Consequences

The set of composite numbers can be expressed, save for 1 and 4, as: $\{n \in \mathbb{N} : n^2 \mid n!\}$.

The prime counting function can be expressed for all $n \geqslant 4$ as:

$$\pi(n) = \frac{4}{3} + \sum_{j=3}^{n-1} \frac{j+1}{j} \left\{ \frac{j!}{j+1} \right\}.$$

Discussion

The number 4, which is the smallest composite number, is exceptional as 'it is the only composite n that does not divide $(n - 1)!$'[2]. This is the reason $C(n)$ indicates all primes and composites apart from 4. If C is used as a measure of *primeness*, then $C(4) = 2/3$ indicates that 4 is the 'least composite' composite in that measure.

The behaviour of $C(n)$ for large n can be considered through the following plausibility argument. First, note that $n \neq 4$ is a prime number only if $n!$ has no divisor that is a multiple of n^2. Erdős et al. [3] have shown that the number of divisors of $n!$ grows faster than any power of n for large n. This entails that the probability that $n!$ has no divisor that is a multiple of n^2, which is the probability that n is prime, tends to 0 as $n \to \infty$.

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107.17 Two curios related to lattice polygons

A *lattice polygon* is a planar polygon in $\mathbb{R}^d$ all of whose vertices have integer coordinates. It is regular if its sides and angles are all equal. The fundamental theorem in this area is:

Theorem A

(i) In $\mathbb{R}^2$, the only regular lattice polygons are squares.

(ii) In $\mathbb{R}^d$, $d \geqslant 3$, the only regular lattice polygons are triangles, squares and hexagons.

Theorem A is blessed with several neat proofs such as the ingenious geometric one in [1] and the one using algebra and trigonometry in [2]. The latter relies on the following result, a very succinct proof of which is given in [3]. This is reprised in [2] and in the recent article, [4].

Theorem B

If q is rational, then the only possible rational values of $\cos(q\pi)$ are 0, $\pm\frac{1}{2}$, ± 1.

Curio 1

Our first curio starts with the striking construction of a regular dodecagon given in [5] and reproduced in Figure 1.

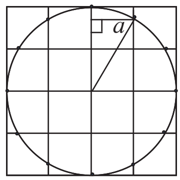

FIGURE 1

The vertices of the dodecagon are the intersection points with the lines of the regular grid. Skipping vertices also constructs a regular hexagon, equilateral triangle and square. Can other regular polygons be realised by such a construction? For an n-sided polygon, the triangle shown in Figure 1 would have $\alpha = \frac{\pi}{2} - \frac{2\pi}{n} = \pi\left(\frac{1}{2} - \frac{2}{n}\right)$ with $\cos\alpha$ rational. In the range $4 \leqslant n < \infty$, Theorem B means that the only possibilities are $\frac{1}{2} - \frac{2}{n} = 0$ with $n = 4$ (which is subsumed in Figure 1) and $\frac{1}{2} - \frac{2}{n} = \frac{1}{3}$ with $n = 12$, as in Figure 1. So Figure 1 is, indeed an isolated curiosity.

Curio 2

Our second curio begins with a lovely observation from [6]: the points whose coordinates are the six permutations of (1, 2, 3) form a regular planar hexagon in $\mathbb{R}^3$. In what follows, the role of the underlying group is important so, for brevity, we will say that: "the elements of the symmetric group S_3, thought of as points in $\mathbb{R}^3$, form a regular planar hexagon". Clearly the points in S_3 lie in the plane $x + y + z = 6$ and, if we look down the normal to the plane in the direction $\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$, we see the hexagon in Figure 2 with side-length $\sqrt{2}$ and short diagonals of length $\sqrt{6}$, meaning that it is indeed regular.

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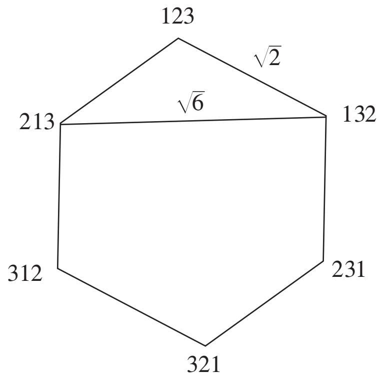

FIGURE 2: For clarity, we use have contracted the labelling of the vertices.

The alternating group A_3 forms the equilateral triangle with vertices 123, 231, 312. The hexagon has area $3\sqrt{3}$ and it is a nice exercise with vectors to show that it can be realised as the mid-plane section through the cube of side-length 2 with vertices as in Figure 3.

