Canad. Math. Bull. Vol. **64** (4), 2023, pp. 1326–1340
http://dx.doi.org/10.4153/S0008439523000395

Linear preservers on idempotents of Fourier algebras

Ying-Fen Lin and Shiho Oi

Abstract. In this article, we give a representation of bounded complex linear operators that preserve idempotent elements on the Fourier algebra of a locally compact group. When such an operator is, moreover, positive or contractive, we show that the operator is induced by either a continuous group homomorphism or a continuous group antihomomorphism. If the groups are totally disconnected, bounded homomorphisms on the Fourier algebra can be realized by the idempotent preserving operators.

1 Introduction

Let G be a locally compact group. The Fourier–Stieltjes $B(G)$ and the Fourier algebras $A(G)$ of G were introduced by Eymard in his celebrating paper [11]. Recall that $B(G)$ is the linear combination of all continuous positive-definite functions on G, and as a Banach space, $B(G)$ is naturally isometric to the predual of $W^*(G)$, the von Neumann algebras generated by the universal representations ω_G of G. Moreover, it is a commutative Banach $*$-algebra with respect to pointwise multiplication and complex conjugation. The Fourier algebra $A(G)$ is the closed ideal of $B(G)$ generated by the functions with compact supports. As a Banach space, $A(G)$ is isometric to the predual of the group von Neumann algebra $\mathrm{VN}(G)$, the von Neumann algebra generated by the left regular representations λ_G of G. It is well known that $A(G)$ is regular and semisimple, and the Fourier and the Fourier–Stieltjes algebras are both subalgebras of $C_b(G)$, the algebra of continuous bounded functions on G.

Takesaki and Tatsuuma in [24] showed that there is a one-to-one correspondence between compact subgroups of G and nonzero right invariant closed self-adjoint subalgebras of $A(G)$. As a refinement, Bekka, Lau, and Schlichting in [2] studied nonzero, closed, invariant $*$-subalgebras of $A(G)$. They showed that these spaces are the Fourier algebras $A(G/K)$ of the quotient group G/K for some compact normal subgroup K of G. On the other hand, Forrest [12] introduced the Fourier algebra $A(G/K)$ of the left coset space G/K, where K is a compact (not necessary to be normal) subgroup of the locally compact group G. This algebra can simultaneously be viewed as an algebra of functions on G/K and as the subalgebra of $A(G)$ consisting of functions in $A(G)$ which are constants on left cosets of K. Note that $A(G/K)$ is

Received by the editors February 2, 2023; revised May 5, 2023; accepted May 7, 2023.
Published online on Cambridge Core May 17, 2023.
The second author was supported by JSPS KAKENHI Grant Number JP21K13804.
AMS subject classification: **47B48**, 47B49, 43A22.
Keywords: Fourier algebras, idempotents, linear preservers, quotient groups.

regular and semisimple, the spectrum $\sigma(A(G/K))$ is G/K, and it is a norm-closed left translation invariant $*$-subalgebra of $A(G)$.

A long-standing question in harmonic analysis is to determine all homomorphisms of Fourier or Fourier–Stieltjes algebras of any locally compact groups. For any pair of locally compact abelian groups G and H, Cohen [6] characterized all bounded homomorphisms from the group algebra $L^1(G)$ to the measure algebra $M(H)$. In doing so, he made use of a profound discovery of his characterization of idempotent measures on the groups. Cohen's results were generalized by Host in [14], where he discovered the general form of idempotents in the Fourier–Stieltjes algebras, and characterized bounded homomorphisms from $A(G)$ to $B(H)$ when the group G has an abelian subgroup of finite index. Further generalizations were made in [15, 16] for any locally compact amenable group G, where completely bounded homomorphisms from $A(G)$ into $B(H)$ were characterized by continuous piecewise affine maps (see also [7]). Most general results were given by Le Pham in [20], and he determined all contractive homomorphisms from $A(G)$ into $B(H)$ for any locally compact groups G and H.

To describe idempotent elements in the Fourier–Stieltjes and the Fourier algebras, we first recall some terminologies. Let G be a group, and let K be a subgroup of G; we see that $Ks = ss^{-1}Ks$ for any $s \in G$, which means that we need not distinguish between left and right cosets of the group G. The *coset ring* of G, denoted $\Omega(G)$, is the smallest ring of subsets of G containing all cosets of subgroups of G. We denote $\Omega_o(G)$ the ring of subsets generated by open cosets of G, and similarly, $\Omega_o^c(G)$ the ring of subsets generated by compact open cosets of G. By [14], idempotents in the Fourier–Stieltjes algebra $B(G)$ are the indicator functions 1_F of an element F of $\Omega_o(G)$. Let $I_B(G)$ be the set of all idempotent elements in $B(G)$. We denote the closure of the span of $I_B(G)$ by $B_I(G)$. From [17, Proposition 1.1], we have that

$$A(G) \cap I_B(G) = \{1_Y : Y \in \Omega_o^c(G)\},$$

which gives rise to idempotents in $A(G)$, denoted by $I(G)$. Let $A_I(G)$ be the subalgebra of $A(G)$ generated by $I(G)$. Note that Ilie and Spronk [16] showed that 1_F is an idempotent in $B(G)$ with $\|1_F\|_{B(G)} = 1$ if and only if F is an open coset in G; however, there are idempotents with small norms [23] or with large norms [1]. Moreover, the existence of idempotents of arbitrarily large norm implies the existence of homomorphisms of arbitrarily large norm (see [1] for details). Thus, idempotent elements play an essential role in the study of homomorphisms on Fourier algebras. It is of its own interest to study the norms of idempotent elements in Fourier–Stieltjes and Fourier algebras, but for our purpose, we will focus solely on operators that preserve idempotents.

In the rich literature of linear preservers, there are many works that study linear maps T on spaces X which preserve some subsets S of X, i.e., $T(S) \subset S$. Dieudonné in [8] studied semilinear maps on $M_n(\mathbb{K})$, the algebra of $n \times n$ matrices over a field $\mathbb{K}$, which preserve the set of all singular matrices. After that, many mathematicians considered linear maps on $M_n(\mathbb{K})$ that preserve subsets of matrices with different properties (e.g., [3, 9, 18, 22] to name a few). In [4], it is shown that every complex linear map T on $M_n(\mathbb{C})$ which preserves the set of all idempotents is either an inner automorphism or an inner antiautomorphism. In addition, in [5], linear maps on

$M_n(\mathbb{C})$ which send potent matrices (that is, matrices A satisfy $A^r = A$ for some integer $r \geq 2$) to potent matrices were characterized. Since then, the studies of idempotent preserving maps have attracted considerable interest (see, e.g., [10, 13]). Recently, in [21], the authors proved that every additive map from the rational span of Hermitian idempotents in a von Neumann algebra into the rational span of Hermitian idempotents in a C$*$-algebra can be extended to a Jordan $*$-homomorphism.

