of the next theorem, we also need the following lemma.

Lemma 4.3 *If $s \leq 4$, then*

$$\operatorname{sdepth}_S(P_k/I) \geq \operatorname{depth}_S(S/I), k = 1, \ldots, s.$$

Proof Assume that $P_k = (x_1, \ldots, x_r)$. Let us recall the argument for $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$. For $s \leq 4$, according to the proof in [12], one must verify each

$$\operatorname{sdepth}_{\overline{S}_\tau}((\cap_{j \in [s] \setminus \tau} P_j) \cap \overline{S}_\tau) + \operatorname{sdepth}_{S''}(S''/(\cap_{j \in \tau} P_j) \cap S'') \geq \operatorname{depth}_S(S/I).$$

On the other hand, from the decomposition of P_k/I in the third paragraph of this section, it turns out also that $\operatorname{sdepth}_S(P_k/I) \geq \operatorname{depth}_S(S/I)$. ■

Lemma 4.4 ([3, Theorem 1.2]) *Let I be a monomial ideal of S. If* $\operatorname{bigsize}_S(I) = \operatorname{size}_S(I)$*, then* $\operatorname{depth}_S(S/I) = \operatorname{size}_S(I)$.

Lemma 4.5 *If* $\operatorname{depth}_S(I) \leq 3$*, then* $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$.

Proof If $\operatorname{depth}_S(I) \leq 2$, that is $\operatorname{depth}_S(S/I) \leq 1$, then, by Theorem 3.2,

$$\operatorname{sdepth}_S(S/I) \geq \operatorname{size}_S(I) \geq 1 \geq \operatorname{depth}_S(S/I).$$

Assume that $\operatorname{depth}_S(I) = 3$, that is, $\operatorname{depth}_S(S/I) = 2$. By [9, Theorem 3.3], $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$ provided S/I is Cohen–Macaulay of dimension 2. Then it follows from [2, Corollary 2.2] that $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$ holds for any monomial ideal I of S with $\operatorname{depth}_S(S/I) = 2$. The lemma follows. ■

Theorem 4.6 *Let I be a squarefree monomial ideal of S. If* $\operatorname{bigsize}_S(I) \leq 2$*, then* $\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I)$.

Proof If $\operatorname{bigsize}_S(I) = \operatorname{size}_S(I)$, then the result follows from Lemma 4.4 and Theorem 3.2. Hence we may assume that $\operatorname{bigsize}_S(I) = 2$ and $\operatorname{size}_S(I) = 1$. By Lemma 4.5, we may also assume that $\operatorname{depth}_S(I) > 3$.

Suppose that $I = P_1 \cap \cdots \cap P_s$ is a reduced intersection of monomial prime ideals. We use induction on s to show that

$$\operatorname{sdepth}_S(S/I) \geq \operatorname{depth}_S(S/I),$$
$$\operatorname{sdepth}_S(P_k/I) \geq \operatorname{depth}_S(S/I), k = 1, \ldots, s.$$

We may assume that $s > 4$ and $\sum_{i=1}^{s} P_i = m$. Applying Lemma 4.2, we get I_1 and I_2 with the property that $\operatorname{depth}_S(I) = \min\{\operatorname{depth}_S(I_1), \operatorname{depth}_S(I_2)\}$. Renumbering (P_i) if necessary, we may assume that $k \le r$.

According to the argument in the proof of [8, Proposition 3.4], there are a decomposition of I as K-spaces

$$I = (I \cap S') \oplus ((x_1, \dots, x_e) \cap I)$$

and $1 \le e < n$ satisfying

$$\begin{gathered}\{x_1, \dots, x_e\} \subseteq P_i, \text{ for } i = 1, \dots, r, \quad S' = K[x_{e+1}, \dots, x_n],\\ \operatorname{depth}_{S'}(I \cap S') = \operatorname{depth}_S(I_1), \quad \operatorname{depth}_S((x_1, \dots, x_e) \cap I) = \operatorname{depth}_S(I_2),\end{gathered}$$

and where

$$I \cap S' = (P_1 \cap S') \cap \cdots \cap (P_r \cap S') \text{ and } (x_1, \dots, x_e) \cap I = (x_1, \dots, x_e) \cap P_{r+1} \cap \cdots \cap P_s$$

are reduced intersections of monomial prime ideals.

On the other hand, we have decompositions of S and P_k as K-spaces:

$$\begin{aligned} S &= S' \oplus (x_1, \dots, x_e),\\ P_k &= (P_k \cap S') \oplus (x_1, \dots, x_e)\end{aligned}$$

from which one get decompositions of S/I and P_k/I as K-spaces

$$\begin{aligned} S/I &= S'/(I \cap S') \oplus (x_1, \dots, x_e)/((x_1, \dots, x_e) \cap I),\\ P_k/I &= (P_k \cap S')/(I \cap S') \oplus (x_1, \dots, x_e)/((x_1, \dots, x_e) \cap I).\end{aligned}$$

Then

$$\operatorname{sdepth}_S(S/I) \ge \min\{\operatorname{sdepth}_{S'}(S'/(I \cap S')), \operatorname{sdepth}_S((x_1, \dots, x_e)/((x_1, \dots, x_e) \cap I))\}$$

and

$$\operatorname{sdepth}_S(P_k/I) \ge \min\{\operatorname{sdepth}_{S'}((P_k \cap S')/(I \cap S')), \operatorname{sdepth}_S((x_1, \dots, x_e)/((x_1, \dots, x_e) \cap I))\}.$$

Applying induction hypothesis to the above summands, we get

$$\begin{aligned} \operatorname{sdepth}_{S'}(S'/(I \cap S')) &\ge \operatorname{depth}_{S'}(S'/(I \cap S'))\\ &= \operatorname{depth}_S(S/I_1),\\ \operatorname{sdepth}_S((x_1, \dots, x_e)/((x_1, \dots, x_e) \cap I)) &\ge \operatorname{depth}_S(S/((x_1, \dots, x_e) \cap I))\\ &= \operatorname{depth}_S(S/I_2),\\ \operatorname{sdepth}_{S'}((P_k \cap S')/(I \cap S')) &\ge \operatorname{depth}_{S'}(S'/(I \cap S'))\\ &= \operatorname{depth}_S(S/I_1).\end{aligned}$$

Note that, when $r = 2$ or $s - r = 2$, there are no induction hypotheses to apply; however, the related results are known by Lemma 4.3. It follows that

$$\begin{aligned}\operatorname{sdepth}_S(S/I) &\geq \min\{\operatorname{depth}_S(S/I_1), \operatorname{depth}_S(S/I_2)\}\\ &= \operatorname{depth}_S(S/I),\\ \operatorname{sdepth}_S(P_k/I) &\geq \min\{\operatorname{depth}_S(S/I_1), \operatorname{depth}_S(S/I_2)\}\\ &= \operatorname{depth}_S(S/I).\end{aligned}$$

This completes the proof. ∎

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