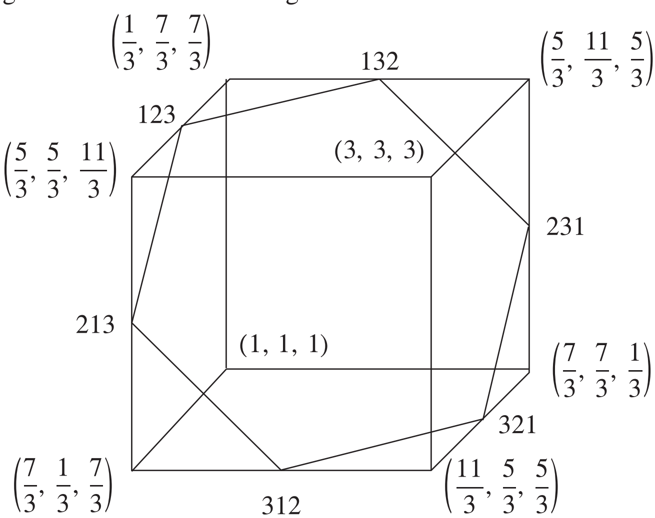

FIGURE 3

What about S_n for $n \geqslant 4$?

Theorem A shows that the elements of S_n, thought of as points in $\mathbb{R}^n$, do not form a regular planar $n!$-gon for $n \geqslant 4$. An alternative proof of this is to show that S_4 is non-planar: it follows, by embedding S_4 in S_n, that S_n is non-planar for $n > 4$ as well.

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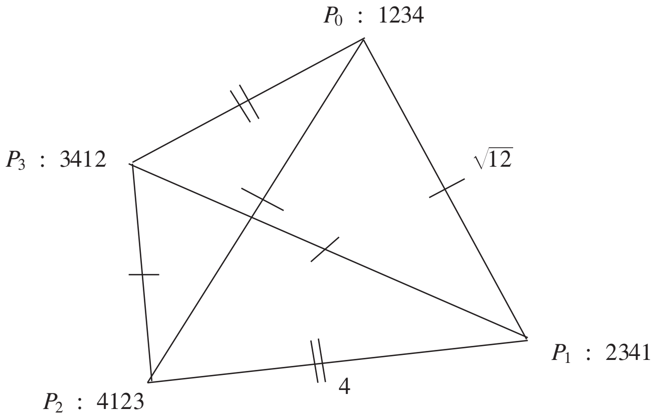

FIGURE 4

To do this, we show that the cyclic subgroup of S_4 of order 4 given by the permutations {1234, 2341, 3412, 4123} is non-planar, Figure 4. One way to see this is to find the volume, V, of $P_0P_1P_2P_3$ from the 6 inter-vertex distances, calculated using Pythagoras in Figure 4. The Cayley-Menger determinant discussed in the Appendix shows that

$$V^2 = \frac{1}{288}\begin{vmatrix} 2P_0P_1^2 & P_0P_1^2 + P_0P_2^2 - P_1P_2^2 & P_0P_1^2 + P_0P_3^2 - P_1P_3^2 \\ P_0P_1^2 + P_0P_2^2 - P_1P_2^2 & 2P_0P_2^2 & P_0P_2^2 + P_0P_3^2 - P_2P_3^2 \\ P_0P_1^2 + P_0P_3^2 - P_1P_3^2 & P_0P_2^2 + P_0P_3^2 - P_2P_3^2 & 2P_0P_3^2 \end{vmatrix}$$

$$= \frac{1}{288}\begin{vmatrix} 24 & 8 & 16 \\ 8 & 24 & 16 \\ 16 & 16 & 32 \end{vmatrix} = \frac{256}{9} \neq 0.$$

Alternatively, we may readily show that the vectors

$$\overrightarrow{P_0P_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}, \overrightarrow{P_0P_3} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \end{pmatrix}, \overrightarrow{P_0P_2} = \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

are linearly independent. This, together with the fact that the points of S_4 lie in the 3-dimensional hyperplane $x_1 + x_2 + x_3 + x_4 = 10$, means that they form a 3-dimensional figure in $\mathbb{R}^4$. The same argument using the cyclic subgroup given by the permutations $\{123\ldots n, 23\ldots n1, 34\ldots n12, \ldots, n12\ldots(n-1)\}$ shows that the points of S_n form an $(n - 1)$-dimensional figure in $\mathbb{R}^n$.