In this article, we study bounded linear operators from $A(G)$ into $B(H)$ which send idempotents to idempotents. We show that such an operator will give rise to an algebraic homomorphism on $A_I(G)$. The algebra $A_I(G)$ will be our main object of study, namely, we will characterize linear mappings defined on the Fourier algebra $A(G)$ or on $A_I(G)$ which preserve $I(G)$. Moreover, we show that when the groups are totally disconnected, idempotent preserving operators will recover algebraic homomorphisms on the Fourier algebra.

2 Main results

Let G be a locally compact group, and let K be a closed subgroup of G. We will denote by G/K the homogeneous space of left cosets of K. Let

$$B(G:K) := \{u \in B(G) : u(xk) = u(x) \text{ for all } x \in G, k \in K\},$$

that is, functions in $B(G)$ which are constant on cosets of K, and

$$A(G:K) := \{u \in B(G:K) : q(\operatorname{supp}(u)) \text{ is compact in } G/K\}^{-B(G)},$$

where $\operatorname{supp}(u)$ is the support of u in G and q is the canonical quotient map from G to G/K. If, furthermore, K is a normal subgroup, by [12, Proposition 3.2], we have that $B(G:K)$ and $A(G:K)$ are isometrically isomorphic to the Fourier–Stieltjes and the Fourier algebras $B(G/K)$ and $A(G/K)$, respectively. Note that $A(G:K) \cap A(G) \neq \{0\}$ if and only if K is compact (see [12, Proposition 3.1] or [24, Theorem 9] and [2, Theorem 2.1] for more details).

Let e be the identity of the group G, and we denote the connected component of e by G_e, which is a closed normal subgroup of G; thus, G/G_e is a totally disconnected locally compact group. The following result about the algebra $A_I(G)$ generated by idempotents of $A(G)$ in relation with $A(G:G_e)$ was given in [17]; for the completion, we give a short proof in the article.

Proposition 2.1 *[17, Proposition 1.1(ii)] If the connected component G_e is compact, then $A_I(G) = A(G:G_e)$, that is, $A_I(G)$ consists of all functions in $A(G)$ that are constant on cosets of G_e. On the other hand, if G_e is not compact, then $A_I(G) = \{0\}$.*

Proof Let $q_G : G \to G/G_e$ be the quotient map onto G/G_e. Since G_e is compact, via $u \mapsto u \circ q_G$, we have that $A(G/G_e)$ is isometrically isomorphic to $A(G:G_e)$, which is a closed subalgebra of $A(G)$. Thus, $A_I(G) = \overline{\operatorname{span}}\{1_Y : Y \in \Omega_0^c(G)\} \subseteq A(G:G_e)$. Conversely, since G/G_e is totally disconnected, we have that the span of the idempotents of $A(G/G_e)$ is dense [12, Theorem 5.3]. Moreover, $A(G/G_e)$ is isomorphic to $A(G:G_e)$; thus, $A(G:G_e)$ is generated by idempotents of $A(G/G_e)$, so $A(G:G_e) \subseteq A_I(G)$. ∎

https://doi.org/10.4153/S0008439523000395 Published online by Cambridge University Press

If the Fourier algebra which contains nontrivial idempotents, that is, the connected component G_e, is compact, then by Proposition 2.1, there is an isometric isomorphism from $A_I(G)$ onto $A(G/G_e)$. More precisely, it induces an isometric isomorphism $\varphi_G : A_I(G) \to A(G/G_e)$ as

$$\varphi_G(f)(q_G(a)) = f(a) \tag{2.1}$$

for any $f \in A_I(G)$ and $a \in G$, where $q_G : G \to G/G_e$ is the quotient map onto G/G_e.

2.1 Idempotent preserving maps with $T(I(G)) \subset I_B(H)$

Let G and H be two locally compact groups. We consider a bounded complex linear map $T : A(G) \to B(H)$ which satisfies

$$T(I(G)) \subset I_B(H). \tag{2.2}$$

For any $f \in \operatorname{span}\{1_Y : Y \in \Omega_o^c(G)\}$, there exist $\alpha_i \in \mathbb{C}$ and $Y_i \in \Omega_o^c(G)$ such that $f = \Sigma_{k=1}^n \alpha_k 1_{Y_k}$. Thus, we have $Tf = \Sigma_{k=1}^n \alpha_k T1_{Y_k} \in \operatorname{span}\{1_Y : Y \in \Omega_o(H)\} \subset B(H)$. Let us recall that $A_I(G) = \overline{\operatorname{span}}\{1_Y : Y \in \Omega_o^c(G)\}$ and $B_I(H) = \overline{\operatorname{span}}\{1_Y : Y \in \Omega_o(H)\}$. Since T is a bounded map, we obtain $T(A_I(G)) \subseteq B_I(H) \subset B(H)$.

Our aim is to obtain a representation of such a map T on $A_I(G)$. If $I(G) = \{0\}$, then $A_I(G) = \{0\}$. Since T is complex linear, we have $T = 0$ on $A_I(G)$. Thus, without loss of generality, we can assume that the Fourier algebra $A(G)$ have nonzero idempotent elements. Hence, the connected component G_e is always a compact normal subgroup of G. On the other hand, we define the following map, which will be used in the sequel.

Definition 2.1 Let G be a locally compact group. Using the axiom of choice, let S be a set of representatives of the cosets of G/G_e, that is, $G = \bigsqcup_{a\in S} aG_e$. Then we define a map $[\,\cdot\,]_{G/G_e}$ from G/G_e onto S by

$$[aG_e]_{G/G_e} = a$$

for any $a \in S$.

We first have the following observations concerning the operator satisfying (2.2).

Lemma 2.2 *The map T preserves the disjointness of idempotents. That is, $Tf \cdot Tg = 0$ for any $f, g \in I(G)$ with $f \cdot g = 0$.*

Proof Let $f, g \in I(G)$ such that $f \cdot g = 0$. Then we have $(f+g)^2 = f + g$. Thus, $f + g \in I(G)$. By the assumption, Tf, Tg, and $T(f+g) \in I_B(H)$. Since we have $(T(f+g))^2 = Tf + Tg$, we get $Tf \cdot Tg = 0$. ■

Definition 2.2 We define $\Phi : A(G/G_e) \to B(H)$ by

$$\Phi(f) = T \circ \varphi_G^{-1}(f)$$

for any $f \in A(G/G_e)$, where φ_G is given in (2.1).

Then Φ is a bounded complex linear operator from $A(G/G_e)$ into $B(H)$. In order to achieve our main result, we consider the dual map $\Phi^* : W^*(H) \to \mathrm{VN}(G/G_e)$ and have the following lemmas.

Lemma 2.3 *Let $\lambda \in \mathrm{VN}(G/G_e)$ and $a \in G/G_e$. Suppose that $a \in \operatorname{supp}\lambda$. Then, for every neighborhood V of a in G/G_e, there exists $h \in I(G/G_e)$ such that* $\operatorname{supp} h \subset V$ *and* $\langle \lambda, h\rangle \neq 0$.

Proof Since G/G_e is totally disconnected, every neighborhood of the identity contains an open compact subgroup. As $a^{-1}V$ is a neighborhood of the identity, there exists an open compact subgroup G_a in G/G_e such that $G_a \subset a^{-1}V$. Thus, $aG_a \subset V$. Since aG_a is a compact open coset in G/G_e, we have that $1_{aG_a} \in A(G/G_e)$ is an idempotent with norm 1. Since $a \in \operatorname{supp}\lambda$, there is $g \in A(G/G_e)$ such that supp $g \subset aG_a$ and $\langle \lambda, g\rangle \neq 0$. Put $\delta = |\langle \lambda, g\rangle|$. As $\varphi_G^{-1}(g) \in A_I(G)$, there are $\alpha_i \in \mathbb{C}$ and $f_i \in I(G)$ such that $\|\varphi_G^{-1}(g) - \sum_{i=1}^n \alpha_i f_i\| < \delta/\|\lambda\|$ for some $n \in \mathbb{N}$. Since φ_G is an isometric isomorphism, we have $\|g - \sum_{i=1}^n \alpha_i \varphi_G(f_i)\| < \delta/\|\lambda\|$ and $\varphi_G(f_i) \in I(G/G_e)$. Then we obtain

$$1_{aG_a}\left(g - \sum_{i=1}^n \alpha_i \varphi_G(f_i)\right) = g - \sum_{i=1}^n \alpha_i 1_{aG_a}\varphi_G(f_i),$$

and thus

$$\left\| g - \sum_{i=1}^n \alpha_i 1_{aG_a}\varphi_G(f_i)\right\| \leq \left\| g - \sum_{i=1}^n \alpha_i \varphi_G(f_i)\right\| < \frac{\delta}{\|\lambda\|}.$$

Suppose that, for every $1 \leq i \leq n$, we have $\langle \lambda, 1_{aG_a}\varphi_G(f_i)\rangle = 0$. Then

$$\begin{aligned} |\langle \lambda, g\rangle| &= \left| \langle \lambda, g\rangle - \sum_{i=1}^n \alpha_i \langle \lambda, 1_{aG_a}\varphi_G(f_i)\rangle \right| \\ &= \left| \left\langle \lambda, \left(g - \sum_{i=1}^n \alpha_i 1_{aG_a}\varphi_G(f_i)\right)\right\rangle \right| \\ &\leq \|\lambda\| \left\| g - \sum_{i=1}^n \alpha_i 1_{aG_a}\varphi_G(f_i)\right\| \\ &< \|\lambda\| \frac{\delta}{\|\lambda\|} = \delta. \end{aligned}$$

This implies that $|\langle \lambda, g\rangle| < \delta$, which is a contradiction. Therefore, there is an $i_0 \in \{1, \ldots, n\}$ such that

$$\langle \lambda, 1_{aG_a}\varphi_G(f_{i_0})\rangle \neq 0.$$

We also have $\operatorname{supp}(1_{aG_a}\varphi_G(f_{i_0})) \subset V$ and $1_{aG_a}\varphi_G(f_{i_0}) \in I(G/G_e)$, and the proof is thus completed. ∎

Proposition 2.4 *For any $a \in H$, there exist unique $b \in G/G_e$ and $\alpha \in \mathbb{C}$ such that $\Phi^*(\omega_H(a)) = \alpha \lambda_{G/G_e}(b)$.*

Proof Suppose that there are $b_1, b_2 \in G/G_e$ such that b_1, b_2 were both in $\operatorname{supp}(\Phi^*(\omega_H(a)))$. Since G_e is a closed subgroup of G, the quotient group G/G_e is Hausdorff. Thus, there are neighborhoods V_{b_1} and V_{b_2} of b_1 and b_2, respectively, in G/G_e such that $V_{b_1} \cap V_{b_2} = \varnothing$. By Lemma 2.3, there are $h_i \in I(G/G_e)$, for $i = 1, 2$, such that $\operatorname{supp} h_i \subset V_{b_i}$ and $\langle \Phi^*(\omega_H(a)), h_i \rangle \neq 0$. As $V_{b_1} \cap V_{b_2} = \varnothing$, we get $h_1 h_2 = 0$. Since φ_G is an isomorphism, we have $\varphi_G^{-1}(h_i) \in I(G)$, for $i = 1, 2$, and $\varphi_G^{-1}(h_1) \cdot \varphi_G^{-1}(h_2) = \varphi_G^{-1}(h_1 h_2) = 0$. By Lemma 2.2, we have $T(\varphi_G^{-1}(h_1)) \cdot T(\varphi_G^{-1}(h_2)) = 0$. On the other hand, we obtain

$$0 \neq \langle \Phi^*(\omega_H(a)), h_1 \rangle = \Phi(h_1)(a) = T \circ \varphi_G^{-1}(h_1)(a) = T(\varphi_G^{-1}(h_1))(a)$$

and

$$0 \neq \langle \Phi^*(\omega_H(a)), h_2 \rangle = \Phi(h_2)(a) = T \circ \varphi_G^{-1}(h_2)(a) = T(\varphi_G^{-1}(h_2))(a).$$

Therefore,

$$T(\varphi_G^{-1}(h_1)) \cdot T(\varphi_G^{-1}(h_2)) \neq 0;$$

this is a contradiction. Since $\operatorname{supp}(\Phi^*(\omega_H(a))) \neq \varnothing$, there is a unique $b \in G/G_e$ such that $\operatorname{supp}(\Phi^*(\omega_H(a))) = \{b\}$. Consequently, by [19, Corollary 2.5.9], there is an $\alpha \in \mathbb{C}$ such that $\Phi^*(\omega_H(a)) = \alpha \lambda_{G/G_e}(b)$. ∎

For any $a \in H$, by Proposition 2.4, there are unique $b \in G/G_e$ and $\alpha \in \mathbb{C}$ such that $\Phi^*(\omega_H(a)) = \alpha \lambda_{G/G_e}(b)$; thus, we have

$$\Phi(f)(a) = \alpha f(b),$$

for any $f \in A(G/G_e)$. We define $\phi : H \to \mathbb{C}$ by $\alpha = \phi(a)$. We also define $\psi : H \to G/G_e$ by $b = \psi(a)$. Then we get

$$\Phi(f)(a) = \phi(a) f(\psi(a)), \tag{2.3}$$

for any $f \in A(G/G_e)$ and $a \in H$.