The tetrahedron in Figure 4 is a 'humbug' shape with two 90° dihedral angles and four congruent isosceles triangle faces. It can be constructed by bringing together two copies of the folded rhombus made from an A4-shaped sheet of paper as shown in Figure 5.

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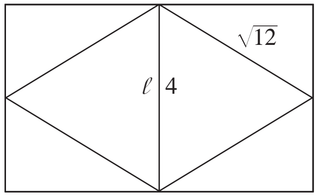

FIGURE 5: Two copies of the rhombus formed from midpoints of an A4-shaped rectangle, when folded through 90° along ℓ, join to form the tetrahedron in Figure 4.

As an aside, we remark that Klein's 4-group can be realised as the subgroup of 4 given by the permutations 1234, 2143, 3412, 4321. This is also the symmetry group of a rectangle with vertices labelled 1, 2, 3, 4 and, rather agreeably, the four permutations involved themselves form a planar rectangle in $\mathbb{R}^4$ as shown in Figure 6.

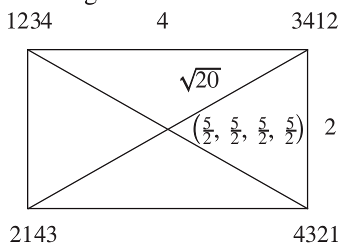

FIGURE 6

Appendix

The Cayley-Menger determinant gives the n-dimensional volume of a simplex formed by $n + 1$ points $P_0, P_1, \ldots, P_n$ in $\mathbb{R}^d$ $(d \geqslant n)$. For $n = 2$, it produces Heron's formula: for $n = 3$ it gives Euler's tetrahedron formula for the volume, V, of a tetrahedron. The vector cognate of the latter is the formula $V = \frac{1}{6}\left|\mathbf{a}.(\mathbf{b} \times \mathbf{c}\right|$, where $\mathbf{a} = \overrightarrow{P_0P_1}$, $\mathbf{b} = \overrightarrow{P_0P_2}$, $\mathbf{c} = \overrightarrow{P_0P_3}$. If the matrix A has columns $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ then $36V^2 = \left|A^tA\right|$, where the elements of A^tA are dot products. Replacing, say, $\mathbf{a}.\mathbf{b}$ by $\frac{1}{2}\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2\right)$, gives Euler's tetrahedron formula and a good feel for the form of the general Cayley-Menger determinant in higher dimensions, as described in [7].

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107.18 A two-variable approach to some standard optimisation problems

Introduction

Problems of the following type are staples of introductory courses on differentiation.

> *A solid right circular cylinder has fixed volume. Show that its total surface area is minimised when the height is twice the radius.*

With standard notation for the attributes of the cylinder, the usual method of solution is to use $\pi r^2 h = V_0$ (fixed) to eliminate h in the expression for total surface area, $S = 2\pi r^2 + 2\pi r \cdot \dfrac{V_0}{\pi r^2}$. The condition $\dfrac{dS}{dr} = 0$ leads to $r = \sqrt[3]{\dfrac{V_0}{2\pi}}$ and routine algebra then identifies $h = \sqrt[3]{\dfrac{4V_0}{\pi}} = 2r$. In such an approach, the pleasant minimising ratio for $\frac{h}{r}$ appears serendipitously and almost as an afterthought. In addition, the fact that the dual problem (of maximising the volume of a solid cylinder of fixed surface area) is solved by the same value of $\frac{h}{r}$ is obscured by the algebra. This point was raised by Prithwijit De and Des MacHale in their Note, [1]; re-reading this stimulated the reflections that follow. Observe that we shall not formally check whether our optimising values correspond to maxima or minima: in any specific situation (such as the above problem), consideration of extreme cases (e.g. $r \to 0$, $r \to \infty$) usually makes this clear.

An alternative approach

We will see here that a multivariable approach yields a rich insight into this type of problem and adds to the repertoire of methods of solution. Suppose then that $V(r, h) = V_0$ is fixed. This equation implicitly defines $h = h(r)$ so, by the chain rule and writing $V_r = \frac{\partial V}{\partial r}$, etc., we have $V_r + h'(r) V_h = 0$.

Similarly, the stationary values of $S(r, h) = S\big(r, h(r)\big)$ occur when $S_r + h'(r) S_h = 0$.

Eliminating $h'(r)$ from these two equations gives what we shall refer to as the Key Equation: $V_r S_h = V_h S_r$.