For any $h \in I(G)$, since we have $\Phi(\varphi_G(h)) = T(h) \in I_B(H)$, we obtain that

$$(\Phi(\varphi_G(h)))^2 = T(h)T(h) = T(h) = \Phi(\varphi_G(h)).$$

On the other hand, since $(\varphi_G(h))^2 = \varphi_G(h^2) = \varphi_G(h)$ in $A(G/G_e)$, we obtain that

$$\phi(a)^2 \varphi_G(h)(\psi(a)) = \phi(a) \varphi_G(h)(\psi(a)) \tag{2.4}$$

for any $h \in I(G)$ and $a \in H$.

Lemma 2.5 *For any $a \in H$, there is an idempotent $1_{\psi(a)G_0}$ of $A(G/G_e)$ where $\psi(a)G_0$ is an open compact neighborhood of $\psi(a)$.*

Proof As G/G_e is totally disconnected, there is an open compact subgroup G_0 in G/G_e. For any $\psi(a) \in G/G_e$, $\psi(a)G_0$ is a compact open coset in G/G_e; hence, $1_{\psi(a)G_0}$ is an idempotent of $A(G/G_e)$ with norm 1. ∎

Lemma 2.6 *The map $\Phi : A(G/G_e) \to B(H)$ is an algebraic homomorphism.*

https://doi.org/10.4153/S0008439523000395 Published online by Cambridge University Press

Proof Let $a \in H$. By Lemma 2.5, there is an idempotent $1_{\psi(a)G_0}$ of $A(G/G_e)$. Since $\varphi_G : A_I(G) \to A(G/G_e)$ is surjective, there is $f \in A_I(G)$ such that $\varphi_G(f) = 1_{\psi(a)G_0}$. Moreover, we have that $f^2 = (\varphi_G^{-1}(1_{\psi(a)G_0}))^2 = \varphi_G^{-1}(1_{\psi(a)G_0}) = f$, and this implies that $f \in I(G)$. Thus, by (2.4), we have

$$\begin{aligned}\phi(a)^2 &= \phi(a)^2 1_{\psi(a)G_0}(\psi(a)) = \phi(a)^2\varphi_G(f)(\psi(a))\\ &= \phi(a)\varphi_G(f)(\psi(a)) = \phi(a)1_{\psi(a)G_0}(\psi(a)) = \phi(a).\end{aligned}$$

Since $a \in H$ is arbitrary, we have

$$\phi^2 = \phi \tag{2.5}$$

on H. Then we get $\phi : H \to \{0,1\}$. In addition, for any $f, g \in A(G/G_e)$ and $a \in H$, we have

$$\begin{aligned}\Phi(fg)(a) &= \phi(a)(fg)(\psi(a)) = \phi(a)^2(fg)(\psi(a))\\ &= \phi(a)f(\psi(a))\phi(a)g(\psi(a)) = (\Phi(f)\Phi(g))(a).\end{aligned}$$

Hence, Φ is an algebraic homomorphism from $A(G/G_e)$ into $B(H)$. ∎

Lemma 2.7 *The map $\psi : \phi^{-1}(1) \to G/G_e$ is continuous.*

Proof For any $a_0 \in \phi^{-1}(1) \subset H$, let U be an open neighborhood of $\psi(a_0)$ in G/G_e. Then there is $f_0 \in A(G/G_e)$ such that

$$f_0(\psi(a_0)) = 1 \quad \text{and} \quad f_0(b) = 0 \ \text{ for } b \in (G/G_e)\backslash U.$$

Let $(a_\lambda)_\lambda \subseteq \phi^{-1}(1)$ be a net such that $a_\lambda \to a_0$. As $\Phi(f_0) \in B(H)$, $\Phi f_0(a_\lambda) \to \Phi f_0(a_0) = f_0(\psi(a_0)) = 1$. There is a λ_0 such that if $\lambda \geq \lambda_0$, then $|\Phi f_0(a_\lambda)| > \frac{1}{2}$. Since $\Phi f_0(a_\lambda) = f_0(\psi(a_\lambda))$, we have $\psi(a_\lambda) \in U$ provided $\lambda \geq \lambda_0$. Thus, ψ is continuous on $\phi^{-1}(1)$. ∎

Lemma 2.8 *The set $\phi^{-1}(1)$ is an open subset of H.*

Proof Let $a \in \phi^{-1}(1)$ be arbitrary. By Lemma 2.5, there is an idempotent $1_{\psi(a)G_0}$ of $A(G/G_e)$ where $\psi(a)G_0$ is an open compact neighborhood of $\psi(a)$. Since $\Phi(1_{\psi(a)G_0}) \in B(H) \subset C_b(H)$, there exists an open neighborhood V of a in H such that if $b \in V$, then

$$|\Phi(1_{\psi(a)G_0})(a) - \Phi(1_{\psi(a)G_0})(b)| \leq \frac{1}{2}.$$

We have

$$|1 - \phi(b)1_{\psi(a)G_0}(\psi(b))| = |\Phi(1_{\psi(a)G_0})(a) - \Phi(1_{\psi(a)G_0})(b)| \leq \frac{1}{2}.$$

Since either $\phi(b)1_{\psi(a)G_0}(\psi(b)) = 1$ or $\phi(b)1_{\psi(a)G_0}(\psi(b)) = 0$, this implies that

$$\phi(b)1_{\psi(a)G_0}(\psi(b)) = 1.$$

Hence, we have $\phi(b) = 1$ for any $b \in V$. Thus, $V \subset \phi^{-1}(1)$. It follows that $\phi^{-1}(1)$ is an open subset of H. ∎

https://doi.org/10.4153/S0008439523000395 Published online by Cambridge University Press

Theorem 2.9 *Let G and H be two locally compact groups, and let $T : A(G) \to B(H)$ be a bounded complex linear operator. Suppose T satisfies that $T(I(G)) \subset I_B(H)$. Then there are an open subset U of H and a continuous map ψ from U into G/G_e such that*

$$Tf(a) = \begin{cases} f([\psi(a)]_{G_e}), & \text{if } a \in U, \\ 0, & \text{if } a \in H \backslash U, \end{cases} \tag{2.6}$$

for any $f \in A_I(G)$ and $a \in H$.

Proof Let $U = \phi^{-1}(1)$. By Lemma 2.8, U is an open subset of H. Moreover, Lemma 2.7 shows that $\psi : U \to G/G_e$ is a continuous map. Applying (2.3), for any $f \in A_I(G)$ and $a \in H$, we have

$$Tf(a) = \Phi(\varphi_G(f))(a) = \phi(a)\varphi_G(f)(\psi(a)).$$

Thus, we get

$$\begin{aligned} Tf(a) &= \begin{cases} \varphi_G(f)(\psi(a)), & \text{if } a \in U, \\ 0, & \text{if } a \in H \backslash U, \end{cases} \\ &= \begin{cases} f([\psi(a)]_{G_e}), & \text{if } a \in U, \\ 0, & \text{if } a \in H \backslash U, \end{cases} \end{aligned}$$

for any $f \in A_I(G)$ and any $a \in H$. ∎

The following example shows that the assumption in Theorem 2.9 does not imply $T(I(G)) \subset I(H)$. This observation is in line with the well-known fact that $f \circ \psi$ may not be in the Fourier algebra $A(H)$ in general (see Remark 2.11).

Example 2.10 Let $G = \{0\}$ be the trivial group. Then we define a bounded linear operator $T : A(G) \to B(\mathbb{Z})$ by $T(1_G) = 1_{\mathbb{Z}}$. Then it satisfies $T(I(G)) \subset I_B(H)$. Note that in this case, we have $U = \mathbb{Z}$ and the continuous map $\psi : \mathbb{Z} \to G/G_e$ is $\psi(n) = 0$ for any $n \in \mathbb{Z}$. On the other hand, since $1_{\mathbb{Z}} \notin A(\mathbb{Z})$, we have $T(I(G)) \not\subset I(H)$.

Remark 2.11 The converse statement of the above theorem may not hold since we do not know if $Tf \in A(H)$ for any $f \in A_I(G)$, even T has a representation of the form (2.6). If we only have $\psi : U \subseteq H \to G$ being continuous, then $f \mapsto f \circ \psi$ maps $A(G)$ into $\ell^\infty(H)$ in general. For abelian groups G and H, Cohen [6] showed that $f \mapsto f \circ \psi$ maps $A(G)$ into $B(H)$ if and only if ψ is a continuous piecewise affine map from a set in the open coset ring of H into G. This characterization was extended by Host [14] to the case when G has an abelian subgroup of finite index and H is arbitrary, and by [20] to general groups.

Under the additional assumptions such as positivity or contractivity on T, we obtain algebraic structures for the open set U and algebraic properties on the map ψ. Let us first recall positive operators on the Fourier algebra.

A bounded linear operator $T : A_I(G) \to B(H)$ is said to be positive if $T(u)$ is positive-definite whenever $u \in A_I(G)$ is a positive-definite function.

https://doi.org/10.4153/S0008439523000395 Published online by Cambridge University Press

Corollary 2.12 *Let G and H be two locally compact groups. Let $T : A_I(G) \to B(H)$ be a positive bounded complex linear operator. If T satisfies that $T(I(G)) \subset I_B(H)$, then there exist an open subgroup U of H and a continuous group homomorphism or antihomomorphism ψ from the open subgroup U of H into G/G_e such that*

$$Tf(a) = \begin{cases} f([\psi(a)]_{G_e}), & \textit{if } a \in U, \\ 0, & \textit{if } a \in H\backslash U, \end{cases}$$

for any $f \in A_I(G)$ and $a \in G$.

Proof Since the isometric isomorphism φ_G preserves positivity, $u \in A_I(G)$ is a positive-definite function if and only if $\varphi_G(u)$ is positive-definite. This implies that T is positive if and only if Φ is positive; thus, $\Phi : A(G/G_e) \to B(H)$ is a positive homomorphism by Lemma 2.6. It follows from [20, Theorem 4.3] that there exist an open subgroup U of H and a continuous group homomorphism or antihomomorphism ψ from U into G/G_e such that for any $f \in A(G/G_e)$, Φf is either equal to $f \circ \psi$ in U, or 0 otherwise. Thus, we have

$$Tf(a) = \begin{cases} f([\psi(a)]_{G_e}), & \text{if } a \in U, \\ 0, & \text{if } a \in H\backslash U, \end{cases}$$

for any $f \in A_I(G)$ and $a \in H$. ■

Corollary 2.13 *Let G and H be two locally compact groups, and let $T : A_I(G) \to B(H)$ be a contractive complex linear operator. If T satisfies that $T(I(G)) \subset I_B(H)$, then there exist an open subgroup U of H, a continuous group homomorphism or antihomomorphism ψ from U into G/G_e, and elements $b \in G$ and $c \in H$ such that*

$$Tf(a) = \begin{cases} f(b[\psi(ca)]_{G_e}), & \textit{if } a \in c^{-1}U, \\ 0, & \textit{if } a \in H\backslash c^{-1}U. \end{cases}$$

Proof Since φ_G is an isometric isomorphism, Φ is a contractive operator provided that so is T. By Lemma 2.6, Φ is a contractive homomorphism from $A(G/G_e)$ into $B(H)$. It follows from [20, Theorem 5.1] that there exist an open subgroup U of H, a continuous group homomorphism or antihomomorphism ψ from U into G/G_e, and elements $bG_e \in G/G_e$ and $c \in H$ such that for any $f \in A(G/G_e)$ and $a \in H$, $\Phi f(a) = f(bG_e\psi(ca))$ provided $a \in c^{-1}U$; otherwise, $\Phi f(a) = 0$. Recalling the definition of Φ, we have the characterization of T. ■

Note that when the group G is totally disconnected, we have $A_I(G) = A(G)$. In such case, positive or contractive complex linear idempotent preserving operators from $A(G)$ to $B(H)$ are algebraic homomorphisms; thus, our results recover Theorems 4.3 and 5.1 in [20].

2.2 Idempotent preserving maps with $T(I(G)) \subset I(H)$

Let us assume that the bounded linear operator $T : A(G) \to B(H)$ satisfies $T(I(G)) \subset I(H)$. Then, naturally, we obtain $T(A_I(G)) \subseteq A_I(H)$.

We define $T_q : A(G/G_e) \to A(H/H_e)$ by

$$T_q(f) = \varphi_H \circ T \circ \varphi_G^{-1}(f) = \varphi_H \circ \Phi(f),$$

for any $f \in A(G/G_e)$, where $\varphi_H : A_I(H) \to A(H/H_e)$ is an isometric isomorphism defined similarly as in (2.1). Note that T_q is an algebraic homomorphism.

Lemma 2.14 *Let $a \in \phi^{-1}(1) \subset H$ and $b \in H$ such that $a^{-1}b \in H_e$. Then $\phi(b) = 1$ and $\psi(a) = \psi(b)$.*

Proof Suppose that $\psi(a) \neq \psi(b)$. By (2.3), we have $\Phi : A(G/G_e) \to B(H)$ such that for any $f \in A(G/G_e)$,

$$\Phi(f)(a) = f(\psi(a))$$

and

$$\Phi(f)(b) = \phi(b)f(\psi(b)).$$

Since G/G_e is Hausdorff, there are disjoint open neighborhoods V_a and V_b of $\psi(a)$ and $\psi(b)$, respectively, in G/G_e. By Lemma 2.3, for $\lambda_{G/G_e}(\psi(a)) \in VN(G/G_e)$, there is $h \in I(G/G_e)$ such that supp $h \subset V_a$ and $h(\psi(a)) \neq 0$. Since $a \in \phi^{-1}(1)$, we get

$$\Phi(h)(a) = h(\psi(a)) \neq 0$$

and

$$\Phi(h)(b) = \phi(b)h(\psi(b)) = 0.$$

By the assumption that $T(I(G)) \subset I(H)$ and $\varphi_G^{-1}(h) \in I(G)$, we have $\Phi(h) = T(\varphi_G^{-1}(h)) \in I(H)$, an idempotent in $A(H)$. Hence, there is $Y \in \Omega_o^c(H)$ such that $1_Y = \Phi(h)$. Since $1_Y(a) = \Phi(h)(a) = h(\psi(a)) \neq 0$, we have $a \in Y$. In addition, Y is a clopen subset of H and H_e is a connected component containing e; thus, $aH_e \subset Y$. This implies that $b = aa^{-1}b \in Y$. It follows that

$$1 = 1_Y(b) = \Phi(h)(b) = 0.$$

This is a contradiction. Thus, we have $\psi(a) = \psi(b)$. Furthermore, suppose that $\phi(b) = 0$. There is an $h \in I(G/G_e)$ such that $h(\psi(b)) \neq 0$. Thus, there is $Y \in \Omega_o^c(H)$ such that $1_Y = \Phi(h)$. By a similar argument, we have $1_Y(a) = \Phi(h)(a) = h(\psi(a)) \neq 0$, $a \in Y$ and $b \in Y$. We obtain that

$$1 = 1_Y(b) = \Phi(h)(b) = \phi(b)h(\psi(b)) = 0.$$

This is a contradiction. Therefore, $\phi(b) = 1$ and $\psi(a) = \psi(b)$. ∎

For any $a, b \in H$, the condition $a^{-1}b \in H_e$ induces an equivalence relation on H. Lemma 2.14 shows that $\phi : H \to \{0,1\}$ and $\psi : H \to G/G_e$ are constant functions on each equivalence class. Thus, these induce maps $\phi' : H/H_e \to \{0,1\}$ and $\psi' : \phi'^{-1}(1) \to G/G_e$ by

$$\phi'(aH_e) = \phi(a) \quad \text{for any } a \in H$$

and

$$\psi'(aH_e) = \psi(a) \quad \text{for any } aH_e \in \phi'^{-1}(1).$$

By Lemma 2.7, the map $\psi : \phi^{-1}(1) \to G/G_e$ is continuous. As we have $\phi'^{-1}(1) = q_H(\phi^{-1}(1))$, we obtain that $\psi' : \phi'^{-1}(1) \to G/G_e$ is continuous.

Theorem 2.15 *Let G and H be two locally compact groups, and let $T : A(G) \to B(H)$ be a bounded complex linear operator. Suppose that T satisfies $T(I(G)) \subset I(H)$. Then there exist an open subset U of H and a continuous map ψ' from an open subset $q_H(U)$ of H/H_e into G/G_e such that*

$$Tf(a) = \begin{cases} f([\psi'(aH_e)]_{G_e}), & \text{if } a \in U, \\ 0, & \text{if } a \in H \backslash U, \end{cases} \tag{2.7}$$

for any $f \in A_I(G)$.

Proof Define $U = \phi^{-1}(1)$. Recall that $q_H : H \to H/H_e$ is the quotient map and U is an open subset of H by Lemma 2.8. By (2.3), for any $f \in A(G/G_e)$ and $a \in H$, we have

$$T_q(f)(aH_e) = \varphi_H \circ \Phi(f)(aH_e) = \Phi(f)(a) = \phi'(aH_e)f(\psi(a)). \tag{2.8}$$

We shall show that $\phi'^{-1}(1)$ is an open subset of H/H_e. Let $a \in \phi'^{-1}(1)$. By Lemma 2.5, there is an idempotent $1_{\psi'(a)G_0}$ of $A(G/G_e)$ where $\psi'(a)G_0$ is an open compact neighborhood of $\psi'(a)$. Since $T_q(1_{\psi'(a)G_0}) \in A(H/H_e) \subset C_0(H/H_e)$, the space of all continuous functions on H/H_e vanishing at infinity, there exists an open neighborhood V of a in H/H_e such that if $b \in q_H^{-1}(V)$, then

$$|T_q(1_{\psi'(a)G_0})(a) - T_q(1_{\psi'(a)G_0})(bH_e)| \le \frac{1}{2}.$$

We have

$$|1 - \phi'(bH_e)1_{\psi'(a)G_0}(\psi(b))| = |T_q(1_{\psi'(a)G_0})(a) - T_q(1_{\psi'(a)G_0})(bH_e)| \le \frac{1}{2}.$$

Since either $\phi'(bH_e)1_{\psi'(a)G_0}(\psi(b)) = 1$ or $\phi'(bH_e)1_{\psi'(a)G_0}(\psi(b)) = 0$, this implies that

$$\phi'(bH_e)1_{\psi'(a)G_0}(\psi(b)) = 1.$$

Hence, we have $\phi'(bH_e) = 1$ for any $b \in q_H^{-1}(V)$. Thus, $V \subset \phi'^{-1}(1)$. It follows that $\phi'^{-1}(1)$ is an open subset of H/H_e. Let us recall that $\psi' : q_H(U) \to G/G_e$ is a continuous map. Applying (2.8), we have

$$\begin{aligned} T_q(f)(aH_e) &= \phi'(aH_e)f(\psi(a)) \\ &= \begin{cases} f(\psi'(aH_e)), & \text{if } a \in U, \\ 0, & \text{if } a \in H \backslash U, \end{cases} \end{aligned}$$

for any $f \in A(G/G_e)$ and $a \in H$. As we have

$$T_q(\varphi_G(f))(aH_e) = \varphi_H \circ T \circ \varphi_G^{-1}(\varphi_G(f))(aH_e) = \varphi_H \circ T(f)(aH_e) = Tf(a)$$

for any $f \in A_I(G)$ and $a \in H$, we get

$$Tf(a) = \begin{cases} \varphi_G(f)(\psi'(aH_e)), & \text{if } a \in U, \\ 0, & \text{if } a \in H \backslash U. \end{cases} \qquad \blacksquare$$

3 Idempotent preserving bijections on $A_I(G)$

In this section, we assume that the bounded linear operator $T : A(G) \to B(H)$ satisfies that $T(I(G)) \subset I(H)$ and $T|_{A_I(G)}$ is a bijection onto $A_I(H)$.

Theorem 3.1 *Let G and H be two locally compact groups, and let $T : A(G) \to B(H)$ be a bounded complex linear operator. Suppose that the operator T satisfies that $T(I(G)) \subset I(H)$ and $T|_{A_I(G)} : A_I(G) \to A_I(H)$ is bijective. Then there exists a homeomorphism $\psi : H/H_e \to G/G_e$ such that*

$$Tf(a) = f([\psi(aH_e)]_{G_e})$$

for all $f \in A_I(G)$ and $a \in H$.

Proof Since $T|_{A_I(G)}$ is a bijective linear map, and φ_G and φ_H are isometric isomorphisms, by the proof of Proposition 2.1, we have $T_q := \varphi_H \circ T|_{A_I(G)} \circ \varphi_G^{-1}$ is an isomorphism from $A(G/G_e)$ onto $A(H/H_e)$.

Applying Theorem 2.15, there are an open subset U of H and a continuous map ψ from an open subset $q_H(U)$ of H/H_e into G/G_e such that

$$T_q(f)(a) = \begin{cases} f(\psi(a)), & \text{if } a \in q_H(U), \\ 0, & \text{if } a \in (H/H_e) \backslash q_H(U), \end{cases}$$

for any $f \in A(G/G_e)$. Since $T_q : A(G/G_e) \to A(H/H_e)$ is surjective and the Fourier algebra $A(H/H_e)$ separates the points in H/H_e, we have $q_H(U) = H/H_e$. Thus, $U = H$ and we have

$$T_q(f)(a) = f(\psi(a))$$

for every $f \in A(G/G_e)$ and $a \in H/H_e$. For any $h \in I(H)$, there exists $h_q \in A(H/H_e)$ with $h_q^2 = h_q$ such that

$$\varphi_H(h) = h_q.$$

Since T_q is bijective, there exists $f_q \in A(G/G_e)$ such that

$$T_q(f_q) = h_q.$$

Moreover, since T_q is an algebraic homomorphism, we have $T_q(f_q^2) = (T_q(f_q))^2 = h_q^2 = h_q = T_q(f_q)$. By the injectivity of T_q, we get $f_q^2 = f_q$. On the other hand, as φ_G is an isometric isomorphism from $A_I(G)$ onto $A(G/G_e)$, there exists $f \in I(G)$ such that

$$\varphi_G(f) = f_q.$$

https://doi.org/10.4153/S0008439523000395 Published online by Cambridge University Press

Hence, we have

$$T(f) = (\varphi_H^{-1} \circ T_q \circ \varphi_G)(f) = \varphi_H^{-1} \circ T_q(f_q) = \varphi_H^{-1}(h_q) = h.$$

This implies that $T(I(G)) = I(H)$. In particular, we have $T^{-1}(I(H)) \subset I(G)$. Thus, we can apply similar arguments to $T|_{A_I(G)}^{-1} : A_I(H) \to A_I(G)$ and to $T_q^{-1} = \varphi_G \circ T|_{A_I(G)}^{-1} \circ \varphi_H^{-1}$ on $A(H/H_e)$, and we can then define a continuous map $\tilde{\psi} : G/G_e \to H/H_e$ such that

$$T_q^{-1}(g)(b) = g(\tilde{\psi}(b))$$

for any $g \in A(H/H_e)$ and $b \in G/G_e$.

For any $g \in A(H/H_e)$ and $a \in H/H_e$, we have

$$g(a) = T_q(T_q^{-1}g)(a) = g(\tilde{\psi}(\psi(a))).$$

Since the Fourier algebra $A(H/H_e)$ separates points in H/H_e, we get

$$a = \tilde{\psi}(\psi(a)) \quad \text{for} \quad a \in H/H_e . \tag{3.1}$$

Moreover, we obtain

$$f(b) = T_q^{-1}(T_q f)(b) = f(\psi(\tilde{\psi}(b))),$$

for any $f \in A(G/G_e)$ and $b \in G/G_e$. Similarly, as $A(G/G_e)$ separates points in G/G_e, we have

$$b = \psi(\tilde{\psi}(b)) \quad \text{for} \quad b \in G/G_e . \tag{3.2}$$

By (3.1) and (3.2), we have that $\psi : H/H_e \to G/G_e$ is a bijection and $\tilde{\psi} = \psi^{-1}$. Let us recall that ψ and $\tilde{\psi}$ are continuous on H/H_e and G/G_e, respectively. As $\tilde{\psi} = \psi^{-1}$, we have that ψ is a homeomorphism. In addition, we obtain

$$T_q(f)(a) = f(\psi(a)) \quad \text{for } f \in A(G/G_e),\ a \in H/H_e .$$

Since $T = \varphi_H^{-1} \circ T_q \circ \varphi_G$, we get

$$Tf(a) = f([\psi(aH_e)]_{G_e})$$

for all $f \in A_I(G)$ and $a \in H$. ∎

Note that the bijectivity in Theorem 3.1 is an essential assumption for the function $\psi : H/H_e \to G/G_e$ to be a homeomorphism.

Example 3.2 Let $G = \{1, 2\}$ be a multiplicative group equipped with the discrete topology. Let $H = \{0\}$ be the trivial group. We define $T : A(G) \to A(H)$ by $Tf(0) = f(1)$ for any $f \in A(G)$. Then T is a bounded complex linear operator on $A(G)$ and for any $1_Y \in A(G)$, $T(1_Y) = 1_H$ if $1 \in Y$; otherwise, $T(1_Y) = 0$. Thus, $T(I(G)) = I(H)$. On the other hand, $T(1_{\{1\}}) = 1_H = T(1_G)$, this implies that $T|_{A_I(G)} : A_I(G) \to A_I(H)$ is not injective. In addition, $\psi : H/H_e = H \to G/G_e = G$ satisfying

$$\psi(0) = 1.$$

is not a homeomorphism.

https://doi.org/10.4153/S0008439523000395 Published online by Cambridge University Press

Under additional assumptions such as positivity or contractivity on T, a characterization of linear idempotent preserving maps between two Fourier algebras follows from Corollaries 2.12 and 2.13. Note that since the continuous map ψ in the following two corollaries is either a group isomorphism or an anti-isomorphism, we naturally have $f \circ \psi \in A_I(H)$ for any $f \in A_I(G)$ (see [25]); thus, we obtain a necessary and sufficient condition for the idempotent preserving operator T on $A_I(G)$.

Corollary 3.3 *Let G and H be two locally compact groups. A surjective complex linear contraction $T : A_I(G) \to A_I(H)$ satisfies $T(I(G)) \subset I(H)$ if and only if there exist a continuous group isomorphism or anti-isomorphism $\psi : H/H_e \to G/G_e$ and an element $b \in G$ such that*

$$Tf(a) = f(b[\psi(aH_e)]_{G_e})$$

for all $f \in A_I(G)$ and $a \in H$.

Proof Since the operator T defined on $A_I(G)$ is contractive and satisfies $T(I(G)) \subset I(H)$, we have a characterization of T by Corollary 2.13, and now the result follows from Theorem 3.1 as T is onto $A_I(H)$. ∎

Similarly, if the operator $T : A_I(G) \to A_I(H)$ is positive and preserves the idempotents, we have a characterization of T by Corollary 2.12 and thus the following corollary follows from Theorem 3.1.

Corollary 3.4 *Let G and H be two locally compact groups. A positive bounded complex linear bijection $T : A_I(G) \to A_I(H)$ satisfies $T(I(G)) \subset I(H)$ if and only if there exists a continuous group isomorphism or anti-isomorphism $\psi : H/H_e \to G/G_e$ such that*

$$Tf(a) = f([\psi(aH_e)]_{G_e})$$

for all $f \in A_I(G)$ and $a \in H$.

We will end our article with a special case when the groups are totally disconnected. In such case, the *idempotent preserving* operators between Fourier algebras recover the results of algebraic homomorphisms. More precisely, Theorem 2.15 and Corollaries 3.3 and 3.4 are followed by the following remark.

Remark 3.5 Suppose that G and H are totally disconnected locally compact groups. Let $T : A(G) \to A(H)$ be a bounded complex linear operator satisfying $T(I(G)) \subset I(H)$. Then there exists a continuous map ψ from an open subset U of H into G such that

$$Tf(a) = \begin{cases} f(\psi(a)), & \text{if } a \in U, \\ 0, & \text{if } a \in H\setminus U, \end{cases}$$

for any $f \in A(G)$ and $a \in H$. In addition, if T is a surjective contraction or T is a positive bijection, then it is equivalent to $Tf = f \circ (b\psi)$ for some $b \in G$ or $Tf = f \circ \psi$,

https://doi.org/10.4153/S0008439523000395 Published online by Cambridge University Press

respectively, where $\psi : H \to G$ is a continuous group isomorphism or group anti-isomorphism, and $b\psi : H \to G$ is defined by $b\psi(\cdot) := b\psi(\cdot)$; in particular, T is an algebraic homomorphism.

References

[1] M. Anoussis, G. K. Eleftherakis, and A. Katavolos, *Idempotents of large norm and homomorphisms of Fourier algebras*. Stud. Math. **267**(2022), no. 1, 109–120.

[2] M. E. B. Bekka, A. T. Lau, and G. Schlichting, *On invariant subalgebras of the Fourier–Stieltjes algebra of a locally compact group*. Math. Ann. **294**(1992), 513–522.

[3] P. Botta, S. Pierce, and W. Watkins, *Linear transformations that preserve the nilpotent matrices*. Pac. J. Math. **104**(1983), 39–46.

[4] M. Brešar and P. Šemrl, *Mappings which preserve idempotents, local automorphisms, and local derivations*. Can. J. Math. **45**(1993), no. 3, 483–496.

[5] M. Brešar and P. Šemrl, *Linear transformations preserving potent matrices*. Proc. Amer. Math. Soc. **119**(1993), no. 1, 81–86.

[6] P. J. Cohen, *On homomorphisms of the group algebras*. Amer. J. Math. **82**(1960), 213–226.

[7] M. Daws, *Completely bounded homomorphisms of the Fourier algebra revisited*. J. Group Theory **25**(2022), no. 3, 579–600.

[8] J. Dieudonné, *Sur Une généralisation du groupe orthogonal à quatre variables*. Arch. Math. **1**(1948), 282–287.

[9] J. Dixon, *Rigid embeddings of simple groups in the general linear group*. Can. J. Math. **29**(1977), 384–391.

[10] G. Dolinar, *Maps on matrix algebras preserving idempotents*. Linear Algebra Appl. **371**(2003), 287–300.

[11] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*. Bull. Soc. Math. France **92**(1964), 181–236.

[12] B. Forrest, *Fourier analysis on coset spaces*. Rocky Mountain J. Math. **28**(1998), no. 1, 173–190.

[13] A. Guterman, C.-K. Li, and P. Šemrl, *Some general techniques on linear preserver problems*. Linear Algebra Appl. **315**(2000), 61–81.

[14] B. Host, *La théorème des idempotents dans $B(G)$*. Bull. Soc. Math. France **114**(1986), 215–223.

[15] M. Ilie, *On Fourier algebra homomorphisms*. J. Funct. Anal. **213**(2004), 88–110.

[16] M. Ilie and N. Spronk, *Completely bounded homomorphisms of the Fourier algebras*. J. Funct. Anal. **225**(2005), no. 2, 480–499.

[17] M. Ilie and N. Spronk, *The algebra generated by idempotents in a Fourier–Stieltjes algebra*. Houst. J. Math. **33**(2007), 1131–1145.

[18] H. G. Jacob, *Coherence invariant mappings on Kronecker products*. Amer. J. Math. **77**(1955), 177–189.

[19] E. Kaniuth and A. T.-M. Lau, *Fourier and Fourier–Stieltjes algebras on locally compact groups*, Mathematical Surveys and Monographs, 231, American Mathematical Society, Providence, RI, 2018.

[20] H. Le Pham, *Contractive homomorphisms of Fourier algebras*. Bull. Lond. Math. Soc. **42**(2010), 937–947.

[21] C.-K. Li, M.-C. Tsai, Y.-S. Wang, and N.-C. Wong, *Additive Hermitian idempotent preservers between operator algebras*. J. Math. Anal. Appl. **505**(2022), no. 2, 125522.

[22] M. Marcus and B. Moyls, *Transformations on tensor product spaces*. Pac. J. Math. **9**(1959), 1215–1221.

[23] J. Mudge and H. Le Pham, *Idempotents with small norms*. J. Funct. Anal. **270**(2016), 4597–4603.

[24] M. Takesaki and N. Tatsuuma, *Duality and subgroups*. Ann. Math. **93**(1971), 344–364.

[25] M. E. Walter, *W^*-algebras and nonabelian harmonic analysis*. J. Funct. Anal. **11**(1972), 17–38.

Mathematical Sciences Research Centre, Queen's University Belfast, Belfast BT7 1NN, UK
e-mail: y.lin@qub.ac.uk

Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181, Japan
e-mail: shiho-oi@math.sc.niigata-u.ac.jp

https://doi.org/10.4153/S0008439523000395 Published online by Cambridge University Press